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MATHEMATICS

eBook included

PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 2

STUDENT BOOK

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PURE MATHEMATICS 2

Student Book

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Endorsement Statement
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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

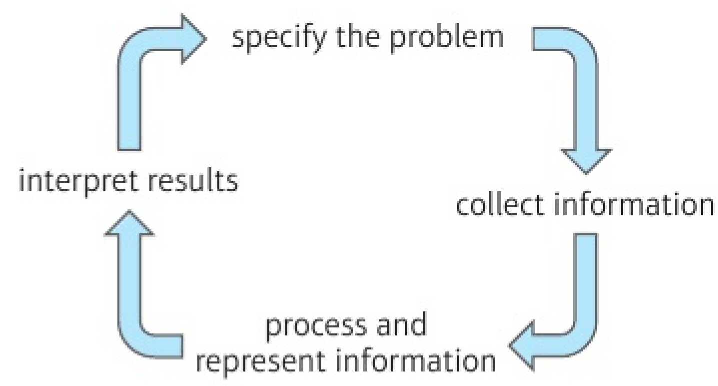
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

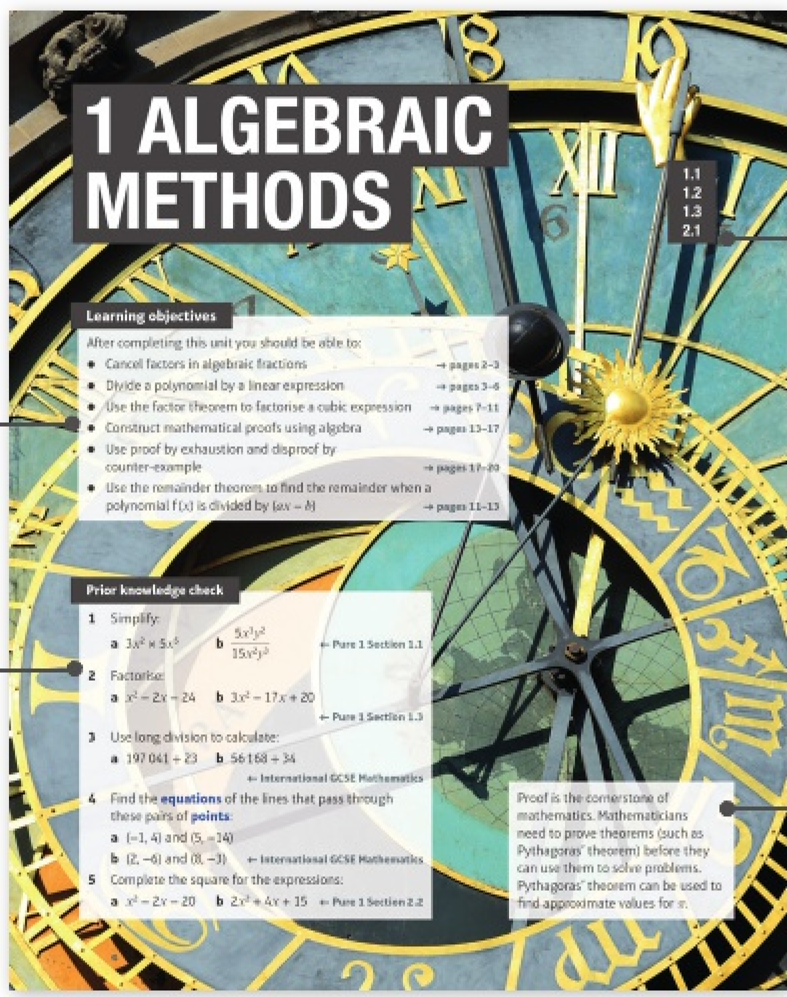
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book



Each chapter starts with a list of Learning objectives

The *Prior knowledge check* helps make sure you are ready to start the chapter

Glossary terms will be identified by bold blue text on their first appearance.

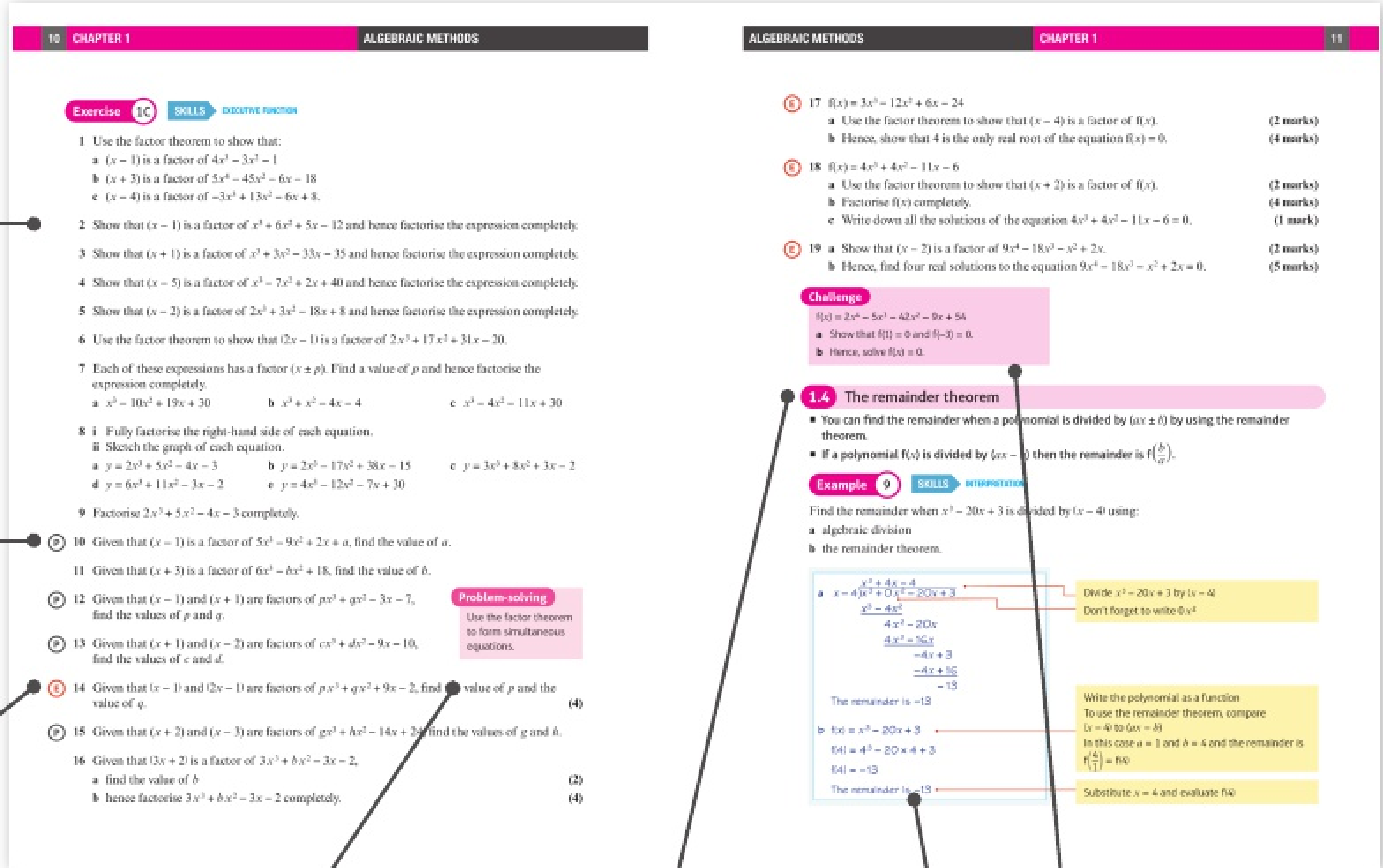
Each chapter is mapped to the specification content for easy reference

The real world applications of the maths you are about to learn are highlighted at the start of the chapter.

Transferable skills are signposted where they naturally occur in the exercises and examples

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams



Problem-solving boxes provide hints, tips and strategies, and Watch out boxes highlight areas where students often lose marks in their exams

Each section begins with explanation and key learning points

Challenge boxes give you a chance to tackle some more difficult questions

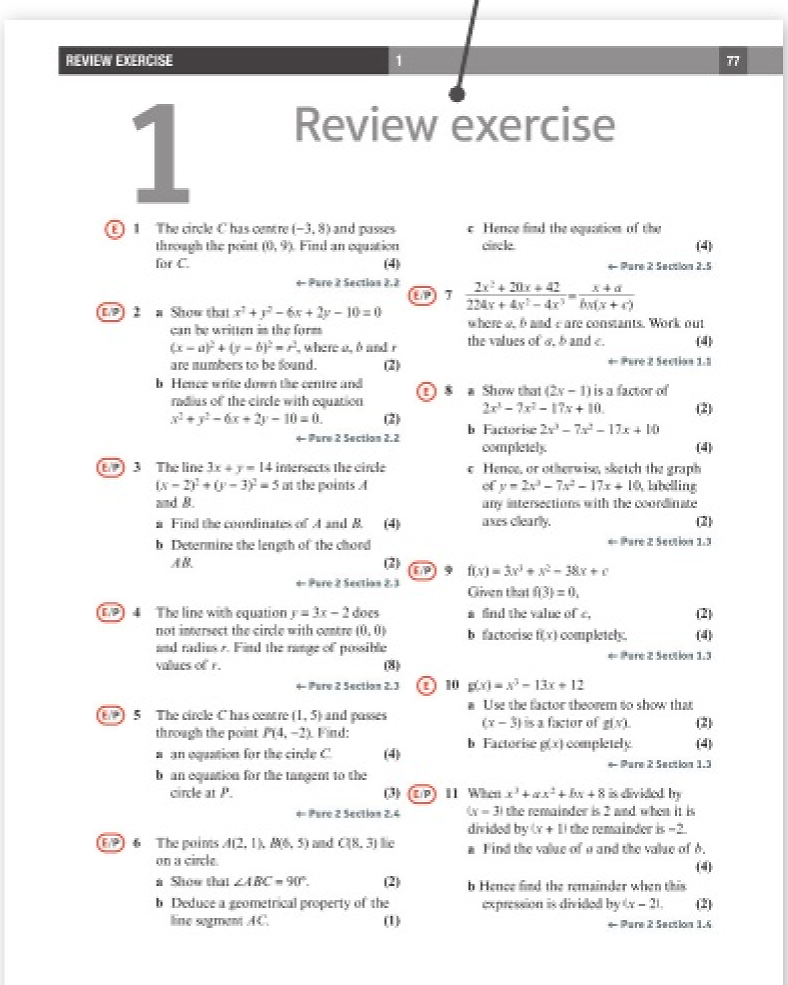
Step-by-step worked examples focus on the key types of questions you'll need to tackle

Exam-style questions are flagged with Ⓔ

Problem-solving questions are flagged with Ⓟ

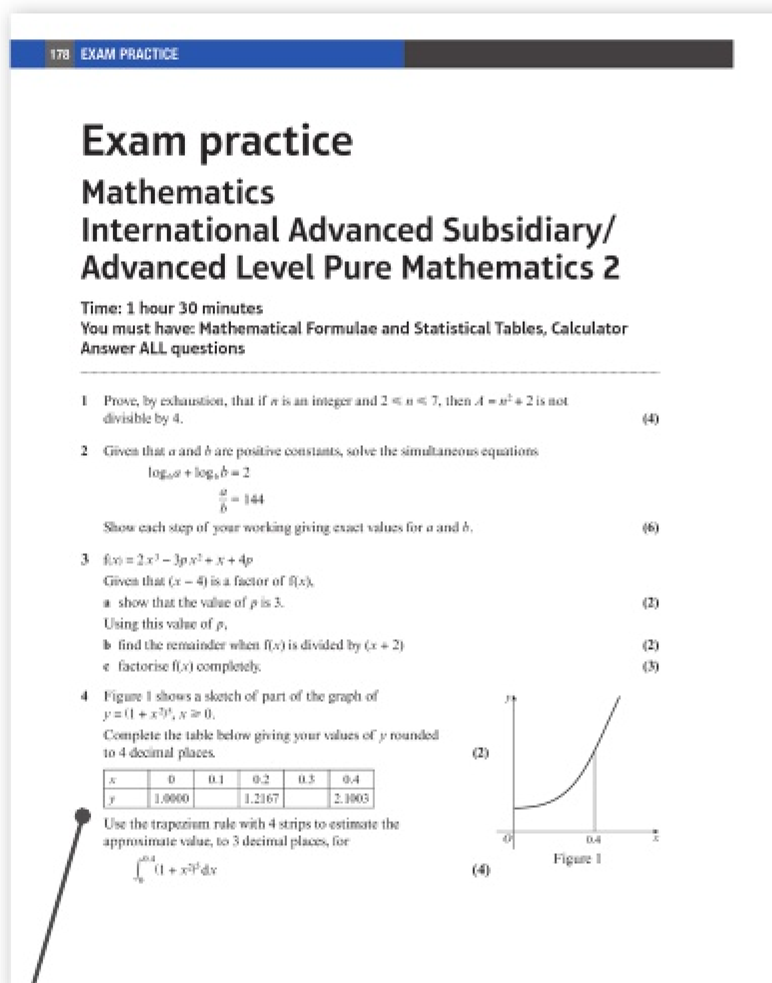
Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions



REVIEW EXERCISE 1 77

1 Review exercise



178 EXAM PRACTICE

Exam practice

Mathematics
International Advanced Subsidiary/
Advanced Level Pure Mathematics 2

Time: 1 hour 30 minutes
You must have: Mathematical Formulae and Statistical Tables, Calculator
Answer ALL questions

A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Pure Mathematics 2 (P2) is a **compulsory** unit in the following qualifications:

International Advanced Subsidiary in Mathematics

International Advanced Subsidiary in Pure Mathematics

International Advanced Level in Mathematics

International Advanced Level in Pure Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
P2: Pure Mathematics 2 Paper code WMA12/01	$33\frac{1}{3}$ % of IAS $16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January, June and October First assessment June 2019

IAS: International Advanced Subsidiary, IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

P2	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	5–10	5–10	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, –, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

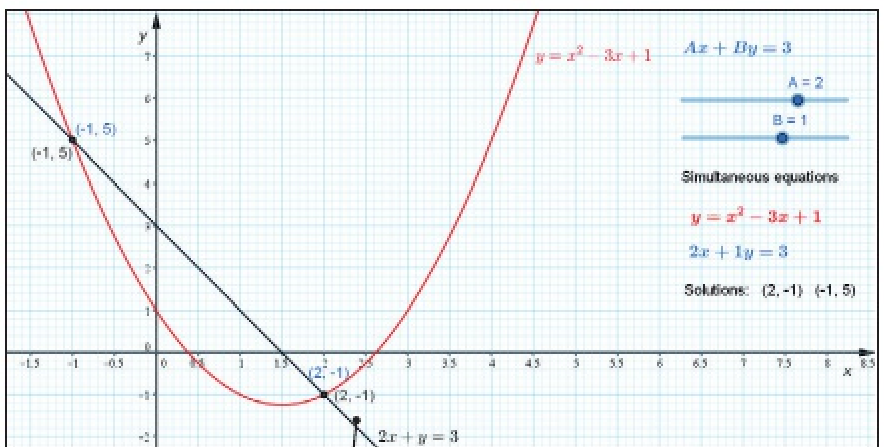
SolutionBank provides a full worked solution for every question in the book. Download all the solutions as a PDF or quickly find the solution you need online.

Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

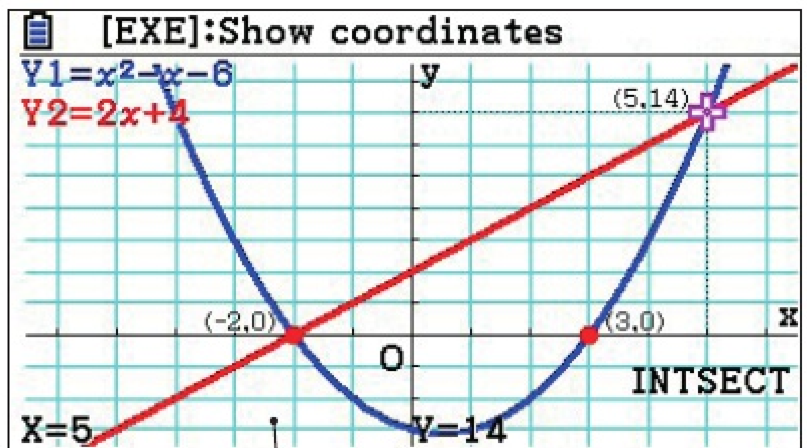
Online Find the point of intersection graphically using technology.

GeoGebra-powered interactives



Interact with the maths you are learning using GeoGebra's easy-to-use tools

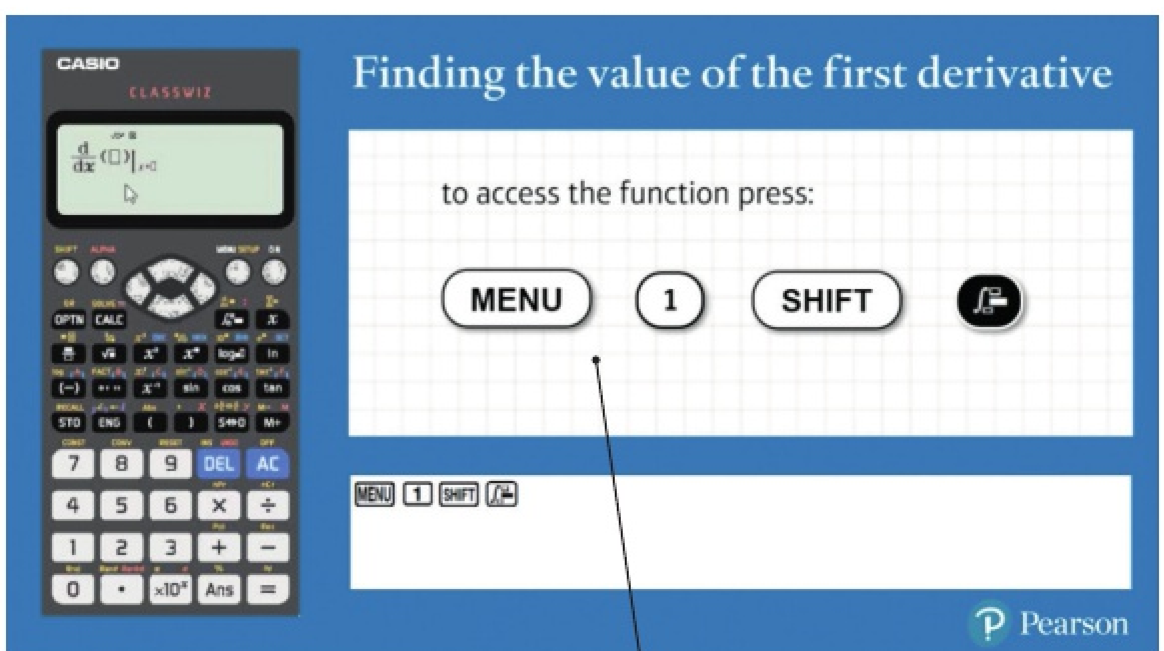
Graphic calculator interactives



Explore the maths you are learning and gain confidence in using a graphic calculator

Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ^nC_r and power functions on your calculator.

1 ALGEBRAIC METHODS

1.1
1.2
1.3
2.1

Learning objectives

After completing this unit you should be able to:

- Cancel factors in algebraic fractions → pages 2–3
- Divide a polynomial by a linear expression → pages 3–6
- Use the factor theorem to factorise a cubic expression → pages 7–11
- Construct mathematical proofs using algebra → pages 13–17
- Use proof by exhaustion and disproof by counter-example → pages 17–20
- Use the remainder theorem to find the remainder when a polynomial $f(x)$ is divided by $(ax - b)$ → pages 11–13

Prior knowledge check

1 Simplify:

a $3x^2 \times 5x^5$ **b** $\frac{5x^3y^2}{15x^2y^3}$ ← Pure 1 Section 1.1

2 Factorise:

a $x^2 - 2x - 24$ **b** $3x^2 - 17x + 20$

← Pure 1 Section 1.3

3 Use long division to calculate:

a 197 041 ÷ 23 **b** 56 168 ÷ 34

← International GCSE Mathematics

4 Find the **equations** of the lines that pass through these pairs of **points**:

a (−1, 4) and (5, −14)

b (2, −6) and (8, −3) ← International GCSE Mathematics

5 Complete the square for the expressions:

a $x^2 - 2x - 20$ **b** $2x^2 + 4x + 15$ ← Pure 1 Section 2.2

Proof is the cornerstone of mathematics. Mathematicians need to prove theorems (such as Pythagoras' theorem) before they can use them to solve problems. Pythagoras' theorem can be used to find approximate values for π.

1.1 Algebraic fractions

You can simplify algebraic fractions using division.

- **When simplifying an algebraic fraction, where possible factorise the numerator and denominator and then cancel common factors.**

Factorise → Cancel common factor

$$\frac{5x^2 - 245}{2x^2 - 15x + 7} = \frac{5(x + 7)(x - 7)}{(2x - 1)(x - 7)} = \frac{5(x + 7)}{2x - 1}$$

Example 1 SKILLS PROBLEM-SOLVING

Simplify these fractions:

a $\frac{7x^4 - 2x^3 + 6x}{x}$ **b** $\frac{(x + 7)(2x - 1)}{(2x - 1)}$ **c** $\frac{x^2 + 7x + 12}{(x + 3)}$ **d** $\frac{x^2 + 6x + 5}{x^2 + 3x - 10}$ **e** $\frac{2x^2 + 11x + 12}{(x + 3)(x + 4)}$

a $\frac{7x^4 - 2x^3 + 6x}{x}$

$= \frac{7x^4}{x} - \frac{2x^3}{x} + \frac{6x}{x}$ — Divide each part of the numerator by x.

$= 7x^3 - 2x^2 + 6$

b $\frac{(x + 7)(2x - 1)}{(2x - 1)} = x + 7$ — Simplify by cancelling the common factor of $(2x - 1)$.

c $\frac{x^2 + 7x + 12}{(x + 3)} = \frac{(x + 3)(x + 4)}{(x + 3)}$ — Factorise: $x^2 + 7x + 12 = (x + 3)(x + 4)$.

$= x + 4$ — Cancel the common factor of $(x + 3)$.

d $\frac{x^2 + 6x + 5}{x^2 + 3x - 10} = \frac{(x + 5)(x + 1)}{(x + 5)(x - 2)}$ — Factorise: $x^2 + 6x + 5 = (x + 5)(x + 1)$ and $x^2 + 3x - 10 = (x + 5)(x - 2)$.

$= \frac{x + 1}{x - 2}$ — Cancel the common factor of $(x + 5)$.

e $2x^2 + 11x + 12 = 2x^2 + 3x + 8x + 12$

$= x(2x + 3) + 4(2x + 3)$

$= (2x + 3)(x + 4)$ — Factorise $2x^2 + 11x + 12$

So $\frac{2x^2 + 11x + 12}{(x + 3)(x + 4)}$

$= \frac{(2x + 3)(x + 4)}{(x + 3)(x + 4)}$

$= \frac{2x + 3}{x + 3}$ — Cancel the common factor of $(x + 4)$.

Exercise 1A SKILLS PROBLEM-SOLVING

1 Simplify these fractions:

a $\frac{4x^4 + 5x^2 - 7x}{x}$ **b** $\frac{7x^5 - 5x^5 + 9x^3 + x^2}{x}$ **c** $\frac{-x^4 + 4x^2 + 6}{x}$

d $\dfrac{7x^5 - x^3 - 4}{x}$ **e** $\dfrac{8x^4 - 4x^3 + 6x}{2x}$ **f** $\dfrac{9x^2 - 12x^3 - 3x}{3x}$

g $\dfrac{7x^3 - x^4 - 2}{5x}$ **h** $\dfrac{-4x^2 + 6x^4 - 2x}{-2x}$ **i** $\dfrac{-x^8 + 9x^4 - 4x^3 + 6}{-2x}$

j $\dfrac{-9x^9 - 6x^6 + 4x^4 - 2}{-3x}$

2 Simplify these fractions as far as possible:

a $\dfrac{(x + 3)(x - 2)}{(x - 2)}$ **b** $\dfrac{(x + 4)(3x - 1)}{(3x - 1)}$ **c** $\dfrac{(x + 3)^2}{(x + 3)}$

d $\dfrac{x^2 + 10x + 21}{(x + 3)}$ **e** $\dfrac{x^2 + 9x + 20}{(x + 4)}$ **f** $\dfrac{x^2 + x - 12}{(x - 3)}$

g $\dfrac{x^2 + x - 20}{x^2 + 2x - 15}$ **h** $\dfrac{x^2 + 3x + 2}{x^2 + 5x + 4}$ **i** $\dfrac{x^2 + x - 12}{x^2 - 9x + 18}$

j $\dfrac{2x^2 + 7x + 6}{(x - 5)(x + 2)}$ **k** $\dfrac{2x^2 + 9x - 18}{(x + 6)(x + 1)}$ **l** $\dfrac{3x^2 - 7x + 2}{(3x - 1)(x + 2)}$

m $\dfrac{2x^2 + 3x + 1}{x^2 - x - 2}$ **n** $\dfrac{x^2 + 6x + 8}{3x^2 + 7x + 2}$ **o** $\dfrac{2x^2 - 5x - 3}{2x^2 - 9x + 9}$

E/P **3** $\dfrac{6x^3 + 3x^2 - 84x}{6x^2 - 33x + 42} = \dfrac{ax(x + b)}{x + c}$, where a, b and c are **constants**.

Work out the values of a, b and c. **(4 marks)**

1.2 Dividing polynomials

A **polynomial** is a **finite** expression with positive whole number **indices**.

- You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.
- You can use long division to divide a polynomial by $(ax \pm b)$, where a and b are constants.

Polynomials	Not polynomials
$2x + 4$	$\sqrt{x}$
$4xy^2 + 3x - 9$	$6x^{-2}$
8	$\dfrac{4}{x}$

Example 2

Divide $x^3 + 2x^2 - 17x + 6$ by $(x - 3)$.

①
$$\begin{array}{r} x^2 \\ x - 3\,\overline{)\,x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 17x \end{array}$$

Start by dividing the first term of the polynomial by x, so that $x^3 \div x = x^2$.

Next multiply $(x - 3)$ by x^2, so that $x^2 \times (x - 3) = x^3 - 3x^2$.

Now subtract, so that $(x^3 + 2x^2) - (x^3 - 3x^2) = 5x^2$.

Copy $-17x$.

②

$$\begin{array}{r} x^2 + 5x \phantom{{}-17x+6} \\ x-3\,\overline{)\,x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \phantom{{}-17x+6} \\ 5x^2 - 17x \phantom{{}+6} \\ \underline{5x^2 - 15x} \phantom{{}+6} \\ -2x + 6 \end{array}$$

Repeat the method. Divide $5x^2$ by x, so that $5x^2 \div x = 5x$.

Multiply $(x - 3)$ by $5x$, so that $5x \times (x - 3) = 5x^2 - 15x$.

Subtract, so that $(5x^2 - 17x) - (5x^2 - 15x) = -2x$.

Copy +6.

③

$$\begin{array}{r} x^2 + 5x - 2 \\ x-3\,\overline{)\,x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \phantom{{}-17x+6} \\ 5x^2 - 17x \phantom{{}+6} \\ \underline{5x^2 - 15x} \phantom{{}+6} \\ -2x + 6 \\ \underline{-2x + 6} \\ 0 \end{array}$$

Repeat the method. Divide $-2x$ by x, so that $-2x \div x = -2$.

Multiply $(x - 3)$ by -2, so that $-2 \times (x - 3) = -2x + 6$.

Subtract, so that $(-2x + 6) - (-2x + 6) = 0$.

No numbers left to copy, so you have finished.

So $\frac{x^3 + 2x^2 - 17x + 6}{x - 3} = x^2 + 5x - 2$

This is called the **quotient**.

Example 3 SKILLS INTERPRETATION

$f(x) = 4x^4 - 17x^2 + 4$

Divide $f(x)$ by $(2x + 1)$, giving your answer in the form $f(x) = (2x + 1)(ax^3 + bx^2 + cx + d)$.

Find $(4x^4 - 17x^2 + 4) \div (2x + 1)$

$$\begin{array}{r} 2x^3 - x^2 - 8x + 4 \\ 2x+1\,\overline{)\,4x^4 + 0x^3 - 17x^2 + 0x + 4} \\ \underline{4x^4 + 2x^3} \phantom{{}-17x^2+0x+4} \\ -2x^3 - 17x^2 \phantom{{}+0x+4} \\ \underline{-2x^3 - x^2} \phantom{{}+0x+4} \\ -16x^2 + 0x \phantom{{}+4} \\ \underline{-16x^2 - 8x} \phantom{{}+4} \\ 8x + 4 \\ \underline{8x + 4} \\ 0 \end{array}$$

So

$4x^4 - 17x^2 + 4 = (2x + 1)(2x^3 - x^2 - 8x + 4)$.

Use long division. Include the terms $0x^3$ and $0x$ when you write out $f(x)$.

You need to multiply $(2x + 1)$ by $2x^3$ to get the $4x^4$ term, so write $2x^3$ in the answer, and write $2x^3(2x + 1) = 4x^4 + 2x^3$ below. Subtract and copy the next term.

You need to multiply $(2x + 1)$ by $-x^2$ to get the $-2x^3$ term, so write $-x^2$ in the answer, and write $-x^2(2x + 1) = -2x^3 - x^2$ below. Subtract and copy the next term.

Repeat the method.

$(4x^4 - 17x^2 + 4) \div (2x + 1) = 2x^3 - x^2 - 8x + 4$.

Example 4

Find the **remainder** when $2x^3 - 5x^2 - 16x + 10$ is divided by $(x - 4)$.

$$\begin{array}{r} 2x^2 + 3x - 4 \\ x - 4 \overline{)\, 2x^3 - 5x^2 - 16x + 10} \\ \underline{2x^3 - 8x^2} \qquad\qquad\quad \\ 3x^2 - 16x \qquad \\ \underline{3x^2 - 12x} \qquad \\ -4x + 10 \\ \underline{-4x + 16} \\ -6 \end{array}$$

So the remainder is −6.

$(x - 4)$ is not a factor of $2x^3 - 5x^2 - 16x + 10$ as the remainder $\neq 0$.

This means you cannot write the expression in the form $(x - 4)(ax^2 + bx + c)$.

Exercise 1B

SKILLS INTERPRETATION

1 Write each polynomial in the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a $x^3 + 6x^2 + 8x + 3$ by $(x + 1)$
b $x^3 + 10x^2 + 25x + 4$ by $(x + 4)$
c $x^3 - x^2 + x + 14$ by $(x + 2)$
d $x^3 + x^2 - 7x - 15$ by $(x - 3)$
e $x^3 - 8x^2 + 13x + 10$ by $(x - 5)$
f $x^3 - 5x^2 - 6x - 56$ by $(x - 7)$

2 Write each polynomial in the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a $6x^3 + 27x^2 + 14x + 8$ by $(x + 4)$
b $4x^3 + 9x^2 - 3x - 10$ by $(x + 2)$
c $2x^3 + 4x^2 - 9x - 9$ by $(x + 3)$
d $2x^3 - 15x^2 + 14x + 24$ by $(x - 6)$
e $-5x^3 - 27x^2 + 23x + 30$ by $(x + 6)$
f $-4x^3 + 9x^2 - 3x + 2$ by $(x - 2)$

3 Divide:

a $x^4 + 5x^3 + 2x^2 - 7x + 2$ by $(x + 2)$
b $4x^4 + 14x^3 + 3x^2 - 14x - 15$ by $(x + 3)$
c $-3x^4 + 9x^3 - 10x^2 + x + 14$ by $(x - 2)$
d $-5x^5 + 7x^4 + 2x^3 - 7x^2 + 10x - 7$ by $(x - 1)$

4 Divide:

a $3x^4 + 8x^3 - 11x^2 + 2x + 8$ by $(3x + 2)$
b $4x^4 - 3x^3 + 11x^2 - x - 1$ by $(4x + 1)$
c $4x^4 - 6x^3 + 10x^2 - 11x - 6$ by $(2x - 3)$
d $6x^5 + 13x^4 - 4x^3 - 9x^2 + 21x + 18$ by $(2x + 3)$
e $6x^5 - 8x^4 + 11x^3 + 9x^2 - 25x + 7$ by $(3x - 1)$
f $8x^5 - 26x^4 + 11x^3 + 22x^2 - 40x + 25$ by $(2x - 5)$
g $25x^4 + 75x^3 + 6x^2 - 28x - 6$ by $(5x + 3)$
h $21x^5 + 29x^4 - 10x^3 + 42x - 12$ by $(7x - 2)$

5 Divide:

a $x^3 + x + 10$ by $(x + 2)$
b $2x^3 - 17x + 3$ by $(x + 3)$
c $-3x^3 + 50x - 8$ by $(x - 4)$

Hint Include $0x^2$ when you write out $f(x)$.

6 Divide:

a $x^3 + x^2 - 36$ by $(x - 3)$
b $2x^3 + 9x^2 + 25$ by $(x + 5)$
c $-3x^3 + 11x^2 - 20$ by $(x - 2)$

7 Show that $x^3 + 2x^2 - 5x - 10 = (x + 2)(x^2 - 5)$

8 Find the remainder when:

a $x^3 + 4x^2 - 3x + 2$ is divided by $(x + 5)$

b $3x^3 - 20x^2 + 10x + 5$ is divided by $(x - 6)$

c $-2x^3 + 3x^2 + 12x + 20$ is divided by $(x - 4)$

9 Show that when $3x^3 - 2x^2 + 4$ is divided by $(x - 1)$ the remainder is 5.

10 Show that when $3x^4 - 8x^3 + 10x^2 - 3x - 25$ is divided by $(x + 1)$ the remainder is -1.

11 Show that $(x + 4)$ is a factor of $5x^3 - 73x + 28$.

12 Simplify $\dfrac{3x^3 - 8x - 8}{x - 2}$

Hint Divide $3x^3 - 8x - 8$ by $(x - 2)$.

13 Divide $x^3 - 1$ by $(x - 1)$.

Hint Write $x^3 - 1$ as $x^3 + 0x^2 + 0x - 1$.

14 Divide $x^4 - 16$ by $(x + 2)$.

E 15 $f(x) = 10x^3 + 43x^2 - 2x - 10$

Find the remainder when $f(x)$ is divided by $(5x + 4)$. **(2 marks)**

E/P SKILLS EXECUTIVE FUNCTION 16 $f(x) = 3x^3 - 14x^2 - 47x - 14$

a Find the remainder when $f(x)$ is divided by $(x - 3)$. **(2 marks)**

b Given that $(x + 2)$ is a factor of $f(x)$, factorise $f(x)$ completely. **(4 marks)**

Problem-solving Write $f(x)$ in the form $(x + 2)(ax^2 + bx + c)$ then factorise the quadratic factor.

E/P SKILLS EXECUTIVE FUNCTION 17 a Find the remainder when $x^3 + 6x^2 + 5x - 12$ is divided by

i $x - 2$,

ii $x + 3$. **(3 marks)**

b Hence, or otherwise, find all the solutions to the equation $x^3 + 6x^2 + 5x - 12 = 0$. **(4 marks)**

E/P SKILLS EXECUTIVE FUNCTION 18 $f(x) = 2x^3 + 3x^2 - 8x + 3$

a Show that $f(x) = (2x - 1)(ax^2 + bx + c)$ where a, b and c are constants to be found. **(2 marks)**

b Hence factorise $f(x)$ completely. **(4 marks)**

c Write down all the real roots of the equation $f(x) = 0$. **(2 marks)**

E/P 19 $f(x) = 12x^3 + 5x^2 + 2x - 1$

a Show that $(4x - 1)$ is a factor of $f(x)$ and write $f(x)$ in the form $(4x - 1)(ax^2 + bx + c)$. **(6 marks)**

b Hence, show that the equation $12x^3 + 5x^2 + 2x - 1 = 0$ has exactly one real solution. **(2 marks)**

1.3 The factor theorem

The factor theorem is a quick way of finding simple linear factors of a polynomial.

- The factor theorem states that if f(x) is a polynomial then:
  - If f(p) = 0, then ($x - p$) is a factor of f(x).
  - If ($x - p$) is a factor of f(x), then f(p) = 0.
  - If $f\left(\frac{b}{a}\right) = 0$, then ($ax - b$) is a factor of f($x$).
  - If ($ax - b$) is a factor of f(x) then $f\left(\frac{b}{a}\right) = 0$.

Watch out

The first two statements are not the same. Here are two similar statements, only one of which is true:

If $x = -2$ then $x^2 = 4$ ✓

If $x^2 = 4$ then $x = -2$ ✗

You can use the factor theorem to quickly factorise a cubic function, g(x):

1 Substitute values into the function until you find a value p such that g(p) = 0.

2 Divide the function by ($x - p$). — The remainder will be 0 because ($x - p$) is a factor of g(x).

3 Write $g(x) = (x - p)(ax^2 + bx + c)$ — The other factor will be quadratic.

4 Factorise the quadratic factor, if possible, to write g(x) as a product of three linear factors.

Example 5 SKILLS ANALYSIS

Show that ($x - 2$) is a factor of $x^3 + x^2 - 4x - 4$ by:

a algebraic division **b** the factor theorem

a

$$\begin{array}{r} x^2 + 3x + 2 \\ x - 2\overline{)x^3 + x^2 - 4x - 4} \\ \underline{x^3 - 2x^2} \\ 3x^2 - 4x \\ \underline{3x^2 - 6x} \\ 2x - 4 \\ \underline{2x - 4} \\ 0 \end{array}$$

Divide $x^3 + x^2 - 4x - 4$ by ($x - 2$).

The remainder is 0, so ($x - 2$) is a factor of $x^3 + x^2 - 4x - 4$.

So ($x - 2$) is a factor of $x^3 + x^2 - 4x - 4$.

b $f(x) = x^3 + x^2 - 4x - 4$ — Write the polynomial as a function.

$f(2) = (2)^3 + (2)^2 - 4(2) - 4$ — Substitute $x = 2$ into the polynomial.

$= 8 + 4 - 8 - 4$

$= 0$

So ($x - 2$) is a factor of $x^3 + x^2 - 4x - 4$.

Use the factor theorem:
If f(p) = 0, then ($x - p$) is a factor of f(x).
Here $p = 2$, so ($x - 2$) is a factor of $x^3 + x^2 - 4x - 4$.

Example 6 SKILLS EXECUTIVE FUNCTION

a Fully factorise $2x^3 + x^2 - 18x - 9$ **b** Hence sketch the graph of $y = 2x^3 + x^2 - 18x - 9$

a $f(x) = 2x^3 + x^2 - 18x - 9$

> Write the polynomial as a function.

$f(-1) = 2(-1)^3 + (-1)^2 - 18(-1) - 9 = 8$
$f(1) = 2(1)^3 + (1)^2 - 18(1) - 9 = -24$
$f(2) = 2(2)^3 + (2)^2 - 18(2) - 9 = -25$
$f(3) = 2(3)^3 + (3)^2 - 18(3) - 9 = 0$

> Try values of x, e.g. −1, 1, 2, 3, … until you find $f(p) = 0$.

> $f(p) = 0$.

So $(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$.

> Use statement 1 from the factor theorem: If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$. Here $p = 3$.

$$\begin{array}{r}
2x^2 + 7x + 3 \\
x - 3\,\overline{)\,2x^3 + x^2 - 18x - 9} \\
\underline{2x^3 - 6x^2} \qquad\qquad \\
7x^2 - 18x \qquad \\
\underline{7x^2 - 21x} \qquad \\
3x - 9 \\
\underline{3x - 9} \\
0
\end{array}$$

> Use long division to find the quotient when dividing by $(x - 3)$.

> You can check your division here: $(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$, so the remainder must be 0.

$2x^3 + x^2 - 18x - 9 = (x - 3)(2x^2 + 7x + 3)$
$= (x - 3)(2x + 1)(x + 3)$

> $2x^2 + 7x + 3$ can also be factorised.

b $0 = (x - 3)(2x + 1)(x + 3)$

> Set $y = 0$ to find the points where the curve crosses the x-axis.

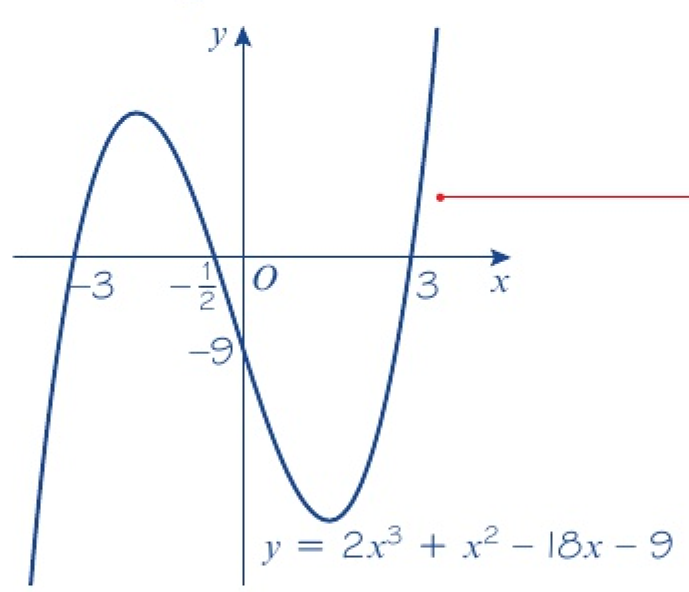
So the curve crosses the x-axis at (3, 0), $(-\frac{1}{2}, 0)$ and (−3, 0).

When $x = 0$, $y = (-3)(1)(3) = -9$

> Set $x = 0$ to find the y-intercept.

The curve crosses the y-axis at (0, −9).

$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



> This is a cubic graph with a positive **coefficient** of x^3 and three distinct roots. You should be familiar with its general shape. ← Pure 1 Section 4.1

Example 7

Given that $(x + 1)$ is a factor of $4x^4 - 3x^2 + a$, find the value of a.

$f(x) = 4x^4 - 3x^2 + a$ — Write the polynomial as a function.

$f(-1) = 0$ — Use statement 2 from the factor theorem. $(x - p)$ is a factor of $f(x)$, so $f(p) = 0$. Here $p = -1$.

$4(-1)^4 - 3(-1)^2 + a = 0$

$4 - 3 + a = 0$

$a = -1$

Substitute $x = -1$ and **solve the equation** for a. Remember $(-1)^4 = 1$.

Example 8

$f(x) = px^3 + x^2 - 19x + p$

Given that $(2x - 3)$ is a factor of $f(x)$

a find the value of p,

b hence factorise $f(x)$ completely.

a $f(x) = px^3 + x^2 - 19x + p$

$f\left(\frac{3}{2}\right) = 0$ — Use the factor theorem. $(2x - 3)$ is a factor of $f(x)$, so $f\left(\frac{3}{2}\right) = 0$

$p\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 - 19 \times \left(\frac{3}{2}\right) + p = 0$ — Substitute $x = \frac{3}{2}$ into $f(x) = 0$

$p\left(\frac{27}{8}\right) + \frac{9}{4} - \frac{57}{2} + p = 0$

$p\left(\frac{27}{8} + 1\right) = \frac{57}{2} - \frac{9}{4}$

$p\left(\frac{35}{8}\right) = \frac{105}{4}$

$p = 6$ — Solve the equation for p

b Now divide $6x^3 + x^2 - 19x + 6$ by $2x - 3$

$$\begin{array}{r} 3x^2 + 5x - 2 \\ 2x - 3\,\overline{)\,6x^3 + x^2 - 19x + 6} \\ \underline{6x^3 - 9x^2} \qquad\qquad \\ 10x^2 - 19x \qquad \\ \underline{10x^2 - 15x} \qquad \\ -4x + 6 \\ \underline{-4x + 6} \\ 0 \end{array}$$

$3x^2 + 5x - 2 = (x + 2)(3x - 1)$ — Factorise the quadratic quotient

$f(x) = (2x - 3)(x + 2)(3x - 1)$ — Remember to write $f(x)$ as a product of its factors

Exercise 1C

SKILLS EXECUTIVE FUNCTION

1 Use the factor theorem to show that:

a $(x - 1)$ is a factor of $4x^3 - 3x^2 - 1$

b $(x + 3)$ is a factor of $5x^4 - 45x^2 - 6x - 18$

c $(x - 4)$ is a factor of $-3x^3 + 13x^2 - 6x + 8$.

2 Show that $(x - 1)$ is a factor of $x^3 + 6x^2 + 5x - 12$ and hence factorise the expression completely.

3 Show that $(x + 1)$ is a factor of $x^3 + 3x^2 - 33x - 35$ and hence factorise the expression completely.

4 Show that $(x - 5)$ is a factor of $x^3 - 7x^2 + 2x + 40$ and hence factorise the expression completely.

5 Show that $(x - 2)$ is a factor of $2x^3 + 3x^2 - 18x + 8$ and hence factorise the expression completely.

6 Use the factor theorem to show that $(2x - 1)$ is a factor of $2x^3 + 17x^2 + 31x - 20$.

7 Each of these expressions has a factor $(x \pm p)$. Find a value of p and hence factorise the expression completely.

a $x^3 - 10x^2 + 19x + 30$ b $x^3 + x^2 - 4x - 4$ c $x^3 - 4x^2 - 11x + 30$

8 i Fully factorise the right-hand side of each equation.

ii Sketch the graph of each equation.

a $y = 2x^3 + 5x^2 - 4x - 3$ b $y = 2x^3 - 17x^2 + 38x - 15$ c $y = 3x^3 + 8x^2 + 3x - 2$

d $y = 6x^3 + 11x^2 - 3x - 2$ e $y = 4x^3 - 12x^2 - 7x + 30$

9 Factorise $2x^3 + 5x^2 - 4x - 3$ completely.

(P) 10 Given that $(x - 1)$ is a factor of $5x^3 - 9x^2 + 2x + a$, find the value of a.

11 Given that $(x + 3)$ is a factor of $6x^3 - bx^2 + 18$, find the value of b.

(P) 12 Given that $(x - 1)$ and $(x + 1)$ are factors of $px^3 + qx^2 - 3x - 7$, find the values of p and q.

(P) 13 Given that $(x + 1)$ and $(x - 2)$ are factors of $cx^3 + dx^2 - 9x - 10$, find the values of c and d.

Problem-solving

Use the factor theorem to form simultaneous equations.

(E) 14 Given that $(x - 1)$ and $(2x - 1)$ are factors of $px^3 + qx^2 + 9x - 2$, find the value of p and the value of q. **(4)**

(P) 15 Given that $(x + 2)$ and $(x - 3)$ are factors of $gx^3 + hx^2 - 14x + 24$, find the values of g and h.

16 Given that $(3x + 2)$ is a factor of $3x^3 + bx^2 - 3x - 2$,

a find the value of b **(2)**

b hence factorise $3x^3 + bx^2 - 3x - 2$ completely. **(4)**

E **17** $f(x) = 3x^3 - 12x^2 + 6x - 24$

a Use the factor theorem to show that $(x - 4)$ is a factor of $f(x)$. **(2 marks)**

b Hence, show that 4 is the only real root of the equation $f(x) = 0$. **(4 marks)**

E **18** $f(x) = 4x^3 + 4x^2 - 11x - 6$

a Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2 marks)**

b Factorise $f(x)$ completely. **(4 marks)**

c Write down all the solutions of the equation $4x^3 + 4x^2 - 11x - 6 = 0$. **(1 mark)**

E **19** **a** Show that $(x - 2)$ is a factor of $9x^4 - 18x^3 - x^2 + 2x$. **(2 marks)**

b Hence, find four real solutions to the equation $9x^4 - 18x^3 - x^2 + 2x = 0$. **(5 marks)**

Challenge

$f(x) = 2x^4 - 5x^3 - 42x^2 - 9x + 54$

a Show that $f(1) = 0$ and $f(-3) = 0$.

b Hence, solve $f(x) = 0$.

1.4 The remainder theorem

- **You can find the remainder when a polynomial is divided by $(ax \pm b)$ by using the remainder theorem.**
- **If a polynomial $f(x)$ is divided by $(ax - b)$ then the remainder is $f\left(\frac{b}{a}\right)$.**

Example 9 SKILLS INTERPRETATION

Find the remainder when $x^3 - 20x + 3$ is divided by $(x - 4)$ using:

a algebraic division

b the remainder theorem.

a
$$\begin{array}{r} x^2 + 4x - 4 \\ x - 4\overline{)x^3 + 0x^2 - 20x + 3} \\ \underline{x^3 - 4x^2} \qquad\qquad \\ 4x^2 - 20x \qquad \\ \underline{4x^2 - 16x} \qquad \\ -4x + 3 \\ \underline{-4x + 16} \\ -13 \end{array}$$

Divide $x^3 - 20x + 3$ by $(x - 4)$

Don't forget to write $0x^2$

The remainder is -13

b $f(x) = x^3 - 20x + 3$

Write the polynomial as a function

To use the remainder theorem, compare $(x - 4)$ to $(ax - b)$

In this case $a = 1$ and $b = 4$ and the remainder is $f\left(\frac{4}{1}\right) = f(4)$

$f(4) = 4^3 - 20 \times 4 + 3$

$f(4) = -13$

The remainder is -13

Substitute $x = 4$ and evaluate $f(4)$

Example 10

When $8x^4 - 4x^3 + ax^2 - 1$ is divided by $(2x + 1)$ the remainder is 3. Find the value of a.

$f(x) = 8x^4 - 4x^3 + ax^2 - 1$

$$f\left(-\frac{1}{2}\right) = 3$$

$$8\left(-\frac{1}{2}\right)^4 - 4\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 - 1 = 3$$

$$8\left(\frac{1}{16}\right) - 4\left(-\frac{1}{8}\right) + a\left(\frac{1}{4}\right) - 1 = 3$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{4}a - 1 = 3$$

$$\frac{1}{4}a = 3$$

$$a = 12$$

Problem-solving

Use the remainder theorem: If $f(x)$ is divided by $(ax - b)$, then the reminder is $f\left(\frac{b}{a}\right)$.
Compare $(2x + 1)$ to $(ax - b)$, so $a = 2$, $b = -1$ and the remainder is $f\left(-\frac{1}{2}\right)$.
Using the fact that the remainder is 3, substitute $x = -\frac{1}{2}$ and solve the equation for a.

Exercise 1D

SKILLS INTERPRETATION

1 Find the remainder when:

a $4x^3 - 5x^2 + 7x + 1$ is divided by $(x - 2)$
b $2x^5 - 32x^3 + x - 10$ is divided by $(x - 4)$
c $-2x^3 + 6x^2 + 5x - 3$ is divided by $(x + 1)$
d $7x^3 + 6x^2 - 45x + 1$ is divided by $(x + 3)$
e $4x^4 - 4x^2 + 8x - 1$ is divided by $(2x - 1)$
f $243x^4 - 27x^3 - 3x + 7$ is divided by $(3x - 1)$
g $64x^3 + 32x^2 - 16x + 9$ is divided by $(4x + 3)$
h $81x^3 - 81x^2 + 9x + 6$ is divided by $(3x - 2)$
i $243x^6 - 780x^2 + 6$ is divided by $(3x + 4)$
j $125x^4 + 5x^3 - 9x$ is divided by $(5x + 3)$.

(P) **2** When $2x^3 - 3x^2 - 2x + a$ is divided by $(x - 1)$ the remainder is -4. Find the value of a.

(P) **3** When $-3x^3 + 4x^2 + bx + 6$ is divided by $(x + 2)$ the remainder is 10. Find the value of b.

(P) **4** When $216x^3 - 32x^2 + cx - 8$ is divided by $(2x - 1)$ the remainder is 1. Find the value of c.

5 Show that $(x - 3)$ is a factor of $x^6 - 36x^3 + 243$.

6 Show that $(2x - 1)$ is a factor of $2x^3 + 17x^2 + 31x - 20$.

(E/P) **7** $f(x) = x^2 + 3x + q$
Given $f(2) = 3$, find $f(-2)$. **(5 marks)**

Hint First find q.

(E/P) **8** $g(x) = x^3 + ax^2 + 3x + 6$
Given $g(-1) = 2$, find the remainder when $g(x)$ is divided by $(3x - 2)$. **(5 marks)**

E/P **9** The expression $2x^3 - x^2 + ax + b$ gives a remainder of 14 when divided by $(x - 2)$ and a remainder of -86 when divided by $(x + 3)$.
Find the value of a and the value of b. **(5 marks)**

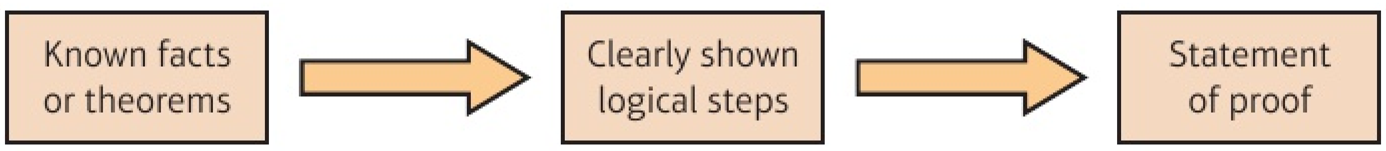
E/P **10** The expression $3x^3 + 2x^2 - px + q$ is divisible by $(x - 1)$ but leaves a remainder of 10 when divided by $(x + 1)$. Find the value of a and the value of b. **(5 marks)**

Problem-solving
Solve simultaneous equations.

1.5 Mathematical proof

A proof is a logical and **structured** argument to show that a mathematical statement (or **conjecture**) is always true.
A mathematical proof usually starts with previously **established** mathematical facts (or **theorems**) and then works through a series of logical steps. The final step in a proof is a **statement** of what has been proven.

Notation A statement that has been proven is called a **theorem**.
A statement that has yet to be proven is called a **conjecture**.



A mathematical proof needs to show that something is true in every case.

- You can prove a mathematical statement is true by **deduction**. This means starting from known facts or definitions, then using logical steps to reach the **desired** conclusion.

Here is an example of proof by deduction:

Statement: The product of two **odd numbers** is odd.

Demonstration: $5 \times 7 = 35$, which is odd

This is demonstration but it is not a proof. You have only shown one case.

Proof: p and q are **integers**, so $2p + 1$ and $2q + 1$ are odd numbers.

You can use $2p + 1$ and $2q + 1$ to represent any odd numbers. If you can show that $(2p + 1) \times (2q + 1)$ is always an odd number then you have proved the statement for all cases.

$$(2p + 1) \times (2q + 1) = 4pq + 2p + 2q + 1$$
$$= 2(2pq + p + q) + 1$$

Since p and q are integers, $2pq + p + q$ is also an integer.

So $2(2pq + p + q) + 1$ is one more than an **even number**.

So the product of two odd numbers is an odd number.

This is the statement of proof.

- In a mathematical proof you must
  - **State any information or assumptions you are using**
  - **Show every step of your proof clearly**
  - **Make sure that every step follows logically from the previous step**
  - **Make sure you have covered all possible cases**
  - **Write a statement of proof at the end of your working**

You need to be able to prove results involving **identities**, such as $(a + b)(a - b) \equiv a^2 - b^2$

- To prove an identity you should
  - **Start with the expression on one side of the identity**
  - **Manipulate that expression algebraically until it matches the other side**
  - **Show every step of your algebraic working**

Notation The symbol $\equiv$ means 'is always equal to'. It shows that two expressions are mathematically **identical**.

Watch out Don't try to 'solve' an identity like an equation. Start from one side and manipulate the expression to match the other side.

Example 11

SKILLS REASONING/ARGUMENTATION

Prove that $(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

$(3x + 2)(x - 5)(x + 7)$
$= (3x + 2)(x^2 + 2x - 35)$
$= 3x^3 + 6x^2 - 105x + 2x^2 + 4x - 70$
$= 3x^3 + 8x^2 - 101x - 70$
So
$(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

Start with the left-hand side and expand the brackets.

In proof questions you need to show all your working.

Left-hand side = right-hand side.

Example 12

Prove that if $(x - p)$ is a factor of $f(x)$ then $f(p) = 0$.

If $(x - p)$ is a factor of $f(x)$ then
$f(x) = (x - p) \times g(x)$
So $f(p) = (p - p) \times g(p)$
i.e. $f(p) = 0 \times g(p)$
So $f(p) = 0$ as required.

$g(x)$ is a polynomial expression.

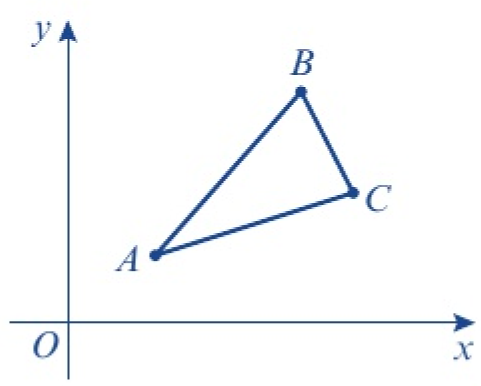
Substitute $x = p$.

$p - p = 0$

Remember $0 \times$ *anything* $= 0$

Example 13

Prove that $A(1, 1)$, $B(3, 3)$ and $C(4, 2)$ are the **vertices** of a right-angled triangle.



The gradient of line $AB = \frac{3 - 1}{3 - 1} = \frac{2}{2} = 1$

The gradient of a line $= \frac{y_2 - y_1}{x_2 - x_1}$

Problem-solving If you need to prove a geometrical result, it can sometimes help to sketch a diagram as part of your working.

The gradient of line $BC = \frac{2-3}{4-3} = \frac{-1}{1} = -1$

The gradient of line $AC = \frac{2-1}{4-1} = \frac{1}{3}$

The gradients are different so the three points are not collinear.

Hence ABC is a triangle.

Gradient of AB × gradient of $BC = 1 \times (-1)$
$= -1$

> If the product of two gradients is –1 then the two lines are **perpendicular**.
> Gradient of line AB × gradient of line $BC = -1$

So AB is perpendicular to BC,
and the triangle is a right-angled triangle.

> Remember to state what you have proved.

Example 14

SKILLS REASONING/ARGUMENTATION

The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots.

Prove that k satisfies the inequality $0 \leqslant k < \frac{8}{9}$

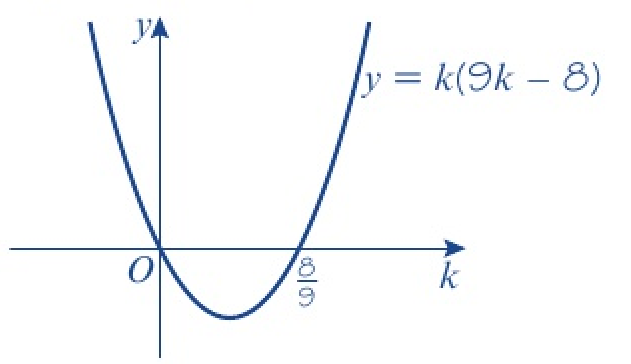
$kx^2 + 3kx + 2 = 0$ has no real roots,
so $b^2 - 4ac < 0$

> State which assumption or information you are using at each stage of your proof.

> Use the discriminant. ← Pure 1 Section 2.5

$(3k)^2 - 4k(2) < 0$
$9k^2 - 8k < 0$
$k(9k - 8) < 0$

> Solve this quadratic inequality by sketching the graph of $y = k(9k - 8)$ ← Pure 1 Section 3.5



> The graph shows that when $k(9k - 8) < 0$, $0 < k < \frac{8}{9}$

$0 < k < \frac{8}{9}$

When $k = 0$:

> Be really careful to consider all the possible situations. You can't use the discriminant if $k = 0$ so look at this case separately.

$(0)x^2 + 3(0)x + 2 = 0$
$2 = 0$

Which is impossible, so no real roots

> Write out all of your conclusions clearly.

So combining these:

$0 \leqslant k < \frac{8}{9}$ as required

> $0 < k < \frac{8}{9}$ together with $k = 0$, gives $0 \leqslant k < \frac{8}{9}$

Exercise 1E

SKILLS REASONING/ARGUMENTATION

> **Hint** The proofs in this exercise are all proofs by deduction.

P **1** Prove that $n^2 - n$ is an even number for all values of n.

P **2** Prove that $\frac{x}{1+\sqrt{2}} \equiv x\sqrt{2} - x$.

P **3** Prove that $(x + \sqrt{y})(x - \sqrt{y}) \equiv x^2 - y$.

P **4** Prove that $(2x - 1)(x + 6)(x - 5) \equiv 2x^3 + x^2 - 61x + 30$.

P **5** Prove that $x^2 + bx \equiv \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

P **6** Prove that the solutions of $x^2 + 2bx + c = 0$ are $x = -b \pm \sqrt{b^2 - c}$.

P **7** Prove that $\left(x - \frac{2}{x}\right)^3 \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

P **8** Prove that $\left(x^3 - \frac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right) \equiv x^{\frac{1}{2}}\left(x^4 - \frac{1}{x^4}\right)$

P **9** Use completing the square to prove that $3n^2 - 4n + 10$ is positive for all values of n.

P **10** Use completing the square to prove that $-n^2 - 2n - 3$ is negative for all values of n.

> **Problem-solving** Any expression that is squared must be $\geqslant 0$.

E/P **11** Prove that $x^2 + 8x + 20 \geqslant 4$ for all values of x. **(3 marks)**

E/P **12** The equation $kx^2 + 5kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leqslant k < \frac{12}{25}$ **(4 marks)**

E/P **13** The equation $px^2 - 5x - 6 = 0$, where p is a constant, has two distinct real roots. Prove that p satisfies the inequality $p > -\frac{25}{24}$ **(4 marks)**

P **14** Prove that $A(3, 1)$, $B(1, 2)$ and $C(2, 4)$ are the vertices of a right-angled triangle.

P **15** Prove that **quadrilateral** $A(1, 1)$, $B(2, 4)$, $C(6, 5)$ and $D(5, 2)$ is a parallelogram.

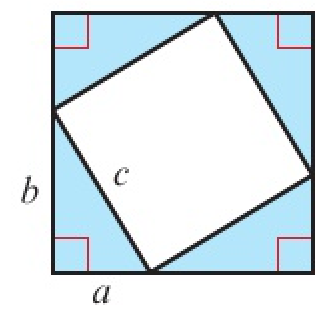
P **16** Prove that quadrilateral $A(2, 1)$, $B(5, 2)$, $C(4, -1)$ and $D(1, -2)$ is a rhombus.

P **17** Prove that $A(-5, 2)$, $B(-3, -4)$ and $C(3, -2)$ are the vertices of an isosceles right-angled triangle.

E/P **18** A **circle** has equation $(x - 1)^2 + y^2 = k$, where $k > 0$.
The straight line L with equation $y = ax$ cuts the circle at two distinct points.
Prove that $k > \frac{a^2}{1 + a^2}$ **(6 marks)**

E/P **19** Prove that the line $4y - 3x + 26 = 0$ is a **tangent** to the circle $(x + 4)^2 + (y - 3)^2 = 100$. **(5 marks)**

P **20** The diagram shows a square and four congruent right-angled triangles.
Use the diagram to prove that $a^2 + b^2 = c^2$.



Problem-solving
Find an expression for the area of the large square in terms of a and b.

Challenge

SKILLS CREATIVITY

1 Prove that $A(7, 8)$, $B(-1, 8)$, $C(6, 1)$ and $D(0, 9)$ are points on the same circle.

2 Prove that any odd number can be written as the **difference** of two squares.

1.6 Methods of proof

A mathematical statement can be proved by **exhaustion**. For example, you can prove that the sum of two **consecutive** square numbers between 100 and 200 is an odd number. The square numbers between 100 and 200 are 121, 144, 169, 196.

$121 + 144 = 265$ which is odd $144 + 169 = 313$ which is odd $169 + 196 = 365$ which is odd

So the sum of two consecutive square numbers between 100 and 200 is an odd number.

- You can prove a mathematical statement is true by exhaustion. This means breaking the statement into smaller cases and proving each case separately.

This method is better suited to a small number of results. You cannot use one example to prove a statement is true, as one example is only one case.

Example 15

Prove that all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

For odd numbers:
$(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$
$4n(n + 1)$ is a multiple of 4, so $4n(n + 1) + 1$ is 1 more than a multiple of 4.

You can write any odd number in the form $2n + 1$ where n is a positive integer.

Problem-solving
Consider the two cases, odd and even numbers, separately.

For even numbers:

$(2n)^2 = 4n^2$

$4n^2$ is a multiple of 4.

All integers are either odd or even, so all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

You can write any even number in the form $2n$ where n is a positive integer.

A mathematical statement can be disproved using a **counter-example**. For example, to prove that the statement '$3n + 3$ is a multiple of 6 for all values of n' is not true you can use the counter-example when $n = 2$, as $3 \times 2 + 3 = 9$ and 9 is not a multiple of 6.

- **You can prove a mathematical statement is not true by a counter-example. A counter-example is an example that does not work for the statement. You do not need to give more than one, as one is sufficient to disprove a statement.**

Example 16

Prove that the following statement is **not** true:

'The sum of two consecutive prime numbers is always even.'

2 and 3 are both prime

$2 + 3 = 5$

5 is odd

So the statement is not true.

You only need one counter-example to show that the statement is false.

Example 17

SKILLS REASONING/ARGUMENTATION

a Prove that for all positive values of x and y:

$$\frac{x}{y} + \frac{y}{x} \geqslant 2$$

b Use a counter-example to show that this is not true when x and y are not both positive.

Watch out You must always start a proof from **known facts**. Never start your proof with the statement you are trying to prove.

a Jottings:

$\frac{x}{y} + \frac{y}{x} \geqslant 2$

$\frac{x^2 + y^2}{xy} \geqslant 2$

$x^2 + y^2 - 2xy \geqslant 0$

$(x - y)^2 \geqslant 0$

Proof:

Consider $(x - y)^2$

$(x - y)^2 \geqslant 0$

$x^2 + y^2 - 2xy \geqslant 0$

$\frac{x^2 + y^2 - 2xy}{xy} \geqslant 0$

This step is valid because x and y are both positive so $xy > 0$.

$\frac{x}{y} + \frac{y}{x} - 2 \geqslant 0$

$\frac{x}{y} + \frac{y}{x} \geqslant 2$

b Try $x = -3$ and $y = 6$

$\frac{-3}{6} + \frac{6}{-3} = -\frac{1}{2} - 2 = -\frac{5}{2}$

This is not $\geqslant 2$ so the statement is not true.

Problem-solving

Use jottings to get some ideas for a good starting point. These don't form part of your proof, but can give you a clue as to what expression you can consider to begin your proof.

Now you are ready to start your proof. You know that any expression squared is $\geqslant 0$. This is a **known fact** so this is a valid way to begin your proof.

State how you have used the fact that x and y are positive in your proof. If $xy = 0$ you couldn't divide the LHS by xy, and if $xy < 0$, then the direction of the inequality would be reversed.

This was what you wanted to prove so you have finished.

Your working for part **a** tells you that the proof fails when $xy < 0$, so try one positive and one negative value.

Exercise 1F

SKILLS REASONING/ARGUMENTATION

(P) **1** Prove that when n is an integer and $1 \leqslant n \leqslant 6$, then $m = n + 2$ is not divisible by 10.

Hint You can try each integer for $1 \leqslant n \leqslant 6$.

(P) **2** Prove that every odd integer between 2 and 26 is either prime or the product of two primes.

(P) **3** Prove that the sum of two consecutive square numbers from 1^2 to 8^2 is an odd number.

(E/P) **4** Prove that all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9. **(4 marks)**

(P) **5** Find a counter-example to disprove each of the following statements:

a If n is a positive integer then $n^4 - n$ is divisible by 4.

b Integers always have an even number of factors.

c $2n^2 - 6n + 1$ is positive for all values of n.

d $2n^2 - 2n - 4$ is a multiple of 3 for all integer values of n.

E/P **6** A student is trying to prove that $x^3 + y^3 < (x + y)^3$.

The student writes:

> $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
> which is less than $x^3 + y^3$ since
> $3x^2y + 3xy^2 > 0$

a Identify the error made in the proof. **(1 mark)**

b Provide a counter-example to show that the statement is not true. **(2 marks)**

Problem-solving

For part **b** you need to write down suitable values of x and y and show that they do not satisfy the inequality.

E/P **7** Prove that for all real values of x

$$(x + 6)^2 \geqslant 2x + 11$$

(3 marks)

E/P **8** Given that a is a positive real number, prove that:

$$a + \frac{1}{a} \geqslant 2$$

Watch out Remember to state how you use the condition that a is positive.

(2 marks)

E/P **9** **a** Prove that for any positive numbers p and q:

$$p + q \geqslant \sqrt{4pq}$$

(3 marks)

b Show, by means of a counter-example, that this inequality does not hold when p and q are both negative. **(2 marks)**

Problem-solving

Use jottings and work backwards to work out what expression to consider.

E/P **10** It is claimed that the following inequality is true for all negative numbers x and y:

$$x + y \geqslant \sqrt{x^2 + y^2}$$

The following proof is offered by a student:

> $x + y \geqslant \sqrt{x^2 + y^2}$
> $(x + y)^2 \geqslant x^2 + y^2$
> $x^2 + y^2 + 2xy \geqslant x^2 + y^2$
> $2xy > 0$ which is true because x and y are both negative, so xy is positive.

a Explain the error made by the student. **(2 marks)**

b By use of a counter-example, verify that the inequality is not satisfied if both x and y are negative. **(1 mark)**

c Prove that this inequality is true if x and y are both positive. **(2 marks)**

Chapter review 1

1 Simplify these fractions as far as possible:

a $\frac{3x^4 - 21x}{3x}$

b $\frac{x^2 - 2x - 24}{x^2 - 7x + 6}$

c $\frac{2x^2 + 7x - 4}{2x^2 + 9x + 4}$

2 Divide $3x^3 + 12x^2 + 5x + 20$ by $(x + 4)$.

3 Simplify $\frac{2x^3 + 3x + 5}{x + 1}$

(E) 4 a Show that $(x - 3)$ is a factor of $2x^3 - 2x^2 - 17x + 15$. **(2 marks)**

b Hence express $2x^3 - 2x^2 - 17x + 15$ in the form $(x - 3)(Ax^2 + Bx + C)$, where the values A, B and C are to be found. **(3 marks)**

(E) 5 Find the remainder when $16x^5 - 20x^4 + 8$ is divided by $(2x - 1)$. **(2 marks)**

(E) 6 a Show that $(x - 2)$ is a factor of $x^3 + 4x^2 - 3x - 18$. **(2 marks)**

b Hence express $x^3 + 4x^2 - 3x - 18$ in the form $(x - 2)(px + q)^2$, where the values p and q are to be found. **(4 marks)**

(E) 7 Factorise completely $2x^3 + 3x^2 - 18x + 8$. **(6 marks)**

(E/P) 8 Find the value of k if $(x - 2)$ is a factor of $x^3 - 3x^2 + kx - 10$. **(4 marks)**

(E/P) 9 $f(x) = 2x^2 + px + q$. Given that $f(-3) = 0$, and $f(4) = 21$:

a find the value of p and q **(6 marks)**

b factorise $f(x)$. **(3 marks)**

(E/P) 10 $h(x) = x^3 + 4x^2 + rx + s$. Given $h(-1) = 0$, and $h(2) = 30$:

a find the values of r and s **(6 marks)**

b factorise $h(x)$. **(3 marks)**

(E) 11 $g(x) = 2x^3 + 9x^2 - 6x - 5$.

a Factorise $g(x)$. **(6 marks)**

b Solve $g(x) = 0$. **(2 marks)**

E **12** **a** Show that $(x - 2)$ is a factor of $f(x) = x^3 + x^2 - 5x - 2$. **(2 marks)**

b Hence, or otherwise, find the exact solutions of the equation $f(x) = 0$. **(4 marks)**

E **13** Given that -1 is a root of the equation $2x^3 - 5x^2 - 4x + 3$, find the two positive roots. **(4 marks)**

E/P **14** The remainder obtained when $x^3 - 5x^2 + px + 6$ is divided by $(x + 2)$ is equal to the remainder obtained when the same expression is divided by $(x - 3)$.

Find the value of p. **(4 marks)**

E **15** $f(x) = x^3 - 2x^2 - 19x + 20$

a Show that $(x + 4)$ is a factor of $f(x)$. **(3 marks)**

b Hence, or otherwise, find all the solutions to the equation $x^3 - 2x^2 - 19x + 20 = 0$. **(4 marks)**

E **16** $f(x) = 6x^3 + 17x^2 - 5x - 6$

a Show that $f(x) = (3x - 2)(ax^2 + bx + c)$, where a, b and c are constants to be found. **(2 marks)**

b Hence factorise $f(x)$ completely. **(4 marks)**

c Write down all the real roots of the equation $f(x) = 0$. **(2 marks)**

17 Prove that $\dfrac{x - y}{\sqrt{x} - \sqrt{y}} \equiv \sqrt{x} + \sqrt{y}$.

P **18** Use completing the square to prove that $n^2 - 8n + 20$ is positive for all values of n.

P **19** Prove that the quadrilateral $A(1, 1)$, $B(3, 2)$, $C(4, 0)$ and $D(2, -1)$ is a square.

P **20** Prove that the sum of two consecutive positive odd numbers less than ten gives an even number.

P **21** Prove that the statement '$n^2 - n + 3$ is a prime number for all values of n' is untrue.

P **22** Prove that $\left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right) \equiv x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$.

P **23** Prove that $2x^3 + x^2 - 43x - 60 \equiv (x + 4)(x - 5)(2x + 3)$.

E **24** The equation $x^2 - kx + k = 0$, where k is a positive constant, has two equal roots. Prove that $k = 4$. **(3 marks)**

P **25** Prove that the distance between opposite edges of a regular hexagon of side length $\sqrt{3}$ is a rational value.

(P) **26** **a** Prove that the difference of the squares of two consecutive even numbers is always divisible by 4.

b Is this statement true for odd numbers? Give a reason for your answer.

(E) **27** A student is trying to prove that $1 + x^2 < (1 + x)^2$.

The student writes:

$(1 + x)^2 = 1 + 2x + x^2$.
So $1 + x^2 < 1 + 2x + x^2$.

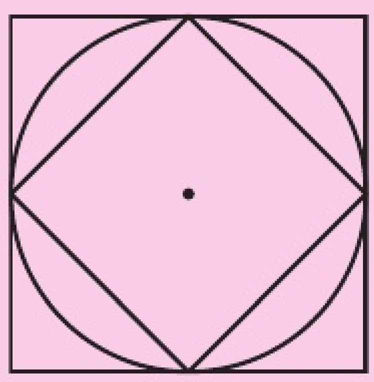
a Identify the error made in the proof. **(1 mark)**

b Provide a counter-example to show that the statement is not true. **(2 marks)**

Challenge

SKILLS INNOVATION

1 The diagram shows two squares and a circle.



a Given that π is defined as the circumference of a circle of **diameter** 1 unit, prove that $2\sqrt{2} < \pi < 4$.

b By similarly constructing regular hexagons inside and outside a circle, prove that $3 < \pi < 2\sqrt{3}$.

2 Prove that if $f(x) = ax^3 + bx^2 + cx + d$ and $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.

Summary of key points

1 When simplifying an algebraic fraction, factorise the numerator and denominator where possible and then cancel common factors.

2 You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.

3 The **factor theorem** states that if f(x) is a polynomial then:

- If f(p) = 0, then $(x - p)$ is a factor of f(x)
- If $(x - p)$ is a factor of f(x), then f(p) = 0
- If $\mathrm{f}\left(\frac{b}{a}\right) = 0$ then $(ax - b)$ is a factor

4 The **remainder theorem** states that if a polynomial f(x) is divided by $(ax - b)$ then the remainder is $\mathrm{f}\left(\frac{b}{a}\right)$

5 You can prove a mathematical statement is true by **deduction**. This means starting from known facts or definitions, then using logical steps to reach the desired conclusion.

6 In a mathematical proof you must

- State any information or assumptions you are using
- Show every step of your proof clearly
- Make sure that every step follows logically from the previous step
- Make sure you have covered all possible cases
- Write a statement of proof at the end of your working

7 To prove an identity you should

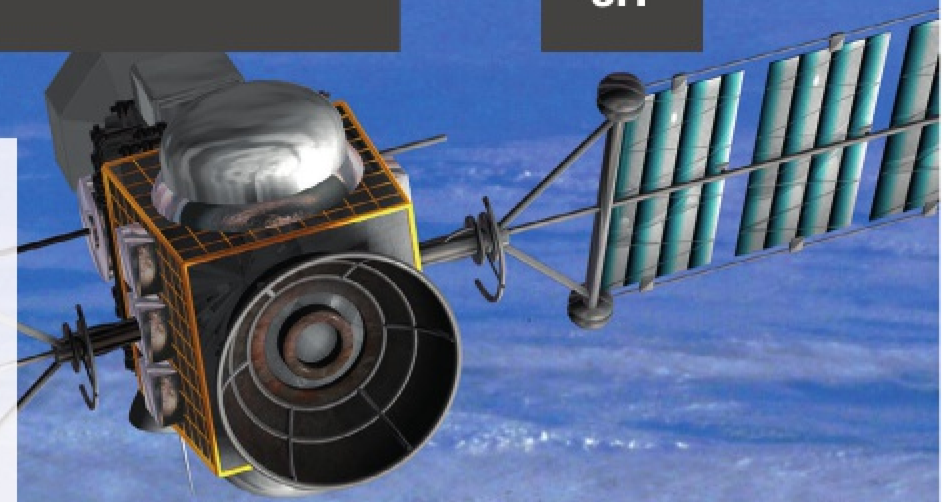
- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step of your algebraic working

8 You can prove a mathematical statement is true by **exhaustion**. This means breaking the statement into smaller cases and proving each case separately.

9 You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one example that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.

2 COORDINATE GEOMETRY IN THE (x, y) PLANE

3.1



Learning objectives

After completing this unit you should be able to:

- Find the mid-point of a line segment → pages 26–27
- Find the equation of the perpendicular bisector to a line segment → pages 28–29
- Know how to find the equation of a circle → pages 29–32
- Solve geometric problems involving straight lines and circles → pages 33–34
- Use circle properties to solve problems on coordinate grids → pages 35–40
- Find the angle in a semicircle and solve other problems involving circles and triangles → pages 40–44

Prior knowledge check

1 Write each of the following in the form $(x + p)^2 + q$:

a $x^2 + 10x + 28$ **b** $x^2 - 6x + 1$

c $x^2 - 12x$ **d** $x^2 + 7x$ ← Pure 1 Section 2.3

2 Find the equation of the line passing through each of the following pairs of points:

a $A(0, -6)$ and $B(4, 3)$

b $P(7, -5)$ and $Q(-9, 3)$

c $R(-4, -2)$ and $T(5, 10)$ ← Pure 1 Section 5.2

3 Use the discriminant to determine whether the following have two real solutions, one real solution or no real solutions.

a $x^2 - 7x + 14 = 0$ **b** $x^2 + 11x + 8 = 0$ **c** $4x^2 + 12x + 9 = 0$ ← Pure 1 Section 2.5

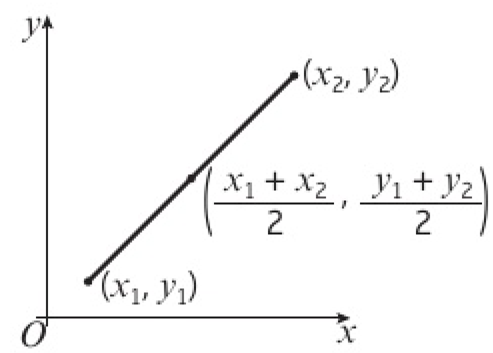
4 Find the equation of the line that passes through the point $(3, -4)$ and is perpendicular to the line with equation $6x - 5y - 1 = 0$ ← Pure 1 Section 5.3

Geostationary orbits are circular orbits around the Earth. Meteorologists use geostationary satellites to provide information about the Earth's surface and atmosphere.

2.1 Midpoints and perpendicular bisectors

You can find the **midpoint** of a line segment by averaging the x- and y-coordinates of its **endpoints**.

- **The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.**

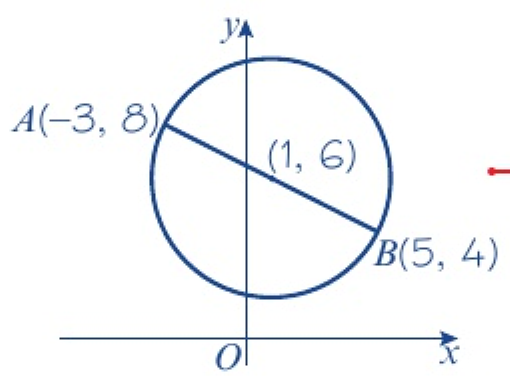


Notation A **line segment** is a finite part of a straight line with two distinct endpoints.

Example 1

SKILLS INTERPRETATION

The line segment AB is a diameter of a circle, where A and B are $(-3, 8)$ and $(5, 4)$ respectively. Find the coordinates of the centre of the circle.



Draw a sketch.

Remember the centre of a circle is the midpoint of a diameter.

The centre of the circle is $\left(\frac{-3 + 5}{2}, \frac{8 + 4}{2}\right)$

$= \left(\frac{2}{2}, \frac{12}{2}\right) = (1, 6)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

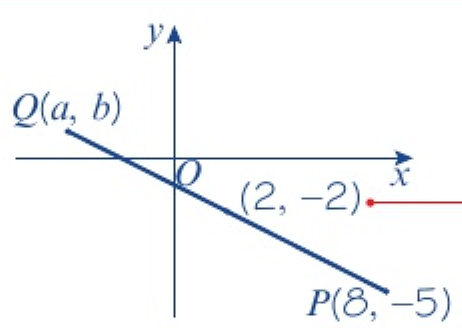
Here $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (5, 4)$.

Example 2

The line segment PQ is a diameter of the circle centre $(2, -2)$. Given that P is $(8, -5)$, find the coordinates of Q.

Problem-solving

In coordinate geometry problems, it is often helpful to draw a sketch showing the information given in the question.



$(2, -2)$ is the midpoint of (a, b) and $(8, -5)$.

Let Q have coordinates (a, b).

$\left(\frac{8 + a}{2}, \frac{-5 + b}{2}\right) = (2, -2)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Here $(x_1, y_1) = (8, -5)$ and $(x_2, y_2) = (a, b)$.

So $\frac{8 + a}{2} = 2$ $\qquad$ $\frac{-5 + b}{2} = -2$

$8 + a = 4$ $\qquad$ $-5 + b = -4$

$a = -4$ $\qquad$ $b = 1$

Compare the x- and y-coordinates separately.

Rearrange the equations to find a and b.

So, Q is $(-4, 1)$.

Exercise 2A

SKILLS INTERPRETATION

1 Find the midpoint of the line segment joining each pair of points:

a (4, 2), (6, 8) **b** (0, 6), (12, 2) **c** (2, 2), (−4, 6)

d (−6, 4), (6, −4) **e** (7, −4), (−3, 6) **f** (−5, −5), (−11, 8)

g $(6a, 4b), (2a, -4b)$ **h** $(-4u, 0), (3u, -2v)$ **i** $(a + b, 2a - b), (3a - b, -b)$

j $(4\sqrt{2}, 1)\ (2\sqrt{2}, 7)$ **k** $(\sqrt{2} - \sqrt{3}, 3\sqrt{2} + 4\sqrt{3}), (3\sqrt{2} + \sqrt{3}, -\sqrt{2} + 2\sqrt{3})$

(P) 2 The line segment AB has endpoints $A(-2, 5)$ and $B(a, b)$. The midpoint of AB is $M(4, 3)$. Find the values of a and b.

3 The line segment PQ is a diameter of a circle, where P and Q are (−4, 6) and (7, 8) respectively. Find the coordinates of the centre of the circle.

(P) 4 The line segment RS is a diameter of a circle, where R and S are $\left(\frac{4a}{5}, -\frac{3b}{4}\right)$ and $\left(\frac{2a}{5}, \frac{5b}{4}\right)$ respectively. Find the coordinates of the centre of the circle.

> **Problem-solving**
> Your answer will be in terms of a and b.

5 The line segment AB is a diameter of a circle, where A and B are (−3, −4) and (6, 10) respectively.

a Find the coordinates of the centre of the circle.

b Show the centre of the circle lies on the line $y = 2x$.

(P) 6 The line segment JK is a diameter of a circle, where J and K are $\left(\frac{3}{4}, \frac{4}{3}\right)$ and $\left(-\frac{1}{2}, 2\right)$ respectively. The centre of the circle lies on the line segment with equation $y = 8x + b$. Find the value of b.

(P) 7 The line segment AB is a diameter of a circle, where A and B are (0, −2) and (6, −5) respectively. Show that the centre of the circle lies on the line $x - 2y - 10 = 0$.

(P) 8 The line segment FG is a diameter of the circle centre (6, 1). Given F is (2, −3), find the coordinates of G.

(P) 9 The line segment CD is a diameter of the circle centre $(-2a, 5a)$. Given D has coordinates $(3a, -7a)$, find the coordinates of C.

(P) 10 The points $M(3, p)$ and $N(q, 4)$ lie on the circle centre (5, 6). The line segment MN is a diameter of the circle. Find the values of p and q.

> **Problem-solving**
> Use the formula for finding the midpoint:
> $\left(\frac{3+q}{2}, \frac{p+4}{2}\right) = (5, 6)$

(P) 11 The points $V(-4, 2a)$ and $W(3b, -4)$ lie on the circle centre $(b, 2a)$. The line segment VW is a diameter of the circle. Find the values of a and b.

Challenge

SKILLS CREATIVITY

A triangle has vertices at $A(3, 5)$, $B(7, 11)$ and $C(p, q)$. The midpoint of side BC is $M(8, 5)$.

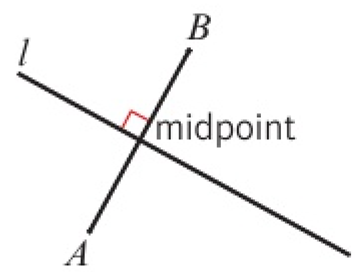
a Find the values of p and q.

b Find the equation of the straight line joining the midpoint of AB to the point M.

c Show that the line in part **b** is parallel to the line AC.

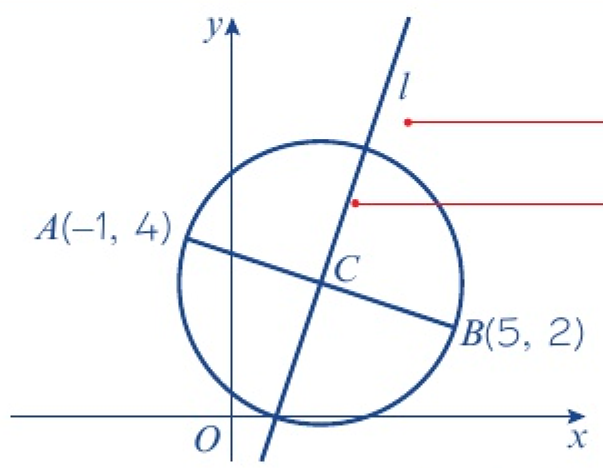
- The perpendicular bisector of a line segment AB is the straight line that is perpendicular to AB and passes through the midpoint of AB.

If the gradient of AB is m then the gradient of its perpendicular bisector, l, will be $-\frac{1}{m}$



Example 3 SKILLS ANALYSIS

The line segment AB is a diameter of the circle centre C, where A and B are $(-1, 4)$ and $(5, 2)$ respectively. The line l passes through C and is perpendicular to AB. Find the equation of l.



Draw a sketch.

l is the perpendicular bisector of AB.

The centre of the circle is $\left(\frac{-1+5}{2}, \frac{4+2}{2}\right)$
$= (2, 3)$

Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
Here $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (5, 2)$.

The gradient of the line segment AB is $\frac{2-4}{5-(-1)} = -\frac{1}{3}$

Use $m = \frac{y_2-y_1}{x_2-x_1}$. Here $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (5, 2)$.

Gradient of $l = 3$.

Remember the product of the gradients of two perpendicular lines is $= -1$, so $-\frac{1}{3} \times 3 = -1$.

The equation of l is

$y - 3 = 3(x - 2)$

$y - 3 = 3x - 6$

So $y = 3x - 3$

The perpendicular line l passes through the point $(2, 3)$ and has gradient 3, so use $y - y_1 = m(x - x_1)$ with $m = 3$ and $(x_1, y_1) = (2, 3)$.
Rearrange the equation into the form $y = mx + c$.

Exercise 2B SKILLS ANALYSIS

1 Find the perpendicular bisector of the line segment joining each pair of points:

a $A(-5, 8)$ and $B(7, 2)$ **b** $C(-4, 7)$ and $D(2, 25)$ **c** $E(3, -3)$ and $F(13, -7)$

d $P(-4, 7)$ and $Q(-4, -1)$ **e** $S(4, 11)$ and $T(-5, -1)$ **f** $X(13, 11)$ and $Y(5, 11)$

E/P **2** The line FG is a diameter of the circle centre C, where F and G are $(-2, 5)$ and $(2, 9)$ respectively. The line l passes through C and is perpendicular to FG. Find the equation of l. **(7 marks)**

P **3** The line JK is a diameter of the circle centre P, where J and K are $(0, -3)$ and $(4, -5)$ respectively. The line l passes through P and is perpendicular to JK. Find the equation of l. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

4 Points A, B, C and D have coordinates $A(-4, -9)$, $B(6, -3)$, $C(11, 5)$ and $D(-1, 9)$.

a Find the equation of the perpendicular bisector of line segment AB.

b Find the equation of the perpendicular bisector of line segment CD.

c Find the coordinates of the point of intersection of the two perpendicular bisectors.

(P) **5** Point X has coordinates $(7, -2)$ and point Y has coordinates $(4, q)$. The perpendicular bisector of XY has equation $y = 4x + b$. Find the value of q and the value of b.

Problem-solving

It is often easier to find unknown values in the order they are given in the question. Find q first, then find b.

Challenge

Triangle PQR has vertices at $P(6, 9)$, $Q(3, -3)$ and $R(-9, 3)$.

a Find the perpendicular bisectors of each side of the triangle.

b Show that all three perpendicular bisectors meet at a single point, and find the coordinates of that point.

Links This point of intersection is called the **circumcentre** of the triangle. ← Pure 2 Section 6.5

2.2 Equation of a circle

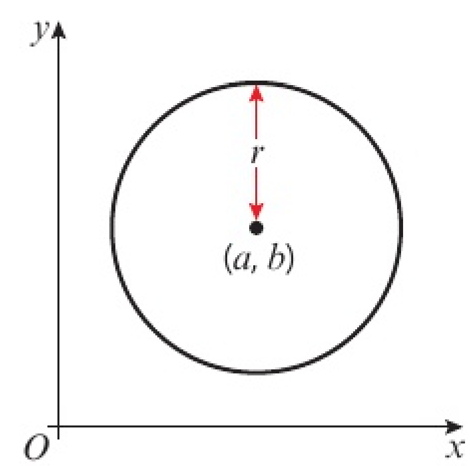
A circle is the set of points that are equidistant from a fixed point. You can use Pythagoras' theorem to derive equations of circles on a coordinate grid.

For any point (x, y) on the circumference of a circle, you can use Pythagoras' theorem to show the relationship between x, y and the **radius** r.

- **The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.**

When a circle has a centre (a, b) and radius r, you can use the following general form of the equation of a circle.

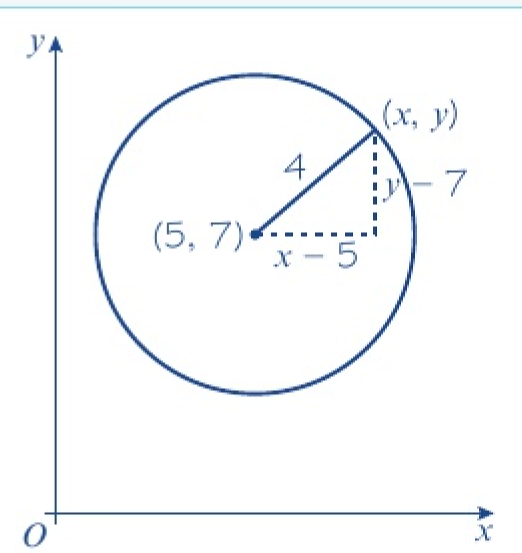
- **The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.**



Links This circle is a translation of the circle $x^2 + y^2 = r^2$ by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. ← Pure 2 Section 4.5

Example 4

Write down the equation of the circle with centre (5, 7) and radius 4.



$(x - 5)^2 + (y - 7)^2 = 4^2$ — Substitute $a = 5$, $b = 7$ and $r = 4$ into the equation.

$(x - 5)^2 + (y - 7)^2 = 16$ — Simplify by calculating $4^2 = 16$.

Online Explore the general form of the equation of a circle using technology.

Example 5

A circle has equation $(x-3)^2+(y+4)^2=20$.

a Write down the centre and radius of the circle.

b Show that the circle passes through $(5, -8)$.

a Centre $(3, -4)$, radius $\sqrt{20} = 2\sqrt{5}$

$r^2 = 20$ so $r = \sqrt{20}$

b $(x-3)^2+(y+4)^2=20$

Substitute $(5, -8)$

Substitute $x = 5$ and $y = -8$ into the equation of the circle.

$(5-3)^2+(-8+4)^2 = 2^2+(-4)^2$

$= 4 + 16$

$= 20$

$(5, -8)$ satisfies the equation of the circle.

So the circle passes through the point $(5, -8)$.

Example 6

SKILLS PROBLEM-SOLVING

The line segment AB is a diameter of a circle, where A and B are $(4, 7)$ and $(-8, 3)$ respectively. Find the equation of the circle.

Length of $AB = \sqrt{(4-(-8))^2+(7-3)^2}$

Use $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Here $(x_1, y_1) = (-8, 3)$ and $(x_2, y_2) = (4, 7)$

$= \sqrt{12^2+4^2}$

$= \sqrt{160}$

$= \sqrt{16} \times \sqrt{10}$

$= 4\sqrt{10}$

So the radius is $2\sqrt{10}$.

The centre is $\left(\frac{4+(-8)}{2}, \frac{7+3}{2}\right) = (-2, 5)$.

Remember the centre of a circle is at the midpoint of a diameter. Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

The equation of the circle is

$(x+2)^2+(y-5)^2 = (2\sqrt{10})^2$

Or $(x+2)^2+(y-5)^2 = 40$.

Problem-solving

You need to work out the steps of this problem yourself:
- Find the radius of the circle by finding the length of the diameter and dividing by 2.
- Find the centre of the circle by finding the midpoint of AB.
- Write down the equation of the circle.

You can multiply out the brackets in the equation of a circle to find it in an alternate form:

$$(x-a)^2+(y-b)^2=r^2$$
$$x^2-2ax+a^2+y^2-2by+b^2=r^2$$
$$x^2+y^2-2ax-2by+b^2+a^2-r^2=0$$

Compare the constant terms with the equation given in the key point: $b^2+a^2-r^2=c$ so $r=\sqrt{f^2+g^2-c}$

- **The equation of a circle can be given in the form:**

  $$x^2+y^2+2fx+2gy+c=0$$

- **This circle has centre $(-f, -g)$ and radius $\sqrt{f^2+g^2-c}$**

If you need to find the centre and radius of a circle with an equation given in expanded form it is usually safest to **complete the square** for the x and y terms.

Example 7 SKILLS INTERPRETATION

Find the centre and the radius of the circle with the equation $x^2 + y^2 - 14x + 16y - 12 = 0$.

Links You need to **complete the square** for the terms in x and for the terms in y.

← Pure 1 Section 2.2

Rearrange into the form $(x - a)^2 + (y - b)^2 = r^2$.

$x^2 + y^2 - 14x + 16y - 12 = 0$

$x^2 - 14x + y^2 + 16y - 12 = 0$ (1)

Group the x terms and y terms together.

Completing the square for x terms and y terms.

$x^2 - 14x = (x - 7)^2 - 49$

$y^2 + 16y = (y + 8)^2 - 64$

Substituting back into (1)

$(x - 7)^2 - 49 + (y + 8)^2 - 64 - 12 = 0$

$(x - 7)^2 + (y + 8)^2 = 125$

Move the number terms to the right-hand side of the equation.

$(x - 7)^2 + (y + 8)^2 = (\sqrt{125})^2$

Write the equation in the form $(x - a)^2 + (y - b)^2 = r^2$.

$\sqrt{125} = \sqrt{25} \times \sqrt{5} = 5\sqrt{5}$

Simplify $\sqrt{125}$.

The equation of the circle is

$(x - 7)^2 + (y + 8)^2 = (5\sqrt{5})^2$

The circle has centre $(7, -8)$ and radius $= 5\sqrt{5}$.

You could also compare the original equation with:

$x^2 + y^2 + 2fx + 2gy + c = 0$

$f = -7$, $g = 8$ and $c = -12$ so the circle has centre $(7, -8)$ and radius $\sqrt{(-7)^2 + 8^2 - (-12)} = 5\sqrt{5}$.

Exercise 2C SKILLS PROBLEM-SOLVING

1 Write down the equation of each circle:

a Centre (3, 2), radius 4 **b** Centre (−4, 5), radius 6 **c** Centre (5, −6), radius $2\sqrt{3}$

d Centre $(2a, 7a)$, radius $5a$ **e** Centre $(-2\sqrt{2}, -3\sqrt{2})$, radius 1

2 Write down the coordinates of the centre and the radius of each circle:

a $(x + 5)^2 + (y - 4)^2 = 9^2$ **b** $(x - 7)^2 + (y - 1)^2 = 16$ **c** $(x + 4)^2 + y^2 = 25$

d $(x + 4a)^2 + (y + a)^2 = 144a^2$ **e** $(x - 3\sqrt{5})^2 + (y + \sqrt{5})^2 = 27$

3 In each case, show that the circle passes through the given point:

a $(x - 2)^2 + (y - 5)^2 = 13$, point (4, 8) **b** $(x + 7)^2 + (y - 2)^2 = 65$, point (0, −2)

c $x^2 + y^2 = 25^2$, point (7, −24) **d** $(x - 2a)^2 + (y + 5a)^2 = 20a^2$, point $(6a, -3a)$

e $(x - 3\sqrt{5})^2 + (y - \sqrt{5})^2 = (2\sqrt{10})^2$ point, $(\sqrt{5}, -\sqrt{5})$

(P) **4** The point (4, −2) lies on the circle centre (8, 1). Find the equation of the circle.

Hint First find the radius of the circle.

E/P **5** The line PQ is the diameter of the circle, where P and Q are (5, 6) and (−2, 2) respectively.
Find the equation of the circle. **(5 marks)**

E/P **6** The point (1, −3) lies on the circle $(x - 3)^2 + (y + 4)^2 = r^2$. Find the value of r. **(3 marks)**

E/P **7** The points $P(2, 2)$, $Q(2 + \sqrt{3}, 5)$ and $R(2 - \sqrt{3}, 5)$ lie on the circle $(x - 2)^2 + (y - 4)^2 = r^2$.

a Find the value of r. **(2 marks)**

b Show that $\triangle PQR$ is equilateral. **(3 marks)**

E/P **8** **a** Show that $x^2 + y^2 - 4x - 11 = 0$ can be written in the form $(x - a)^2 + y^2 = r^2$, where a and r are numbers to be found. **(2 marks)**

b Hence write down the centre and radius of the circle with equation $x^2 + y^2 - 4x - 11 = 0$ **(2 marks)**

Problem-solving

Start by writing $(x^2 - 4x)$ in the form $(x - a)^2 - b$.

E/P **9** **a** Show that $x^2 + y^2 - 10x + 4y - 20 = 0$ can be written in the form $(x - a)^2 + (y - b)^2 = r^2$, where a, b and r are numbers to be found. **(2 marks)**

b Hence write down the centre and radius of the circle with equation $x^2 + y^2 - 10x + 4y - 20 = 0$. **(2 marks)**

10 Find the centre and radius of the circle with each of the following equations.

a $x^2 + y^2 - 2x + 8y - 8 = 0$

b $x^2 + y^2 + 12x - 4y = 9$

c $x^2 + y^2 - 6y = 22x - 40$

d $x^2 + y^2 + 5x - y + 4 = 2y + 8$

e $2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$

Hint Start by writing the equation in one of the following forms:
$(x - a)^2 + (y - b)^2 = r^2$
$x^2 + y^2 + 2fx + 2gy + c = 0$

E/P **11** A circle C has equation $x^2 + y^2 + 12x + 2y = k$, where k is a constant.

a Find the coordinates of the centre of C. **(2 marks)**

b State the range of possible values of k. **(2 marks)**

Problem-solving

A circle must have a positive radius.

E/P **12** The point $P(7, -14)$ lies on the circle with equation $x^2 + y^2 + 6x - 14y = 483$.
The point Q also lies on the circle such that PQ is a diameter.
Find the coordinates of point Q. **(4 marks)**

E/P **13** The circle with equation $(x - k)^2 + y^2 = 41$ passes through the point (3, 4).
Find the two possible values of k. **(5 marks)**

Challenge

SKILLS CREATIVITY

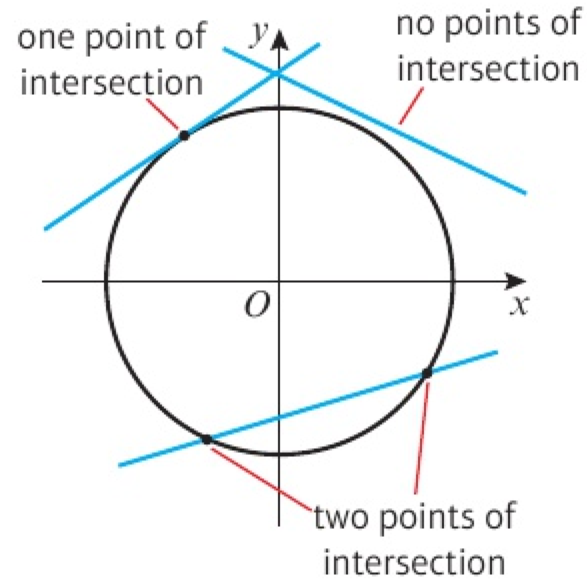
1 A circle with equation $(x - k)^2 + (y - 2)^2 = 50$ passes through the point (4, −5).
Find the possible values of k and the equation of each circle.

2 By completing the square for x and y, show that the equation $x^2 + y^2 + 2fx + 2gy + c = 0$ describes a circle with centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$.

2.3 Intersections of straight lines and circles

You can use algebra to find the coordinates of intersection of a straight line and a circle.

- **A straight line can intersect a circle once, by just touching the circle, or twice. Not all straight lines will intersect a given circle.**



Example 8

Find the coordinates of the points where the line $y = x + 5$ meets the circle $x^2 + (y - 2)^2 = 29$.

$$x^2 + (y - 2)^2 = 29$$
$$x^2 + (x + 5 - 2)^2 = 29$$
$$x^2 + (x + 3)^2 = 29$$
$$x^2 + x^2 + 6x + 9 = 29$$
$$2x^2 + 6x - 20 = 0$$
$$x^2 + 3x - 10 = 0$$
$$(x + 5)(x - 2) = 0$$

So $x = -5$ and $x = 2$.

$x = -5$: $y = -5 + 5 = 0$

$x = 2$: $y = 2 + 5 = 7$

The line meets the circle at $(-5, 0)$ and $(2, 7)$.

Online Explore intersections of straight lines and circles using GeoGebra.

Solve the equations simultaneously, so substitute $y = x + 5$ into the equation of the circle. ← Pure 1 Section 3.2

Simplify the equation to form a quadratic equation.

The resulting quadratic equation has two distinct solutions, so the line intersects the circle at two distinct points.

Now find the y-coordinates, so substitute the values of x into the equation of the line.

Remember to write the answers as coordinates.

Example 9

SKILLS EXECUTIVE FUNCTION

Show that the line $y = x - 7$ does not meet the circle $(x + 2)^2 + y^2 = 33$.

$$(x + 2)^2 + y^2 = 33$$
$$(x + 2)^2 + (x - 7)^2 = 33$$
$$x^2 + 4x + 4 + x^2 - 14x + 49 = 33$$
$$2x^2 - 10x + 20 = 0$$
$$x^2 - 5x + 10 = 0$$

Now $b^2 - 4ac = (-5)^2 - 4 \times 1 \times 10$
$= 25 - 40$
$= -15$

$b^2 - 4ac < 0$, so the line does not meet the circle.

Try to solve the equations simultaneously, so substitute $y = x - 7$ into the equation of the circle.

Use the discriminant $b^2 - 4ac$ to test for roots of the quadratic equation. ← Pure 1 Section 2.5

Problem-solving

If $b^2 - 4ac > 0$ there are two distinct roots.
If $b^2 - 4ac = 0$ there is a repeated root.
If $b^2 - 4ac < 0$ there are no real roots.

Exercise 2D

SKILLS EXECUTIVE FUNCTION

1 Find the coordinates of the points where the circle $(x-1)^2 + (y-3)^2 = 45$ meets the x-axis.

Hint Substitute $y = 0$ into the equation.

2 Find the coordinates of the points where the circle $(x-2)^2 + (y+3)^2 = 29$ meets the y-axis.

3 The line $y = x + 4$ meets the circle $(x-3)^2 + (y-5)^2 = 34$ at A and B. Find the coordinates of A and B.

4 Find the coordinates of the points where the line $x + y + 5 = 0$ meets the circle $x^2 + 6x + y^2 + 10y - 31 = 0$.

P 5 Show that the line $x - y - 10 = 0$ does not meet the circle $x^2 - 4x + y^2 = 21$.

Problem-solving
Attempt to solve the equations simultaneously. Use the discriminant to show that the resulting quadratic equation has no solutions.

E/P 6 **a** Show that the line $x + y = 11$ meets the circle with equation $x^2 + (y-3)^2 = 32$ at only one point. **(4 marks)**

b Find the coordinates of the point of intersection. **(1 mark)**

E/P 7 The line $y = 2x - 2$ meets the circle $(x-2)^2 + (y-2)^2 = 20$ at A and B.

a Find the coordinates of A and B. **(5 marks)**

b Show that AB is a diameter of the circle. **(2 marks)**

E/P 8 The line $x + y = a$ meets the circle $(x-p)^2 + (y-6)^2 = 20$ at $(3, 10)$, where a and p are constants.

a Work out the value of a. **(1 mark)**

b Work out the two possible values of p. **(5 marks)**

E/P 9 The circle with equation $(x-4)^2 + (y+7)^2 = 50$ meets the straight line with equation $x - y - 5 = 0$ at points A and B.

a Find the coordinates of the points A and B. **(5 marks)**

b Find the equation of the perpendicular bisector of line segment AB. **(3 marks)**

c Show that the perpendicular bisector of AB passes through the centre of the circle. **(1 mark)**

d Find the area of triangle OAB. **(2 marks)**

E/P 10 The line with equation $y = kx$ intersects the circle with equation $x^2 - 10x + y^2 - 12y + 57 = 0$ at two distinct points.

a Show that $21k^2 - 60k + 32 < 0$. **(5 marks)**

b Determine the range of possible values for k. Round your answer to 2 decimal places. **(3 marks)**

E/P 11 The line with equation $y = 4x - 1$ does not intersect the circle with equation $x^2 + 2x + y^2 = k$. Find the range of possible values of k.

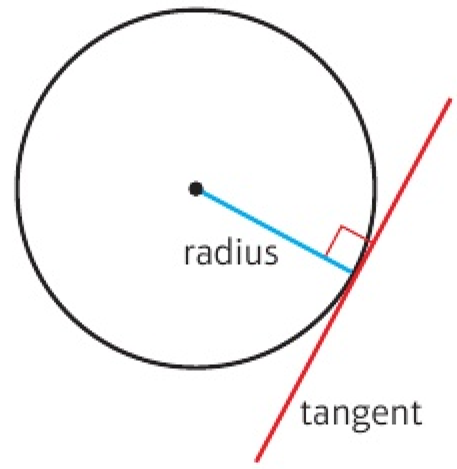
E/P 12 The line with equation $y = 2x + 5$ meets the circle with equation $x^2 + kx + y^2 = 4$ at exactly one point. Find two possible values of k. **(7 marks)**

Problem-solving
If you are solving a problem where there are 0, 1 or 2 solutions (or points of intersection), you might be able to use the discriminant.

2.4 Use tangent and chord properties

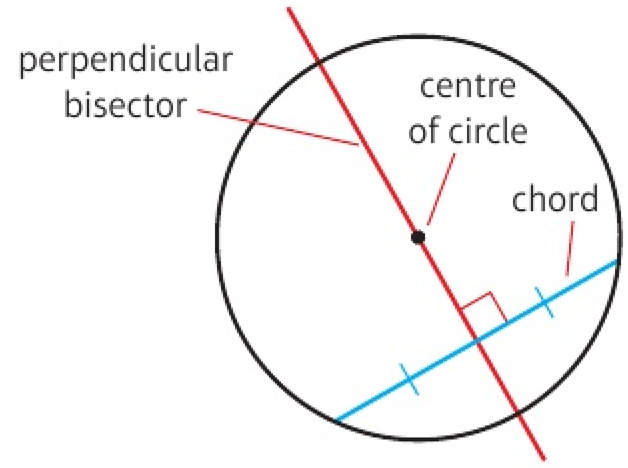
You can use the properties of tangents and **chords** within circles to solve geometric problems. A tangent to a circle is a straight line that intersects the circle at only one point.

- **A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.**



A chord is a line segment that joins two points on the circumference of a circle.

- **The perpendicular bisector of a chord will go through the centre of a circle.**



Online Explore the circle theorems using GeoGebra.

Example 10

The circle C has equation $(x - 2)^2 + (y - 6)^2 = 100$.

a Verify that the point $P(10, 0)$ lies on C.

b Find an equation of the tangent to C at the point $(10, 0)$, giving your answer in the form $ax + by + c = 0$.

a $(x - 2)^2 + (y - 6)^2 = (10 - 2)^2 + (0 - 6)^2$

$= 8^2 + (-6)^2$

$= 64 + 36$

$= 100$ ✓

Substitute $(x, y) = (10, 0)$ into the equation for the circle.

The point $P(10, 0)$ satisfies the equation, so P lies on C.

b The centre of circle C is $(2, 6)$. Find the gradient of the line between $(2, 6)$ and P.

A circle with equation $(x - a)^2 + (y - b)^2 = r^2$ has centre (a, b).

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 0}{2 - 10} = \frac{6}{-8} = -\frac{3}{4}$

Use the gradient formula with $(x_1, y_1) = (10, 0)$ and $(x_2, y_2) = (2, 6)$.

The gradient of the tangent is $\frac{4}{3}$

The tangent is perpendicular to the radius at that point. If the gradient of the radius is m then the gradient of the tangent will be $-\frac{1}{m}$

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{4}{3}(x - 10)$

Substitute $(x_1, y_1) = (10, 0)$ and $m = \frac{4}{3}$ into the equation for a straight line.

$3y = 4x - 40$

Simplify.

$4x - 3y - 40 = 0$

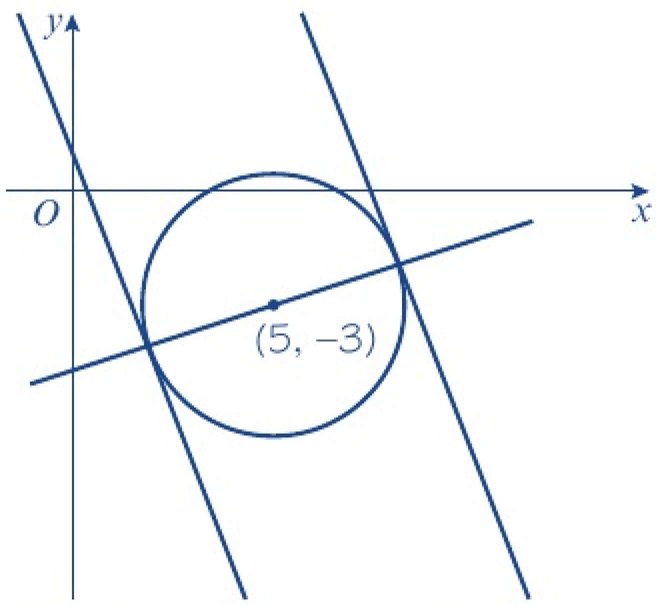
Leave the answer in the correct form.

Example 11 SKILLS ANALYSIS

A circle C has equation $(x-5)^2 + (y+3)^2 = 10$.

The line l is a tangent to the circle and has gradient -3.

Find two possible equations for l, giving your answers in the form $y = mx + c$.



Problem-solving

Draw a sketch showing the circle and the two possible tangents with gradient −3. If you are solving a problem involving tangents and circles there is a good chance you will need to use the radius at the point of intersection, so draw this on your sketch.

Find a line that passes through the centre of the circle that is perpendicular to the tangents.

This line will intersect the circle at the same points where the tangent intersects the circle.

The gradient of this line is $\frac{1}{3}$

The gradient of the tangents is −3, so the gradient of a perpendicular line will be $\frac{-1}{-3} = \frac{1}{3}$

The coordinates of the centre of circle are (5, −3)

A circle with equation $(x-a)^2 + (y-b)^2 = r^2$ has centre (a, b).

$$y - y_1 = m(x - x_1)$$

$$y + 3 = \frac{1}{3}(x-5)$$

$$y + 3 = \frac{1}{3}x - \frac{5}{3}$$

Substitute $(x_1, y_1) = (5, -3)$ and $m = \frac{1}{3}$ into the equation for a straight line.

$$y = \frac{1}{3}x - \frac{14}{3}$$

This is the equation of the line passing through the circle.

$$(x-5)^2 + (y+3)^2 = 10$$

$$(x-5)^2 + \left(\frac{1}{3}x - \frac{14}{3} + 3\right)^2 = 10$$

Substitute $y = \frac{1}{3}x - \frac{14}{3}$ into the equation for a circle to find the points of intersection.

$$(x-5)^2 + \left(\frac{1}{3}x - \frac{5}{3}\right)^2 = 10$$

$$x^2 - 10x + 25 + \frac{1}{9}x^2 - \frac{10}{9}x + \frac{25}{9} = 10$$

Simplify the expression.

$$\frac{10}{9}x^2 - \frac{100}{9}x + \frac{250}{9} = 10$$

$$10x^2 - 100x + 250 = 90$$

$$10x^2 - 100x + 160 = 0$$

$$x^2 - 10x + 16 = 0$$

$$(x-8)(x-2) = 0$$

Factorise to find the values of x.

$$x = 8 \text{ or } x = 2$$

$$y = -\frac{6}{3} = -2 \text{ or } y = -4$$

Substitute $x = 8$ and $x = 2$ into $y = \frac{1}{3}x - \frac{14}{3}$.

So the tangents will intersect the circle at (8, −2) and (2, −4)

$y - y_1 = m(x - x_1)$

$y + 2 = -3(x - 8)$ — Substitute $(x_1, y_1) = (8, -2)$ and $m = -3$ into the equation for a straight line.

$y = -3x + 22$ — This is one possible equation for the tangent.

$y - y_1 = m(x - x_1)$

$y + 4 = -3(x - 2)$ — Substitute $(x_1, y_1) = (2, -4)$ and $m = -3$ into the equation for a straight line.

$y = -3x + 2$ — This is the other possible equation for the tangent.

Example 12

SKILLS EXECUTIVE FUNCTION

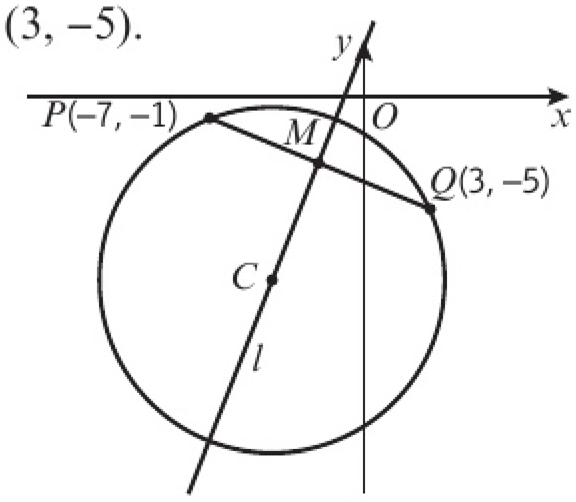
The points P and Q lie on a circle with centre C, as shown in the diagram.
The point P has coordinates (−7, −1) and the point Q has coordinates (3, −5).
M is the midpoint of the line segment PQ.
The line l passes through the points M and C.

a Find an equation for l.

Given that the y-coordinate of C is −8,

b show that the x-coordinate of C is −4

c find an equation of the circle.



a The midpoint M of line segment PQ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-7 + 3}{2}, \frac{-1 + (-5)}{2}\right)$$

Use the midpoint formula with $(x_1, y_1) = (-7, -1)$ and $(x_1, y_1) = (3, -5)$.

$$= (-2, -3)$$

$$\text{Gradient of } PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - (-1)}{3 - (-7)}$$

Use the gradient formula with $(x_1, y_1) = (-7, -1)$ and $(x_1, y_1) = (3, -5)$.

$$= \frac{-4}{10} = -\frac{2}{5}$$

The gradient of a line perpendicular to PQ is $\frac{5}{2}$

Problem-solving

If a gradient is given as a fraction, you can find the perpendicular gradient quickly by turning the fraction upside down and changing the sign.

$y - y_1 = m(x - x_1)$

$y + 3 = \frac{5}{2}(x + 2)$ — Substitute $(x_1, y_1) = (-2, -3)$ and $m = \frac{5}{2}$ into the equation of a straight line.

$y + 3 = \frac{5}{2}x + 5$

$y = \frac{5}{2}x + 2$ — Simplify and leave in the form $y = mx + c$.

b $y = \frac{5}{2}x + 2$

$-8 = \frac{5}{2}x + 2$ — The perpendicular bisector of any chord passes through the centre of the circle. Substitute $y = -8$ into the equation of the straight line to find the corresponding x-coordinate.

$\frac{5}{2}x = -10$

$x = -4$ — Solve the equation to find x.

c The centre of the circle is $(-4, -8)$.
To find the radius of the circle:

$$CQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(3 - (-4))^2 + (-5 - (-8))^2}$$
$$= \sqrt{49 + 9} = \sqrt{58}$$

So the circle has a radius of $\sqrt{58}$.

The equation of the circle is:

$$(x - a)^2 + (y - b)^2 = r^2$$
$$(x + 4)^2 + (y + 8)^2 = 58$$

The radius is the length of the line segment CP or CQ.

Substitute $(x_1, y_1) = (-4, -8)$ and $(x_2, y_2) = (3, -5)$.

Substitute $(a, b) = (-4, -8)$ and $r = \sqrt{58}$ into the equation of a circle.

Exercise 2E

SKILLS PROBLEM-SOLVING

1 The line $x + 3y - 11 = 0$ touches the circle $(x + 1)^2 + (y + 6)^2 = r^2$ at $(2, 3)$.

a Find the radius of the circle.

b Show that the radius at $(2, 3)$ is perpendicular to the line.

2 The point $P(1, -2)$ lies on the circle centre $(4, 6)$.

a Find the equation of the circle.

b Find the equation of the tangent to the circle at P.

3 The points A and B with coordinates $(-1, -9)$ and $(7, -5)$ lie on the circle C with equation $(x - 1)^2 + (y + 3)^2 = 40$.

a Find the equation of the perpendicular bisector of the line segment AB.

b Show that the perpendicular bisector of AB passes through the centre of the circle C.

P **4** The points P and Q with coordinates $(3, 1)$ and $(5, -3)$ lie on the circle C with equation $x^2 - 4x + y^2 + 4y = 2$.

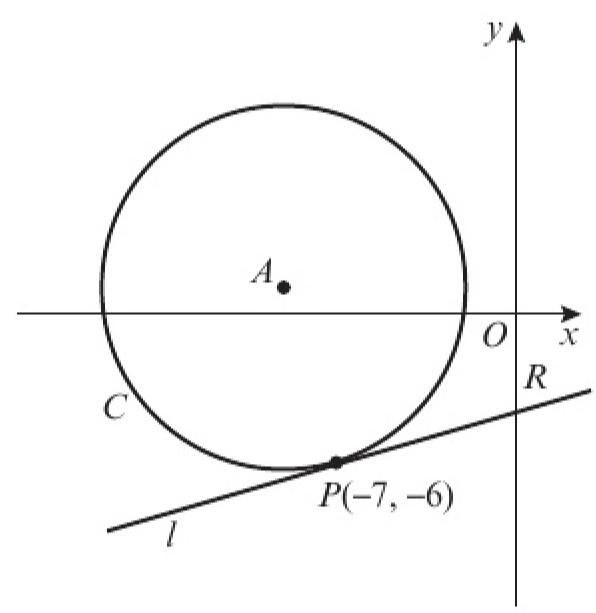
a Find the equation of the perpendicular bisector of the line segment PQ.

b Show that the perpendicular bisector of PQ passes through the centre of the circle C.

E **5** The circle C has equation $x^2 + 18x + y^2 - 2y + 29 = 0$.

a Verify the point $P(-7, -6)$ lies on C. **(2 marks)**

b Find an equation for the tangent to C at the point P, giving your answer in the form $y = mx + b$. **(4 marks)**



c Find the coordinates of R, the point of intersection of the tangent and the y-axis. **(2 marks)**

d Find the area of the triangle APR. **(2 marks)**

E/P **6** The tangent to the circle $(x + 4)^2 + (y - 1)^2 = 242$ at $(7, -10)$ meets the y-axis at S and the x-axis at T.

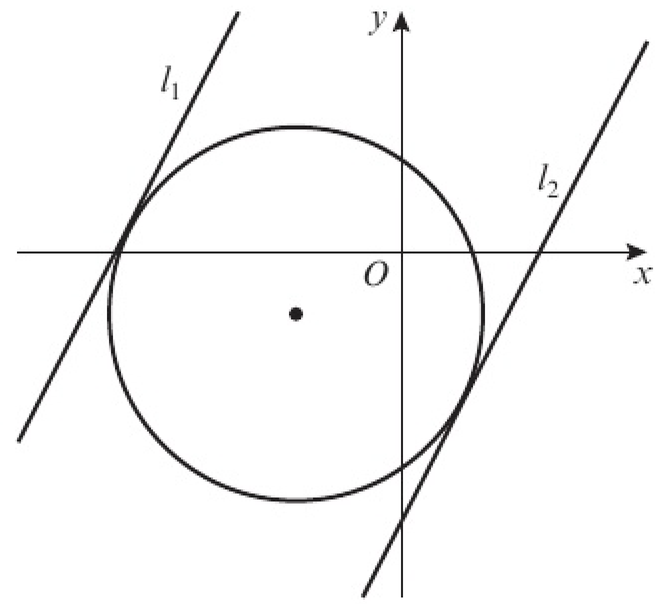
a Find the coordinates of S and T. **(5 marks)**

b Hence, find the area of $\triangle OST$, where O is the origin. **(3 marks)**

E/P **7** The circle C has equation $(x + 5)^2 + (y + 3)^2 = 80$.

The line l is a tangent to the circle and has gradient 2.

Find two possible equations for l giving your answers in the form $y = mx + c$. **(8 marks)**



E/P **8** The line with equation $2x + y - 5 = 0$ is a tangent to the circle with equation

$(x - 3)^2 + (y - p)^2 = 5$

a Find the two possible values of p. **(8 marks)**

b Write down the coordinates of the centre of the circle in each case. **(2 marks)**

Problem-solving

The line is a tangent to the circle so it must intersect at exactly one point. You can use the discriminant to determine the values of p for which this occurs.

E/P **9** The circle C has centre $P(11, -5)$ and passes through the point $Q(5, 3)$.

a Find an equation for C. **(3 marks)**

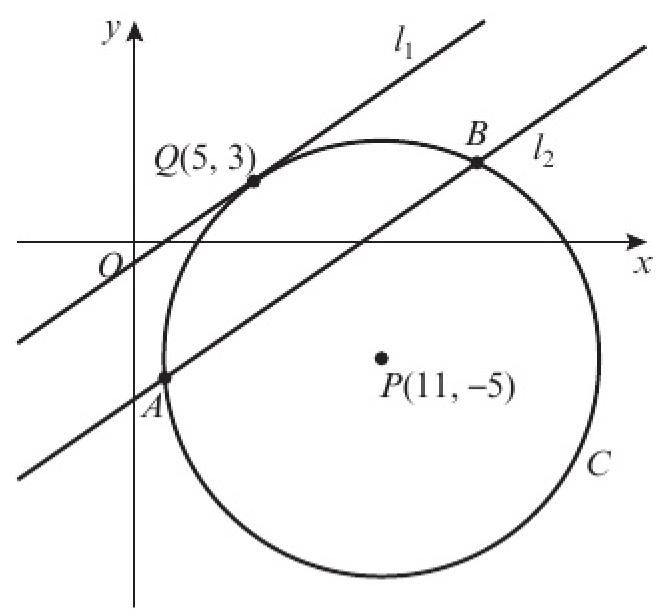
The line l_1 is a tangent to C at the point Q.

b Find an equation for l_1. **(4 marks)**

The line l_2 is parallel to l_1 and passes through the midpoint of PQ. Given that l_2 intersects C at A and B

c find the coordinates of points A and B **(4 marks)**

d find the length of the line segment AB, leaving your answer in its simplest surd form. **(3 marks)**



E/P **10** The points R and S lie on a circle with centre $C(a, -2)$, as shown in the diagram.

The point R has coordinates $(2, 3)$ and the point S has coordinates $(10, 1)$.

M is the midpoint of the line segment RS.

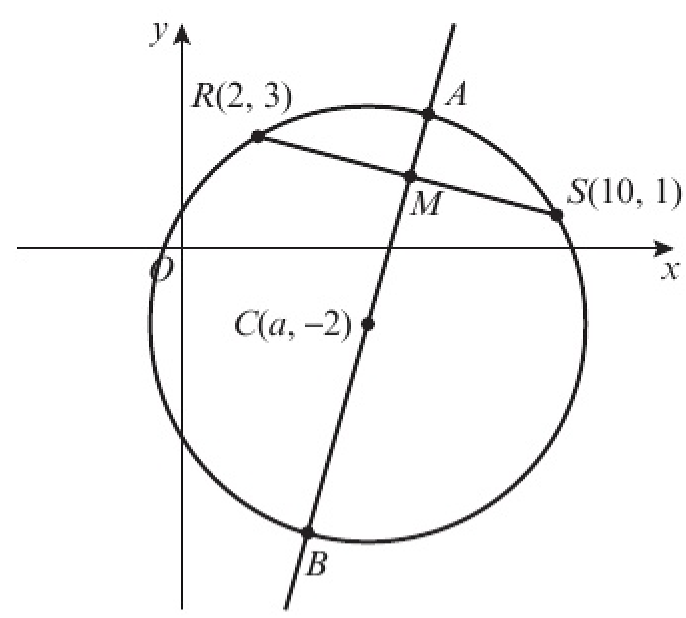
The line l passes through M and C.

a Find an equation for l. **(4 marks)**

b Find the value of a. **(2 marks)**

c Find the equation of the circle. **(3 marks)**

d Find the points of intersection, A and B, of the line l and the circle. **(5 marks)**



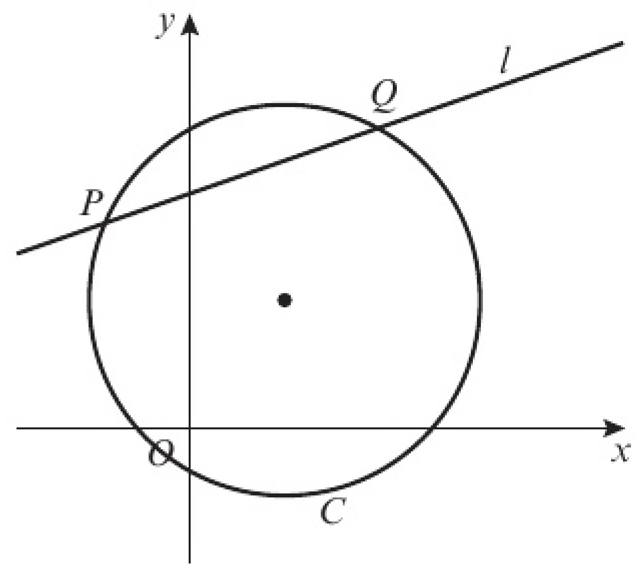
E/P **11** The circle C has equation $x^2 - 4x + y^2 - 6y = 7$.

The line l with equation $x - 3y + 17 = 0$ intersects the circle at the points P and Q.

a Find the coordinates of the point P and the point Q. **(4 marks)**

b Find the equation of the tangent at the point P and the point Q. **(4 marks)**

c Find the equation of the perpendicular bisector of the chord PQ. **(3 marks)**



d Show that the two tangents and the perpendicular bisector intersect at a single point and find the coordinates of the point of intersection. **(2 marks)**

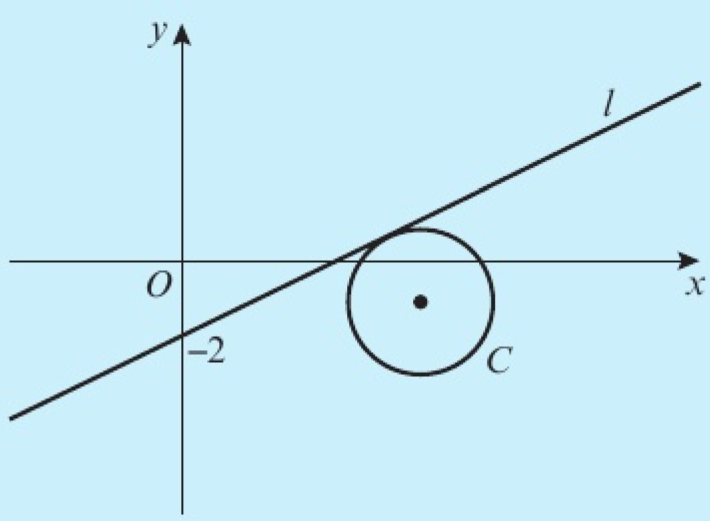
Challenge

SKILLS INNOVATION

1 The circle C has equation $(x - 7)^2 + (y + 1)^2 = 5$.

The line l with positive gradient passes through $(0, -2)$ and is a tangent to the circle.

Find an equation of l, giving your answer in the form $y = mx + c$.



2 The circle with centre C has equation $(x - 2)^2 + (y - 1)^2 = 10$.

The tangents to the circle at points P and Q meet at the point R with coordinates $(6, -1)$.

a Show that $CPRQ$ is a square.

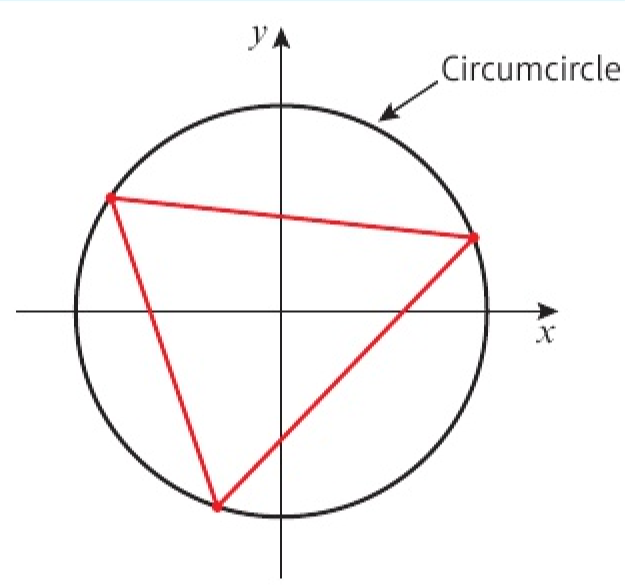
b Hence find the equations of both tangents.

Problem-solving

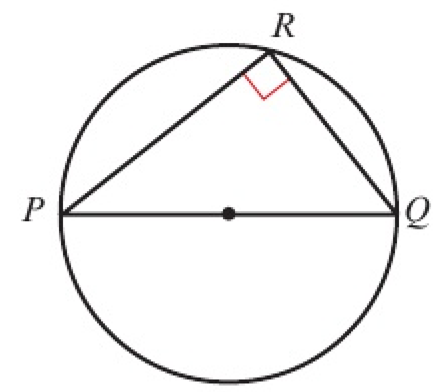
Use the point $(0, -2)$ to write an equation for the tangent in terms of m. Substitute this equation into the equation for the circle.

2.5 Circles and triangles

A triangle consists of three points, called vertices.
It is always possible to draw a unique circle through the three vertices of any triangle. This circle is called the **circumcircle** of the triangle. The centre of the circle is called the **circumcentre** of the triangle and is the point where the perpendicular bisectors of each side intersect.



For a right-angled triangle, the **hypotenuse** of the triangle is a diameter of the circumcircle.

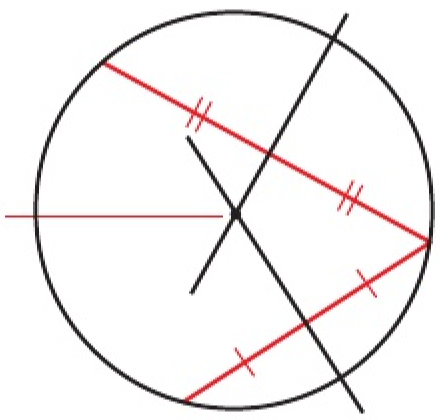


You can state this result in two other ways:

- **If $\angle PRQ = 90°$ then R lies on the circle with diameter PQ.**
- **The angle in a semicircle is always a right angle.**

To find the centre of a circle given any three points on the circumference:

- **Find the equations of the perpendicular bisectors of two different chords.**
- **Find the coordinates of the point of intersection of the perpendicular bisectors.**

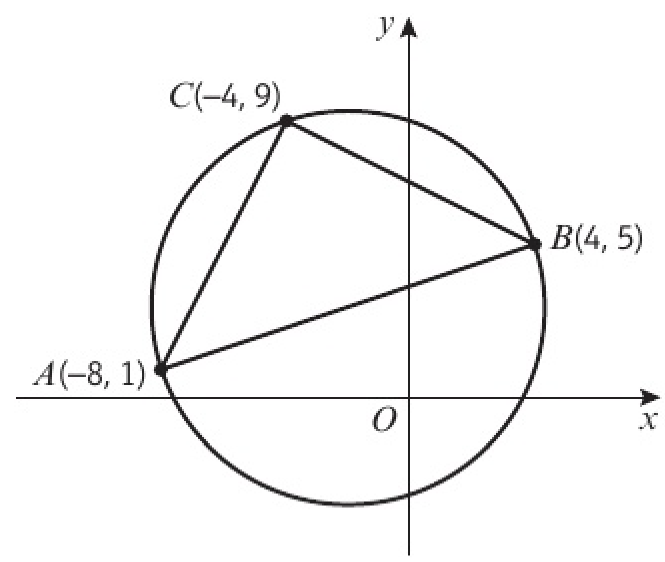


Example 13

The points $A(-8, 1)$, $B(4, 5)$ and $C(-4, 9)$ lie on the circle, as shown in the diagram.

a Show that AB is a diameter of the circle.

b Find an equation of the circle.



a Test triangle ABC to see if it is a right-angled triangle.

$AB^2 = (4-(-8))^2 + (5 - 1)^2$
$= 12^2 + 4^2 = 160$
$AC^2 = (-4-(-8))^2 + (9 - 1)^2$
$= 4^2 + 8^2 = 80$
$BC^2 = (-4 - 4)^2 + (9 - 5)^2$
$= (-8)^2 + 4^2 = 80$

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ to determine the length of each side of the triangle ABC.

Now, $80 + 80 = 160$ so $AC^2 + BC^2 = AB^2$

Use Pythagoras' theorem to test if triangle ABC is a right-angled triangle.

So triangle ABC is a right-angled triangle and AB is the diameter of the circle.

If ABC is a right-angled triangle, its longest side must be a diameter of the circle that passes through all three points.

b Find the midpoint M of AB.

The centre of the circle is the midpoint of AB.

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-8 + 4}{2}, \frac{1 + 5}{2}\right)$
$= (-2, 3)$

Substitute $(x_1, y_1) = (-8, 1)$ and $(x_2, y_2) = (4, 5)$.

The diameter is $\sqrt{160} = 4\sqrt{10}$

From part **a**, $AB^2 = 160$.

The radius is $2\sqrt{10}$

The radius is half the diameter.

$(x - a)^2 + (y - b)^2 = r^2$
$(x + 2)^2 + (y - 3)^2 = (2\sqrt{10})^2$
$(x + 2)^2 + (y - 3)^2 = 40$

Substitute $(a, b) = (-2, 3)$ and $r = 2\sqrt{10}$ into the equation for a circle.

Example 14

SKILLS EXECUTIVE FUNCTION

The points $P(3, 16)$, $Q(11, 12)$ and $R(-7, 6)$ lie on the circumference of a circle. The equation of the perpendicular bisector of PQ is $y = 2x$.

a Find the equation of the perpendicular bisector of PR.

b Find the centre of the circle.

c Work out the equation of the circle.

Online Explore triangles and their circumcircles using GeoGebra.

a The midpoint of PR is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{3 + (-7)}{2}, \frac{16 + 6}{2}\right) = (-2, 11)$

> The perpendicular bisector of PR passes through the midpoint of PR.
> Substitute $(x_1, y_1) = (3, 16)$ and $(x_2, y_2) = (-7, 6)$ into the midpoint formula.

The gradient of PR is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 16}{-7 - 3}$

$= \frac{-10}{-10} = 1$

> Substitute $(x_1, y_1) = (3, 16)$ and $(x_2, y_2) = (-7, 6)$ into the gradient formula.

The gradient of a line perpendicular to PR is -1.

$y - y_1 = m(x - x_1)$

$y - 11 = -1(x - (-2))$

> Substitute $m = -1$ and $(x_1, y_1) = (-2, 11)$ into the equation for a straight line.

$y - 11 = -x - 2$

$y = -x + 9$

> Simplify and leave in the form $y = mx + c$.

b Equation of perpendicular bisector to PQ: $y = 2x$

Equation of perpendicular bisector to PR: $y = -x + 9$

> Solve these two equations simultaneously to find the point of intersection. The two perpendicular bisectors intersect at the centre of the circle.

$2x = -x + 9$

$3x = 9$

$x = 3$

> This is the x-coordinate of the centre of the circle.

$y = 2x$

$y = 2(3) = 6$

> Substitute $x = 3$ to find the y-coordinate of the centre of the circle.

The centre of the circle is at $(3, 6)$.

c Find the distance between $(3, 6)$ and $Q(11, 12)$.

> The radius of the circle is the distance from the centre to a point on the circumference of the circle.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(11 - 3)^2 + (12 - 6)^2}$

> Substitute $(x_1, y_1) = (3, 6)$ and $(x_2, y_2) = (11, 12)$ into the distance formula.

$d = \sqrt{64 + 36}$

$d = \sqrt{100} = 10$

> Simplify to find the radius of the circle.

The circle through the points P, Q and R has a radius of 10.

The centre of the circle is $(3, 6)$.

The equation for the circle is

$(x - 3)^2 + (y - 6)^2 = 100$

> Substitute $(a, b) = (3, 6)$ and $r = 10$ into $(x - a)^2 + (y - b)^2 = r^2$

Exercise 2F

SKILLS EXECUTIVE FUNCTION

1 The points $U(-2, 8)$, $V(7, 7)$ and $W(-3, -1)$ lie on a circle.

a Show that triangle UVW has a right angle.

b Find the coordinates of the centre of the circle.

c Write down an equation for the circle.

2 The points $A(2, 6)$, $B(5, 7)$ and $C(8, -2)$ lie on a circle.

a Show that AC is a diameter of the circle.

b Write down an equation for the circle.

c Find the area of the triangle ABC.

3 The points $A(-3, 19)$, $B(9, 11)$ and $C(-15, 1)$ lie on the circumference of a circle.

a Find the equation of the perpendicular bisector of

i AB **ii** AC

b Find the coordinates of the centre of the circle.

c Write down an equation for the circle.

4 The points $P(-11, 8)$, $Q(-6, -7)$ and $R(4, -7)$ lie on the circumference of a circle.

a Find the equation of the perpendicular bisector of

i PQ **ii** QR

b Find an equation for the circle.

(P) 5 The points $R(-2, 1)$, $S(4, 3)$ and $T(10, -5)$ lie on the circumference of a circle C. Find an equation for the circle.

Problem-solving

Use headings in your working to keep track of what you are working out at each stage.

(P) 6 Consider the points $A(3, 15)$, $B(-13, 3)$, $C(-7, -5)$ and $D(8, 0)$.

a Show that ABC is a right-angled triangle.

b Find the equation of the circumcircle of triangle ABC.

c Hence show that A, B, C and D all lie on the circumference of this circle.

(P) 7 The points $A(-1, 9)$, $B(6, 10)$, $C(7, 3)$ and $D(0, 2)$ lie on a circle.

a Show that $ABCD$ is a square.

b Find the area of $ABCD$.

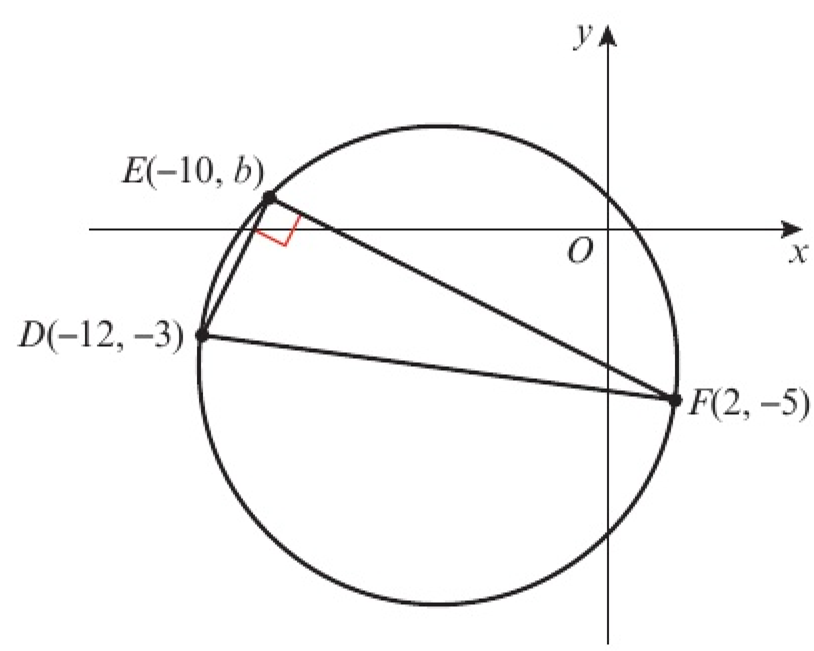
c Find the centre of the circle.

(E/P) 8 The points $D(-12, -3)$, $E(-10, b)$ and $F(2, -5)$ lie on the circle C as shown in the diagram.

Given that $\angle DEF = 90°$ and $b > 0$

a show that $b = 1$ **(5 marks)**

b find an equation for C. **(4 marks)**



E/P **9** A circle has equation $x^2 + 2x + y^2 - 24y - 24 = 0$

a Find the centre and radius of the circle. **(3 marks)**

b The points $A(-13, 17)$ and $B(11, 7)$ both lie on the circumference of the circle. Show that AB is a diameter of the circle. **(3 marks)**

c The point C lies on the negative x-axis and the angle $ACB = 90°$. Find the coordinates of C. **(3 marks)**

Chapter review 2

P **1** The line segment QR is a diameter of the circle centre C, where Q and R have coordinates (11, 12) and (−5, 0) respectively. The point P has coordinates (13, 6).

a Find the coordinates of C.

b Find the radius of the circle.

c Write down the equation of the circle.

d Show that P lies on the circle.

P **2** Show that (0, 0) lies inside the circle $(x - 5)^2 + (y + 2)^2 = 30$.

E/P **3** The circle C has equation $x^2 + 3x + y^2 + 6y = 3x - 2y - 7$.

a Find the centre and radius of the circle. **(4 marks)**

b Find the points of intersection of the circle and the y-axis. **(3 marks)**

c Show that the circle does not intersect the x-axis. **(2 marks)**

4 The centres of the circles $(x - 8)^2 + (y - 8)^2 = 117$ and $(x + 1)^2 + (y - 3)^2 = 106$ are P and Q respectively.

a Show that P lies on $(x + 1)^2 + (y - 3)^2 = 106$.

b Find the length of PQ.

P **5** The points $A(-1, 0)$, $B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $C\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ are the vertices of a triangle.

a Show that the circle $x^2 + y^2 = 1$ passes through the vertices of the triangle.

b Show that $\triangle ABC$ is equilateral.

E/P **6** A circle with equation $(x - k)^2 + (y - 3k)^2 = 13$ passes through the point (3, 0).

a Find two possible values of k. **(6 marks)**

b Given that $k > 0$, write down the equation of the circle. **(1 mark)**

E/P **7** The line with $3x - y - 9 = 0$ does not intersect the circle with equation $x^2 + px + y^2 + 4y = 20$. Show that $42 - 10\sqrt{10} < p < 42 + 10\sqrt{10}$. **(6 marks)**

P **8** The line $y = 2x - 8$ meets the coordinate axes at A and B. The line segment AB is a diameter of the circle. Find the equation of the circle.

P **9** The circle centre (8, 10) meets the x-axis at (4, 0) and $(a, 0)$.

a Find the radius of the circle.

b Find the value of a.

10 The circle $(x - 5)^2 + y^2 = 36$ meets the x-axis at P and Q. Find the coordinates of P and Q.

11 The circle $(x + 4)^2 + (y - 7)^2 = 121$ meets the y-axis at $(0, m)$ and $(0, n)$. Find the values of m and n.

E **12** The circle C with equation $(x + 5)^2 + (y + 2)^2 = 125$ meets the positive coordinate axes at $A(a, 0)$ and $B(0, b)$.

a Find the values of a and b. **(2 marks)**

b Find the equation of the line AB. **(2 marks)**

c Find the area of the triangle OAB, where O is the origin. **(2 marks)**

P **13** The circle, centre (p, q) radius 25, meets the x-axis at $(-7, 0)$ and $(7, 0)$, where $q > 0$.

a Find the values of p and q.

b Find the coordinates of the points where the circle meets the y-axis.

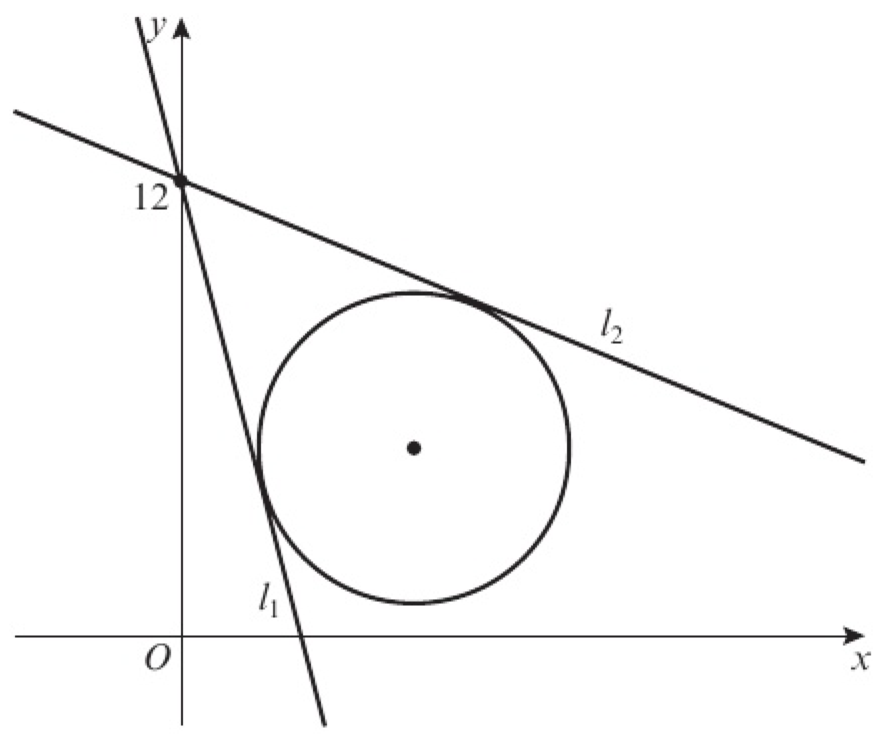
P **14** The point $A(-3, -7)$ lies on the circle centre $(5, 1)$. Find the equation of the tangent to the circle at A.

P **15** The line segment AB is a chord of a circle centre $(2, -1)$, where A and B are $(3, 7)$ and $(-5, 3)$ respectively. AC is a diameter of the circle. Find the area of $\triangle ABC$.

E/P **16** The circle C has equation $(x - 6)^2 + (y - 5)^2 = 17$.

The lines l_1 and l_2 are each a tangent to the circle and intersect at the point $(0, 12)$.

Find the equations of l_1 and l_2, giving your answers in the form $y = mx + c$. **(8 marks)**



E/P **17** The points A and B lie on a circle with centre C, as shown in the diagram.

The point A has coordinates $(3, 7)$ and the point B has coordinates $(5, 1)$.

M is the midpoint of the line segment AB.

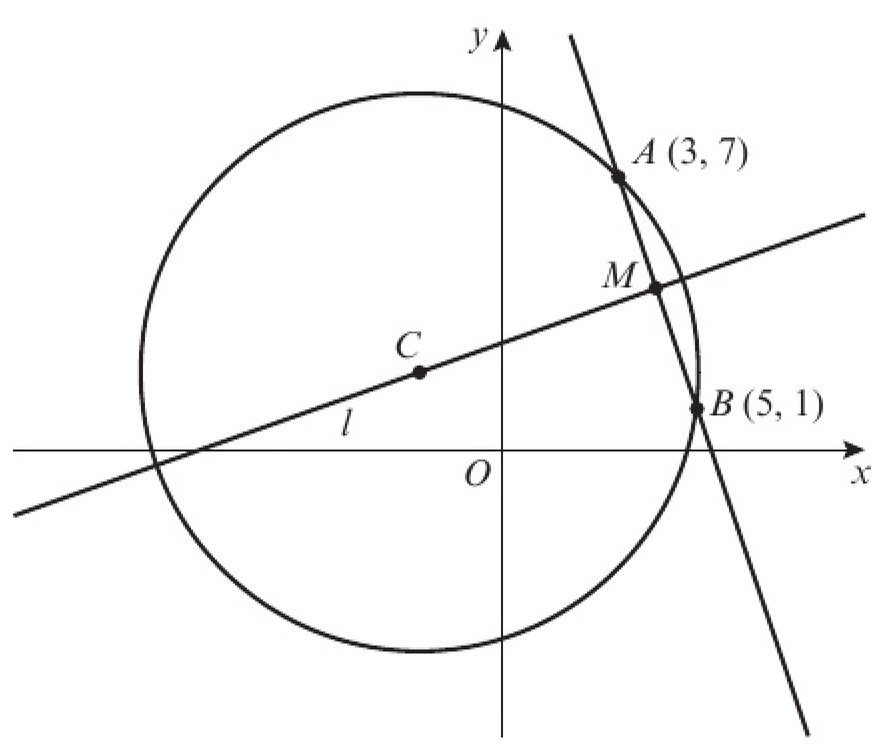
The line l passes through the points M and C.

a Find an equation for l. **(4 marks)**

Given that the x-coordinate of C is -2:

b find an equation of the circle **(4 marks)**

c find the area of the triangle ABC. **(3 marks)**

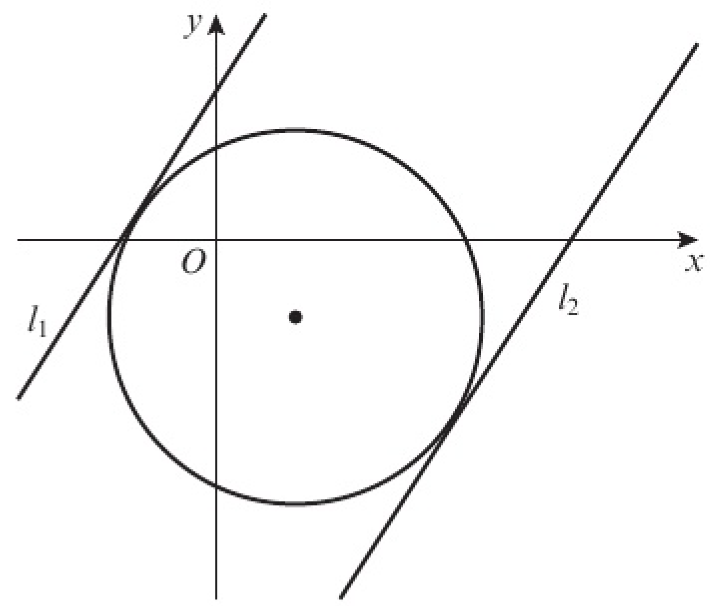


E/P **18** The circle C has equation $(x - 3)^2 + (y + 3)^2 = 52$.

The baselines l_1 and l_2 are tangents to the circle and have gradient $\frac{3}{2}$

a Find the points of intersection, P and Q, of the tangents and the circle. **(6 marks)**

b Find the equations of lines l_1 and l_2, giving your answers in the form $ax + by + c = 0$. **(2 marks)**



E/P **19** The circle C has equation $x^2 + 6x + y^2 - 2y = 7$.

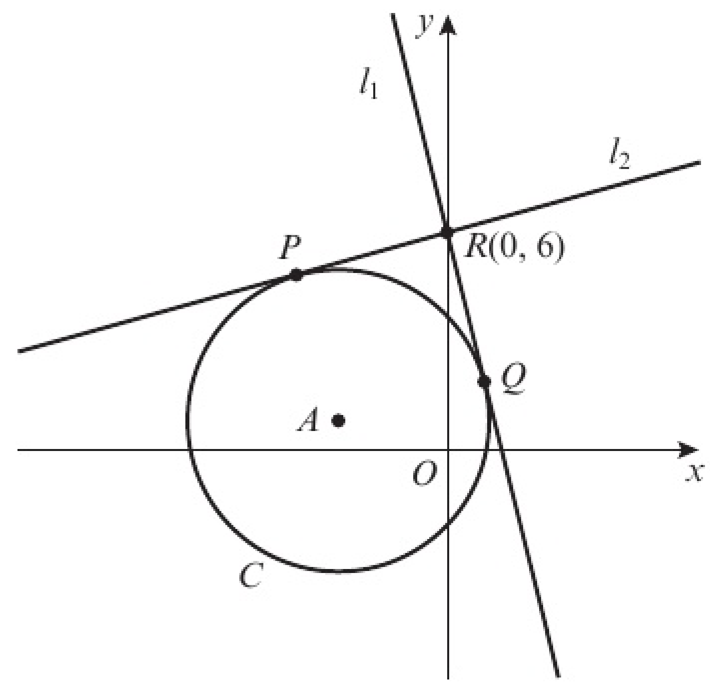
The lines l_1 and l_2 are tangents to the circle.

They intersect at the point $R(0, 6)$.

a Find the equations of lines l_1 and l_2, giving your answers in the form $y = mx + b$. **(6 marks)**

b Find the points of intersection, P and Q, of the tangents and the circle. **(4 marks)**

c Find the area of quadrilateral $APRQ$. **(2 marks)**



E/P **20** The circle C has a centre at $(6, 9)$ and a radius of $\sqrt{50}$.

The line l_1 with equation $x + y - 21 = 0$ intersects the circle at the points P and Q.

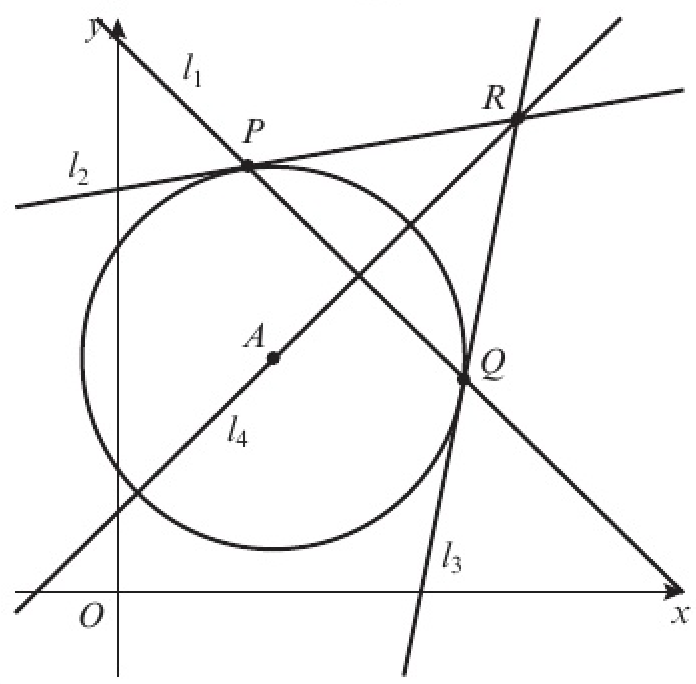
a Find the coordinates of the point P and the point Q. **(5 marks)**

b Find the equations of l_2 and l_3, the tangents at the points P and Q respectively. **(4 marks)**

c Find the equation of l_4, the perpendicular bisector of the chord PQ. **(4 marks)**

d Show that the two tangents and the perpendicular bisector intersect and find the coordinates of R, the point of intersection. **(2 marks)**

e Calculate the area of the kite $APRQ$. **(3 marks)**



P **21** The line $y = -3x + 12$ meets the coordinate axes at A and B.

a Find the coordinates of A and B.

b Find the coordinates of the midpoint of AB.

c Find the equation of the circle that passes through A, B and O, where O is the origin.

E/P **22** The points $A(-3, -2)$, $B(-6, 0)$ and $C(1, q)$ lie on the circumference of a circle such that $\angle BAC = 90°$.

a Find the value of q. **(4 marks)**

b Find the equation of the circle. **(4 marks)**

E/P **23** The points $R(-4, 3)$, $S(7, 4)$ and $T(8, -7)$ lie on the circumference of a circle.

a Show that RT is the diameter of the circle. **(4 marks)**

b Find the equation of the circle. **(4 marks)**

P **24** The points $A(-4, 0)$, $B(4, 8)$ and $C(6, 0)$ lie on the circumference of circle C.
Find the equation of the circle.

P **25** The points $A(-7, 7)$, $B(1, 9)$, $C(3, 1)$ and $D(-7, 1)$ lie on a circle.

a Find the equation of the perpendicular bisector of:
i AB **ii** CD

b Find the equation of the circle.

Challenge

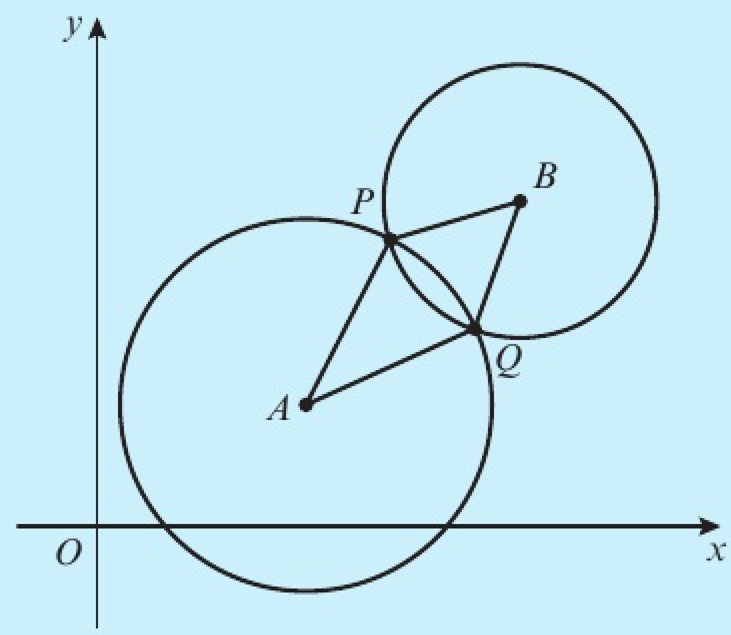
SKILLS CREATIVITY

The circle with equation $(x - 5)^2 + (y - 3)^2 = 20$ with centre A intersects the circle with equation $(x - 10)^2 + (y - 8)^2 = 10$ with centre B at the points P and Q.

a Find the equation of the line containing the points P and Q in the form $ax + by + c = 0$.

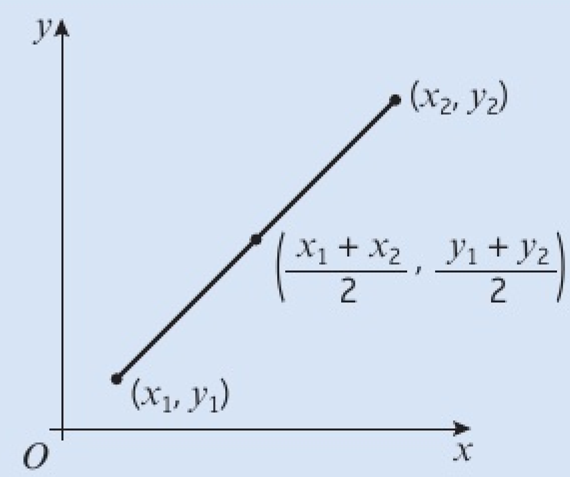
b Find the coordinates of the points P and Q.

c Find the area of the kite $APBQ$.

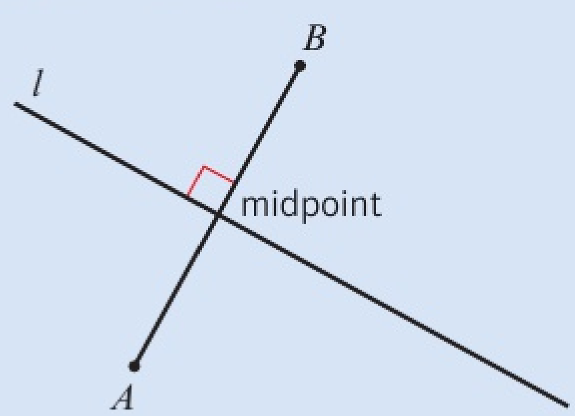


Summary of key points

1 The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



2 The perpendicular bisector of a line segment AB is the straight line that is perpendicular to AB and passes through the midpoint of AB.



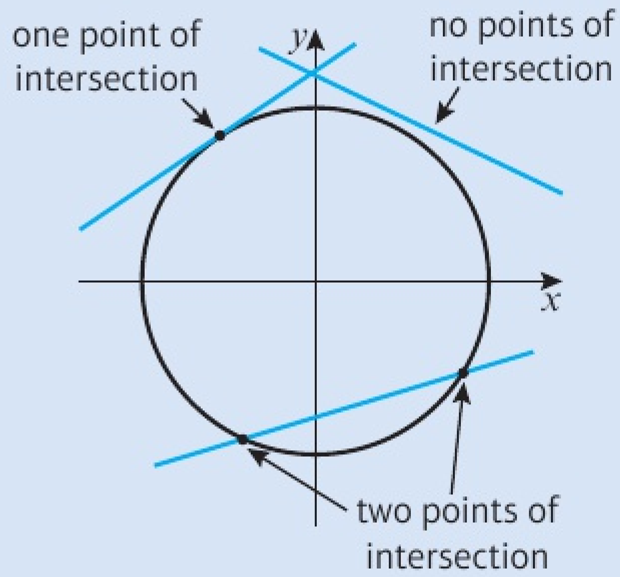
If the gradient of AB is m then the gradient of its perpendicular bisector, l, will be $-\frac{1}{m}$

3 The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

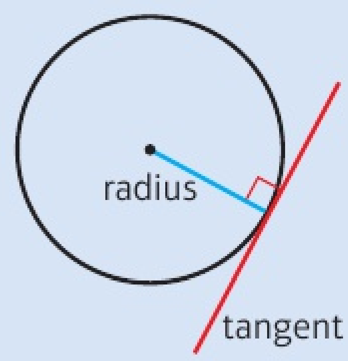
4 The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

5 The equation of a circle can be given in the form: $x^2 + y^2 + 2fx + 2gy + c = 0$
This circle has centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$

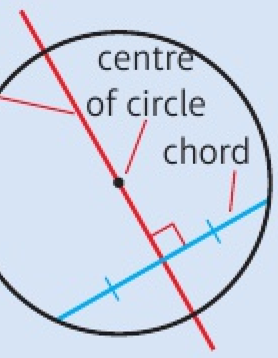
6 A straight line can intersect a circle once, by just touching the circle, or twice. Not all straight lines will intersect a given circle.



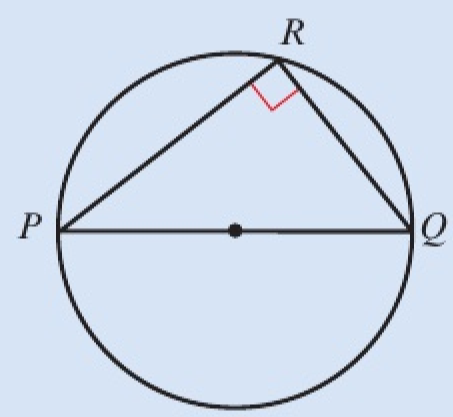
7 A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.



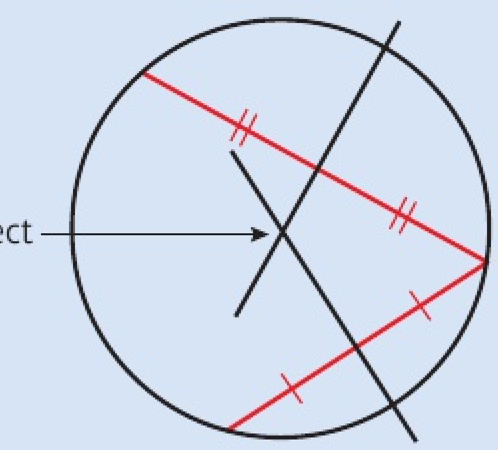
8 The perpendicular bisector of a chord will go through the centre of a circle.

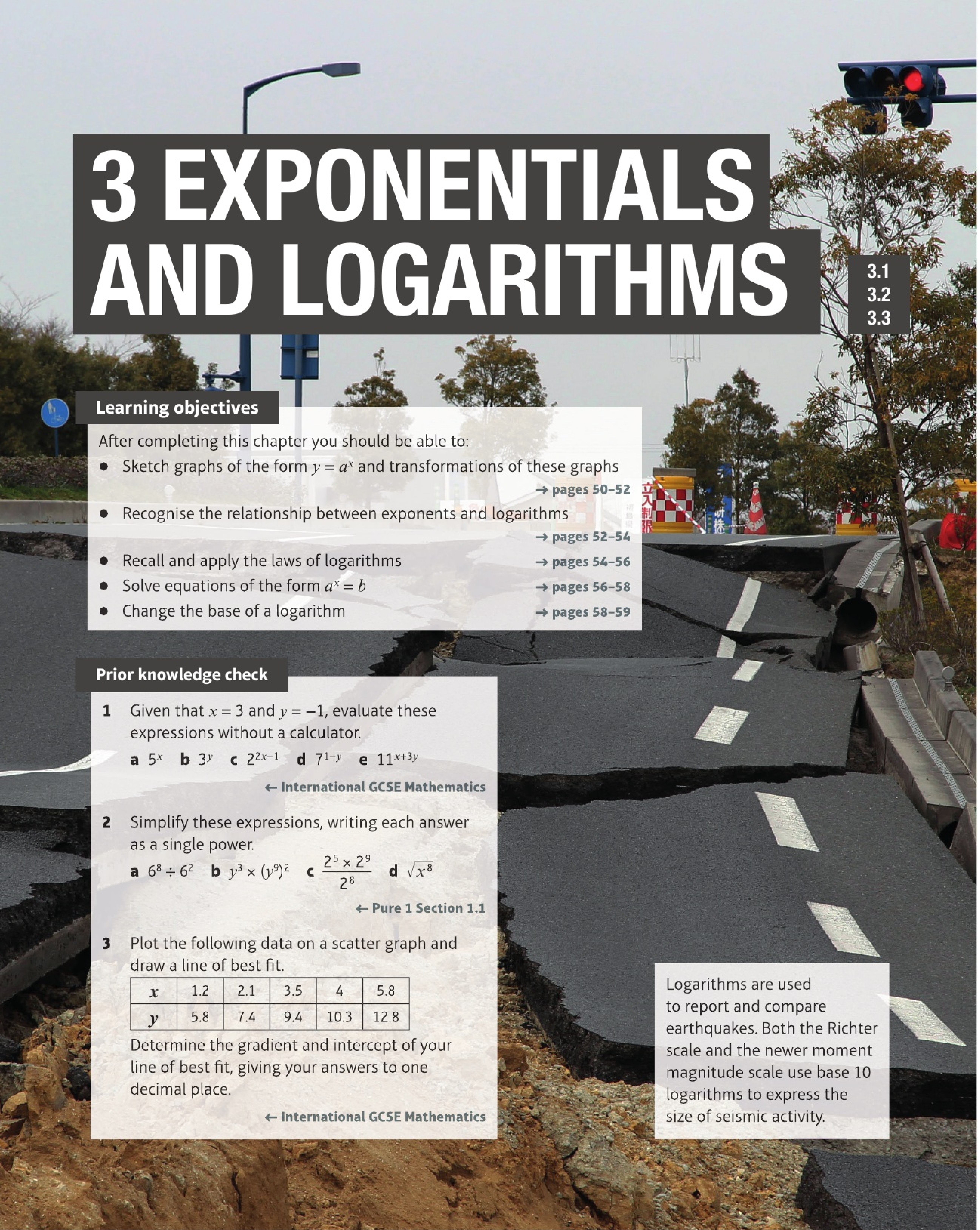


9
- If $\angle PRQ = 90°$ then R lies on the circle with diameter PQ.
- The angle in a semicircle is always a right angle.



10 To find the centre of a circle given any three points:
- Find the equations of the perpendicular bisectors of two different chords.
- Find the coordinates of intersection of the perpendicular bisectors.





3 EXPONENTIALS AND LOGARITHMS

3.1
3.2
3.3

Learning objectives

After completing this chapter you should be able to:

- Sketch graphs of the form $y = a^x$ and transformations of these graphs → pages 50–52
- Recognise the relationship between exponents and logarithms → pages 52–54
- Recall and apply the laws of logarithms → pages 54–56
- Solve equations of the form $a^x = b$ → pages 56–58
- Change the base of a logarithm → pages 58–59

Prior knowledge check

1 Given that $x = 3$ and $y = -1$, evaluate these expressions without a calculator.

a 5^x **b** 3^y **c** 2^{2x-1} **d** 7^{1-y} **e** 11^{x+3y}

← International GCSE Mathematics

2 Simplify these expressions, writing each answer as a single power.

a $6^8 \div 6^2$ **b** $y^3 \times (y^9)^2$ **c** $\frac{2^5 \times 2^9}{2^8}$ **d** $\sqrt{x^8}$

← Pure 1 Section 1.1

3 Plot the following data on a scatter graph and draw a line of best fit.

x	1.2	2.1	3.5	4	5.8
y	5.8	7.4	9.4	10.3	12.8

Determine the gradient and intercept of your line of best fit, giving your answers to one decimal place.

← International GCSE Mathematics

Logarithms are used to report and compare earthquakes. Both the Richter scale and the newer moment magnitude scale use base 10 logarithms to express the size of seismic activity.

3.1 Exponential functions

Functions of the form $f(x) = a^x$, where a is a constant, are called **exponential functions**. You should become familiar with these functions and the shapes of their graphs.

For an example, look at a table of values of $y = 2^x$.

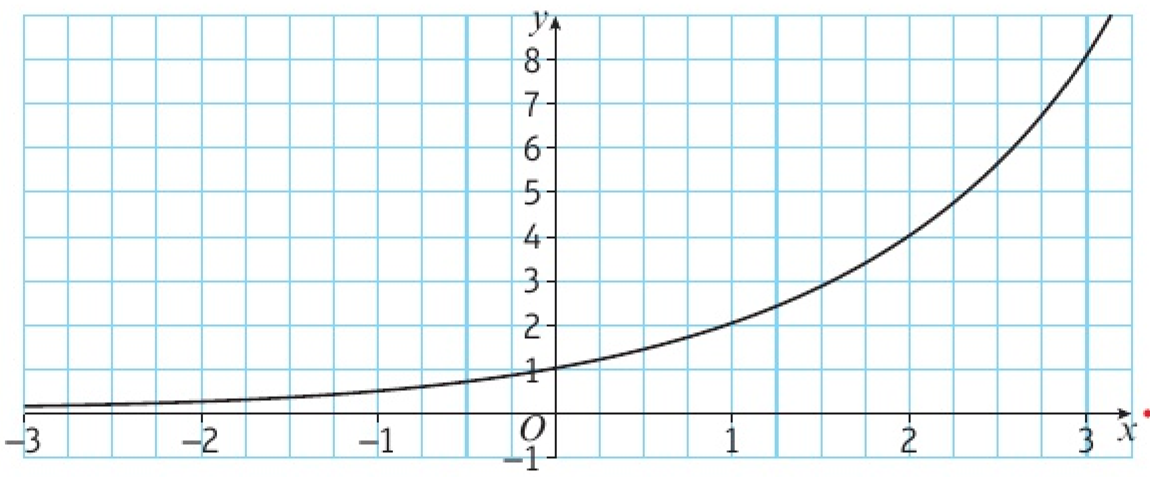
x	−3	−2	−1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notation In the expression 2^x, x can be called an **index**, a **power** or an **exponent.**

The value of 2^x tends towards 0 as x decreases, and grows without limit as x increases.

Links Recall that $2^0 = 1$ and that $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ ← Pure 1 Section 1.4

The graph of $y = 2^x$ is a smooth curve that looks like this:



The x-axis is an **asymptote** to the curve.

Example 1

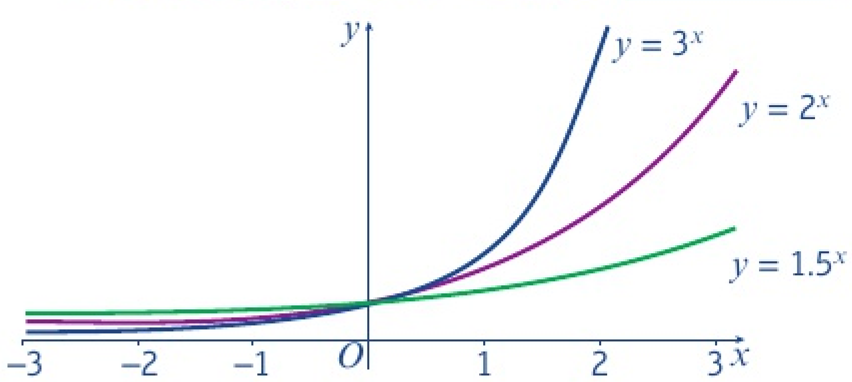
a On the same axes sketch the graphs of $y = 3^x$, $y = 2^x$ and $y = 1.5^x$.

b On another set of axes sketch the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$.

a For all three graphs, $y = 1$ when $x = 0$.
When $x > 0$, $3^x > 2^x > 1.5^x$.
When $x < 0$, $3^x < 2^x < 1.5^x$.

$a^0 = 1$

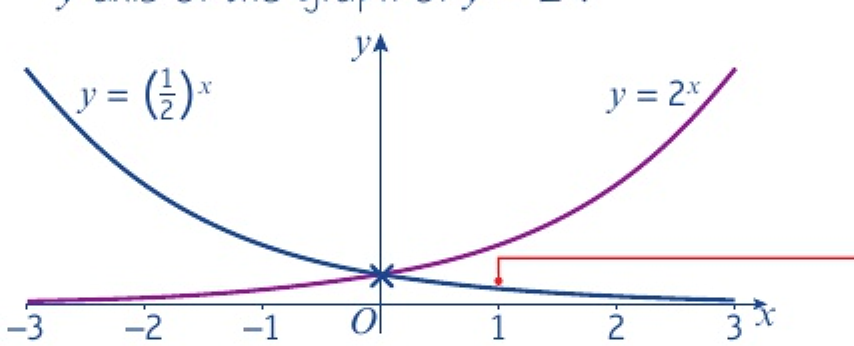
Work out the relative positions of the three graphs.



Whenever $a > 1$, $f(x) = a^x$ is an **increasing function**. In this case, the value of a^x grows without limit as x **increases**, and tends towards 0 as x **decreases**.

b The graph of $y = \left(\frac{1}{2}\right)^x$ is a reflection in the y-axis of the graph of $y = 2^x$.

Since $\frac{1}{2} = 2^{-1}$, $y = \left(\frac{1}{2}\right)^x$ is the same as $y = (2^{-1})^x = 2^{-x}$.



Whenever $0 < a < 1$, $f(x) = a^x$ is a **decreasing function**. In this case, the value of a^x tends towards 0 as x **increases**, and grows without limit as x **decreases**.

Example 2 SKILLS ANALYSIS

Sketch the graph of $y = \left(\frac{1}{2}\right)^{x-3}$. Give the coordinates of the point where the graph crosses the y-axis.

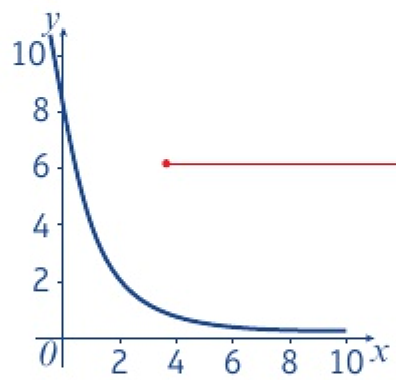
If $f(x) = \left(\frac{1}{2}\right)^x$ then $y = f(x - 3)$.
The graph is a translation of the graph $y = \left(\frac{1}{2}\right)^x$ by the vector $\begin{pmatrix}3\\0\end{pmatrix}$.

The graph crosses the y-axis when $x = 0$.

$y = \left(\frac{1}{2}\right)^{0-3}$

$y = 8$

The graph crosses the y-axis at (0, 8).



Problem-solving

If you have to sketch the graph of an unfamiliar function, try writing it as a transformation of a familiar function. ← Pure 1 Section 4.4

You can also consider this graph as a stretch of the graph $y = \left(\frac{1}{2}\right)^x$

$$y = \left(\frac{1}{2}\right)^{x-3}$$
$$= \left(\frac{1}{2}\right)^x \times \left(\frac{1}{2}\right)^{-3}$$
$$= \left(\frac{1}{2}\right)^x \times 8$$
$$= 8\left(\frac{1}{2}\right)^x = 8f(x)$$

So the graph of $y = \left(\frac{1}{2}\right)^{x-3}$ is a vertical stretch of the graph of $y = \left(\frac{1}{2}\right)^x$ with scale factor 8.

Exercise 3A SKILLS ANALYSIS

1 a Draw an accurate graph of $y = (1.7)^x$, for $-4 \leqslant x \leqslant 4$.
b Use your graph to solve the equation $(1.7)^x = 4$.

2 a Draw an accurate graph of $y = (0.6)^x$, for $-4 \leqslant x \leqslant 4$.
b Use your graph to solve the equation $(0.6)^x = 2$.

3 Sketch the graph of $y = 1^x$.

(P) 4 For each of these statements, decide whether it is true or false, justifying your answer or offering a counter-example.
a The graph of $y = a^x$ passes through (0, 1) for all positive real numbers a.
b The function $f(x) = a^x$ is always an increasing function for $a > 0$.
c The graph of $y = a^x$, where a is a positive real number, never crosses the x-axis.

5 The function f(x) is defined as $f(x) = 3^x$, $x \in \mathbb{R}$. On the same axes, sketch the graphs of:
a $y = f(x)$ b $y = 2f(x)$ c $y = f(x) - 4$ d $y = f\left(\frac{1}{2}x\right)$

Write down the coordinates of the point where each graph crosses the y-axis, and give the equations of any asymptotes.

(P) 6 The graph of $y = ka^x$ passes through the points (1, 6) and (4, 48). Find the values of the constants k and a.

Problem-solving

Substitute the coordinates into $y = ka^x$ to create two simultaneous equations. Use division to eliminate one of the two unknowns.

(P) **7** The graph of $y = pq^x$ passes through the points (−3, 150) and (2, 0.048).

a By drawing a sketch or otherwise, explain why $0 < q < 1$.

b Find the values of the constants p and q.

Challenge

Sketch the graph of $y = 2^{x-2} + 5$. Give the coordinates of the point where the graph crosses the y-axis.

3.2 Logarithms

The inverses of exponential functions are called **logarithms**. A relationship which is expressed using an exponent can also be written in terms of logarithms.

- **$\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$**

Notation a is called the base of the logarithm.

Example 3

Write each statement as a logarithm.

a $3^2 = 9$ **b** $2^7 = 128$ **c** $64^{\frac{1}{2}} = 8$

a $3^2 = 9$, so $\log_3 9 = 2$

b $2^7 = 128$, so $\log_2 128 = 7$

c $64^{\frac{1}{2}} = 8$, so $\log_{64} 8 = \frac{1}{2}$

In words, you would say 'the logarithm of 9 to the base 3 is 2'.

Logarithms can take fractional or negative values.

Example 4

SKILLS ANALYSIS

Rewrite each statement using a power.

a $\log_3 81 = 4$ **b** $\log_2 \left(\frac{1}{8}\right) = -3$

a $\log_3 81 = 4$, so $3^4 = 81$

b $\log_2 \left(\frac{1}{8}\right) = -3$, so $2^{-3} = \frac{1}{8}$

Example 5

Without using a calculator, find the value of:

a $\log_3 81$ **b** $\log_4 0.25$ **c** $\log_{0.5} 4$ **d** $\log_a (a^5)$

a $\log_3 81 = 4$ — Because $3^4 = 81$.

b $\log_4 0.25 = -1$ — Because $4^{-1} = \frac{1}{4} = 0.25$.

c $\log_{0.5} 4 = -2$ — Because $0.5^{-2} = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$.

d $\log_a (a^5) = 5$ — Because $a^5 = a^5$.

You can use your calculator to find logarithms of any base. Some calculators have a specific [log$_\square\square$] key for this function. Most calculators also have separate buttons for logarithms to the base 10 (usually written as [log] and logarithms to the base e (usually written as [ln], which you will meet in Pure Mathematics 3).

Notation Logarithms to the base e are typically called **natural logarithms**. This is why the calculator key is labelled [ln].

Online Use the logarithm buttons on your calculator.

Example 6

Use your calculator to find the following logarithms to 3 decimal places.

a $\log_3 40$ **b** $\log_{10} 75$

a 3.358

For part **a** use [log$_\square\square$].

b 1.875

For part **b** you can use either [log] or [log$_\square\square$].

Exercise 3B

SKILLS ANALYSIS

1 Rewrite using a logarithm.

a $4^4 = 256$ **b** $3^{-2} = \frac{1}{9}$ **c** $10^6 = 1\,000\,000$

d $11^1 = 11$ **e** $(0.2)^3 = 0.008$

2 Rewrite using a power.

a $\log_2 16 = 4$ **b** $\log_5 25 = 2$ **c** $\log_9 3 = \frac{1}{2}$

d $\log_5 0.2 = -1$ **e** $\log_{10} 100\,000 = 5$

3 Without using a calculator, find the value of

a $\log_2 8$ **b** $\log_5 25$ **c** $\log_{10} 10\,000\,000$ **d** $\log_{12} 12$

e $\log_3 729$ **f** $\log_{10} \sqrt{10}$ **g** $\log_4 (0.25)$ **h** $\log_{0.25} 16$

i $\log_a (a^{10})$ **j** $\log_{\frac{2}{3}} \left(\frac{9}{4}\right)$

4 Without using a calculator, find the value of x for which

a $\log_5 x = 4$ **b** $\log_x 81 = 2$ **c** $\log_7 x = 1$

d $\log_2 (x - 1) = 3$ **e** $\log_3 (4x + 1) = 4$ **f** $\log_x (2x) = 2$

5 Use your calculator to evaluate these logarithms to three decimal places.

a $\log_9 230$ **b** $\log_5 33$ **c** $\log_{10} 1020$ **d** $\log_e 3$

(P) **6** **a** Without using a calculator, justify why the value of $\log_2 50$ must be between 5 and 6.

b Use a calculator to find the exact value of $\log_2 50$ to 4 significant figures.

Hint Use corresponding statements involving powers of 2.

7 **a** Find the values of:

i $\log_2 2$ **ii** $\log_3 3$ **iii** $\log_{17} 17$

b Explain why $\log_a a$ has the same value for all positive values of a ($a \neq 1$).

8 **a** Find the values of:

i $\log_2 1$ **ii** $\log_3 1$ **iii** $\log_{17} 1$

b Explain why $\log_a 1$ has the same value for all positive values of a ($a \neq 1$).

3.3 Laws of logarithms

Expressions involving more than one logarithm can often be rearranged or simplified. For instance:

$\log_a x = m$ and $\log_a y = n$ — Take two logarithms with the same base

$x = a^m$ and $y = a^n$ — Rewrite these expressions using powers

$xy = a^m \times a^n = a^{m+n}$ — Multiply these powers

$\log_a xy = m + n = \log_a x + \log_a y$ — Rewrite your result using logarithms

This result is one of the **laws of logarithms**.

You can use similar methods to prove two further laws.

- **The laws of logarithms:**
  - $\log_a x + \log_a y = \log_a xy$ **(the multiplication law)**
  - $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ **(the division law)**
  - $\log_a (x^k) = k \log_a x$ **(the power law)**

Watch out You need to learn these three laws of logarithms, and the special cases below.

- **You should also learn to recognise the following special cases:**
  - $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$ **(the power law when $k = -1$)**
  - $\log_a a = 1$ **($a > 0$, $a \neq 1$)**
  - $\log_a 1 = 0$ **($a > 0$, $a \neq 1$)**

Example 7

Write as a single logarithm.

a $\log_3 6 + \log_3 7$ **b** $\log_2 15 - \log_2 3$ **c** $2\log_5 3 + 3\log_5 2$ **d** $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

a $\log_3 (6 \times 7)$ — Use the multiplication law.

$= \log_3 42$

b $\log_2 (15 \div 3)$

$= \log_2 5$ — Use the division law.

c $2\log_5 3 = \log_5 (3^2) = \log_5 9$
$3\log_5 2 = \log_5 (2^3) = \log_5 8$
$\log_5 9 + \log_5 8 = \log_5 72$

First apply the power law to both parts of the expression.
Then use the multiplication law.

d $4\log_{10}\left(\frac{1}{2}\right) = \log_{10}\left(\frac{1}{2}\right)^4 = \log_{10}\left(\frac{1}{16}\right)$

$\log_{10} 3 - \log_{10}\left(\frac{1}{16}\right) = \log_{10}\left(3 \div \frac{1}{16}\right)$

$= \log_{10} 48$

Use the power law first.
Then use the division law.

Example 8

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a (x^2yz^3)$ **b** $\log_a\left(\frac{x}{y^3}\right)$ **c** $\log_a\left(\frac{x\sqrt{y}}{z}\right)$ **d** $\log_a\left(\frac{x}{a^4}\right)$

a $\log_a (x^2yz^3)$
$= \log_a (x^2) + \log_a y + \log_a (z^3)$
$= 2\log_a x + \log_a y + 3\log_a z$

b $\log_a\left(\frac{x}{y^3}\right)$
$= \log_a x - \log_a (y^3)$
$= \log_a x - 3\log_a y$

c $\log_a\left(\frac{x\sqrt{y}}{z}\right)$
$= \log_a (x\sqrt{y}) - \log_a z$
$= \log_a x + \log_a \sqrt{y} - \log_a z$
$= \log_a x + \frac{1}{2}\log_a y - \log_a z$

Use the power law ($\sqrt{y} = y^{\frac{1}{2}}$).

d $\log_a\left(\frac{x}{a^4}\right)$
$= \log_a x - \log_a (a^4)$
$= \log_a x - 4\log_a a$
$= \log_a x - 4$

$\log_a a = 1$.

Example 9

SKILLS PROBLEM-SOLVING

Solve the equation $\log_{10} 4 + 2\log_{10} x = 2$.

$\log_{10} 4 + 2\log_{10} x = 2$
$\log_{10} 4 + \log_{10} x^2 = 2$
$\log_{10} 4x^2 = 2$
$4x^2 = 10^2$
$4x^2 = 100$
$x^2 = 25$
$x = 5$

Use the power law.

Use the multiplication law.

Rewrite the logarithm using powers.

Watch out $\log_{10} x$ is only defined for **positive** values of x, so $x = -5$ cannot be a solution of the equation.

Example 10

Solve the equation $\log_3(x + 11) - \log_3(x - 5) = 2$

$\log_3(x + 11) - \log_3(x - 5) = 2$

$\log_3\left(\frac{x + 11}{x - 5}\right) = 2$ — Use the division law.

$\frac{x + 11}{x - 5} = 3^2$ — Rewrite the logarithm using powers.

$x + 11 = 9(x - 5)$

$x + 11 = 9x - 45$

$56 = 8x$

$x = 7$

Exercise 3C

SKILLS PROBLEM-SOLVING

1 Write as a single logarithm.

a $\log_2 7 + \log_2 3$ **b** $\log_2 36 - \log_2 4$ **c** $3\log_5 2 + \log_5 10$

d $2\log_6 8 - 4\log_6 3$ **e** $\log_{10} 5 + \log_{10} 6 - \log_{10}\left(\frac{1}{4}\right)$

2 Write as a single logarithm, then simplify your answer.

a $\log_2 40 - \log_2 5$ **b** $\log_6 4 + \log_6 9$ **c** $2\log_{12} 3 + 4\log_{12} 2$

d $\log_8 25 + \log_8 10 - 3\log_8 5$ **e** $2\log_{10} 2 - (\log_{10} 5 + \log_{10} 8)$

3 Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a(x^3y^4z)$ **b** $\log_a\left(\frac{x^5}{y^2}\right)$ **c** $\log_a(a^2x^2)$

d $\log_a\left(\frac{x}{z\sqrt{y}}\right)$ **e** $\log_a\sqrt{ax}$

4 Solve the following equations:

a $\log_2 3 + \log_2 x = 2$ **b** $\log_6 12 - \log_6 x = 3$

c $2\log_5 x = 1 + \log_5 6$ **d** $2\log_9(x + 1) = 2\log_9(2x - 3) + 1$

Hint Move the logarithms onto the same side if necessary and use the division law.

E/P **5** **a** Given that $\log_3(x + 1) = 1 + 2\log_3(x - 1)$, show that $3x^2 - 7x + 2 = 0$. **(5 marks)**

b Hence, or otherwise, solve $\log_3(x + 1) = 1 + 2\log_3(x - 1)$. **(2 marks)**

P **6** Given that a and b are positive constants, and that $a > b$, solve the simultaneous equations

$a + b = 13$

$\log_6 a + \log_6 b = 2$

Problem-solving Pay careful attention to the conditions on a and b given in the question.

Challenge

SKILLS CREATIVITY

By writing $\log_a x = m$ and $\log_a y = n$, prove that $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$.

3.4 Solving equations using logarithms

You can use logarithms and your calculator to solve equations of the form $a^x = b$.

Example 11

Solve the following equations, giving your answers to 3 decimal places.

a $3^x = 20$ **b** $5^{4x-1} = 61$

a $3^x = 20$,
so $x = \log_3 20 = 2.727$

b $5^{4x-1} = 61$, so $4x - 1 = \log_5 61$
$4x = \log_5 61 + 1$
$x = \dfrac{\log_5 61 + 1}{4}$
$= 0.889$

Use the $\log_{\square}\square$ button on your calculator.

You can evaluate the final answer in one step on your calculator.

Example 12

Solve the equation $5^{2x} - 12(5^x) + 20 = 0$, giving your answer to 3 significant figures.

$5^{2x} - 12(5^x) + 20$ is a quadratic function of 5^x
$(5^x - 10)(5^x - 2) = 0$
$5^x = 10$ or $5^x = 2$
$5^x = 10 \Rightarrow x = \log_5 10 \Rightarrow x = 1.43$
$5^x = 2 \Rightarrow x = \log_5 2 \Rightarrow x = 0.431$

An alternative method is to rewrite the equation using the substitution $y = 5^x$: $y^2 - 12y + 20 = 0$.

Watch out Solving the quadratic equation gives you two possible values for 5^x. Make sure you calculate both corresponding values of x for your final answer.

You can solve more complicated equations by 'taking logs' of both sides.

■ **Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$**

Example 13 SKILLS EXECUTIVE FUNCTION

Find the solution to the equation $3^x = 2^{x+1}$, giving your answer to four decimal places.

$3^x = 2^{x+1}$
$\log 3^x = \log 2^{x+1}$
$x \log 3 = (x + 1) \log 2$
$x \log 3 = x \log 2 + \log 2$
$x \log 3 - x \log 2 = \log 2$
$x(\log 3 - \log 2) = \log 2$
$x = \dfrac{\log 2}{\log 3 - \log 2} = 1.7095$

This step is called 'taking logs of both sides'. The logs on both sides must be to the **same base**. Here 'log' is used to represent $\log_{10}$.

Use the power law.

Move all the terms in x to one side then factorise.

Exercise 3D SKILLS EXECUTIVE FUNCTION

1 Solve, giving your answers to 3 significant figures.

a $2^x = 75$ **b** $3^x = 10$ **c** $5^x = 2$ **d** $4^{2x} = 100$

e $9^{x+5} = 50$ **f** $7^{2x-1} = 23$ **g** $11^{3x-2} = 65$ **h** $2^{3-2x} = 88$

2 Solve, giving your answers to 3 significant figures.

a $2^{2x} - 6(2^x) + 5 = 0$ **b** $3^{2x} - 15(3^x) + 44 = 0$

c $5^{2x} - 6(5^x) - 7 = 0$ **d** $3^{2x} + 3^{x+1} - 10 = 0$

e $7^{2x} + 12 = 7^{x+1}$ **f** $2^{2x} + 3(2^x) - 4 = 0$

g $3^{2x+1} - 26(3^x) - 9 = 0$ **h** $4(3^{2x+1}) + 17(3^x) - 7 = 0$

Hint $3^{x+1} = 3^x \times 3^1 = 3(3^x)$

Problem-solving Consider these equations as functions of functions. Part **a** is equivalent to $u^2 - 6u + 5 = 0$, with $u = 2^x$.

(E) **3** Solve the following equations, giving your answers to 3 significant figures where appropriate.

a $3^{x+1} = 2000$ **(2 marks)**

b $\log_5(x - 3) = -1$ **(2 marks)**

(E/P) **4** **a** Sketch the graph of $y = 4^x$, stating the coordinates of any points where the graph crosses the axes. **(2 marks)**

b Solve the equation $4^{2x} - 10(4^x) + 16 = 0$. **(4 marks)**

Hint Attempt this question without a calculator.

5 Solve the following equations, giving your answers to four decimal places.

a $5^x = 2^{x+1}$ **b** $3^{x+5} = 6^x$ **c** $7^{x+1} = 3^{x+2}$

Hint Take logs of both sides.

3.5 Changing the base of a logarithm

It is sometimes convenient to rewrite logarithms using a different base.

Working in base a, suppose that: $\log_a x = m$

Writing this as a power: $a^m = x$

Taking logs to a different base b: $\log_b a^m = \log_b x$

Using the power law: $m\log_b a = \log_b x$

Writing m as $\log_a x$: $\log_a x \times \log_b a = \log_b x$

This can be written as:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Watch out Some older calculators do not have a $\log_\square \square$ key to calculate logs to any base.

Using this rule, notice in particular that $\log_a b = \frac{\log_b b}{\log_b a}$, but $\log_b b = 1$, so:

$$\log_a b = \frac{1}{\log_b a}$$

Example 14

Find, to 3 significant figures, the value of $\log_8 11$.

$$\log_8 11 = \frac{\log_{10} 11}{\log_{10} 8}$$
$$= 1.15$$

One method is to use the change of base rule to change to base 10

$$8^x = 11$$

Another method is to solve $8^x = 11$

$$\log_{10}(8^x) = \log_{10} 11$$

Take logs to base 10 of each side

$$x \log_{10} 8 = \log_{10} 11$$

Use the power law

$$x = \frac{\log_{10} 11}{\log_{10} 8}$$

Divide by $\log_{10} 8$

$$x = 1.15$$

Example 15 SKILLS EXECUTIVE FUNCTION

Solve the equation $\log_5 x + 6\log_x 5 = 5$

$$\log_5 x + \frac{6}{\log_5 x} = 5$$

Use change of base rule (special case)

Let $\log_5 x = y$

$$y + \frac{6}{y} = 5$$
$$y^2 + 6 = 5y$$

Multiply through by y

$$y^2 - 5y + 6 = 0$$
$$(y - 3)(y - 2) = 0$$

So $y = 3$ or $y = 2$

$\log_5 x = 3$ or $\log_5 x = 2$

$x = 5^3$ or $x = 5^2$

Write as powers

$x = 125$ or $x = 25$

Exercise 3E SKILLS EXECUTIVE FUNCTION

1 Find, to 3 decimal places:

a $\log_7 120$ b $\log_3 45$ c $\log_2 19$ d $\log_{11} 3$

2 Solve, giving your answer to 3 significant figures:

a $8^x = 14$ b $9^x = 99$ c $12^x = 6$

3 Solve, giving your answer to 3 significant figures:

a $\log_2 x = 8 + 9\log_x 2$ b $\log_4 x + 2\log_x 4 + 3 = 0$ c $\log_2 x + \log_4 x = 2$

Chapter review 3

1 Sketch a graph, labelling all intersections and asymptotes, for $y = 2^{-x}$.

Hint Recall that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

P 2 **a** Express $\log_a(p^2q)$ in terms of $\log_a p$ and $\log_a q$.
b Given that $\log_a(pq) = 5$ and $\log_a(p^2q) = 9$, find the values of $\log_a p$ and $\log_a q$.

P 3 Given that $p = \log_q 16$, express in terms of p,
a $\log_q 2$
b $\log_q(8q)$

4 Solve these equations, giving your answers to 3 significant figures.
a $4^x = 23$ **b** $7^{2x+1} = 1000$ **c** $10^x = 6^{x+2}$

E/P 5 **a** Using the substitution $u = 2^x$, show that the equation $4^x - 2^{x+1} - 15 = 0$ can be written in the form $u^2 - 2u - 15 = 0$. **(2 marks)**
b Hence solve the equation $4^x - 2^{x+1} - 15 = 0$, giving your answer to 2 decimal places. **(3 marks)**

E 6 Solve the equation $\log_2(x + 10) - \log_2(x - 5) = 4$. **(4 marks)**

E/P 7 Given that $y = 3x^2$,
a show that $\log_3 y = 1 + 2\log_3 x$;
b hence, or otherwise, solve the equation
$$1 + 2\log_3 x = \log_3(28x - 9).$$ **(6 marks)**

E/P 8 Find the values of x if $2\log_3 x - \log_3(x - 2) = 2$. **(5 marks)**

E/P 9 Find, giving your answer to 3 significant figures where appropriate, the value of x for which
a $5^x = 10$
b $\log_9(x - 2) = -1$. **(4 marks)**

E/P 10 Given that $0 < x < 4$ and $\log_5(4 - x) - 2\log_5 x = 1$, find the value of x. **(6 marks)**

E/P 11 **a** Find the positive value of x such that $\log_x 64 = 2$.
b Solve for x
$$\log_2(11 - 6x) = 2\log_2(x - 1) + 3.$$ **(8 marks)**

E/P **12 a** Find the value of y such that

$$\log_2 y = -3.$$

b Find the values of x such that

$$\frac{\log_2 32 + \log_2 16}{\log_2 x} = \log_2 x$$

(7 marks)

E/P **13 a** Given that $2\log_3(x - 5) - \log_3(2x - 13) = 1$, show that $x^2 - 16x + 64 = 0$.

b Hence, or otherwise, solve the equation $2\log_3(x - 5) - \log_3(2x - 13) = 1$. **(7 marks)**

E/P **14 a** Find the exact value of x for which

$$\log_2(2x) = \log_2(5x + 4) - 3.$$

b Given that

$$\log_a y + 3\log_a 2 = 5$$

express y in terms of a.

Give your answer in its simplest form. **(7 marks)**

Summary of key points

1 $\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$

2 **The laws of logarithms:**

- $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ (the division law)
- $\log_a (x^k) = k \log_a x$ (the power law)

3 You should also learn to recognise the following special cases:

- $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$ (the power law when $k = -1$)
- $\log_a a = 1$ $(a > 0, a \neq 1)$
- $\log_a 1 = 0$ $(a > 0, a \neq 1)$

4 You can change the base of a logarithm using the following rule:

- $\log_a x = \frac{\log_b x}{\log_b a}$
- $\log_a b = \frac{1}{\log_b a}$

4 THE BINOMIAL EXPANSION

Learning objectives

After completing this chapter you should be able to:

- Use Pascal's triangle to identify binomial coefficients and use them to expand simple binomial expressions → pages 63–65
- Use combinations and factorial notation → pages 65–67
- Use the binomial expansion to expand brackets → pages 67–68
- Find individual coefficients in a binomial expansion → pages 69–71
- Make approximations using the binomial expansion → pages 71–73

Prior knowledge check

1 Expand and simplify where possible:

a $(2x - 3y)^2$ **b** $(x - y)^3$ **c** $(2 + x)^3$

← Pure 1 Section 1.2

2 Simplify

a $(-2x)^3$ **b** $(3x)^{-4}$

c $\left(\frac{2}{5}x\right)^2$ **d** $\left(\frac{1}{3}x\right)^{-3}$

← Pure 1 Section 1.1, 1.4

3 Simplify

a $(25x)^{\frac{1}{2}}$ **b** $(64x)^{-\frac{2}{3}}$

c $\left(\frac{9}{100}x\right)^{-\frac{1}{2}}$ **d** $\left(\frac{8}{27}x\right)^{\frac{4}{3}}$

← Pure 1 Section 1.1

The binomial expansion can be used to expand brackets raised to large powers.
It can be used to simplify probability models with a large number of trials, such as those used by manufacturers to predict faults.

4.1 Pascal's triangle

You can use **Pascal's triangle** to quickly expand expressions such as $(x + 2y)^3$.

Consider the expansions of $(a + b)^n$ for $n = 0, 1, 2, 3$ and 4:

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Each coefficient is the sum of the two coefficients immediately above it.

Every term in the expansion of $(a + b)^n$ has total index n:
In the $6a^2b^2$ term the total index is $2 + 2 = 4$.
In the $4ab^3$ term the total index is $1 + 3 = 4$.

The coefficients in the expansions form a pattern that is known as Pascal's triangle.

- **Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.**

Here are the first 7 rows of Pascal's triangle:

```
                  1
               1     1
            1  +  2     1
         1     3  +  3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

The third row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^2$.

- **The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.**

Example 1

Use Pascal's triangle to find the expansions of:

a $(x + 2y)^3$ **b** $(2x - 5)^4$

a $(x + 2y)^3$

The coefficients are 1, 3, 3, 1 so:

$$(x + 2y)^3 = 1x^3 + 3x^2(2y) + 3x(2y)^2 + 1(2y)^3$$
$$= x^3 + 6x^2y + 12xy^2 + 8y^3$$

Index = 3 so look at the 4th row of Pascal's triangle to find the coefficients.

This is the expansion of $(a + b)^3$ with $a = x$ and $b = 2y$. Use brackets to make sure you don't make a mistake. $(2y)^2 = 4y^2$.

b $(2x - 5)^4$

The coefficients are 1, 4, 6, 4, 1 so:

$(2x - 5)^4 = 1(2x)^4 + 4(2x)^3(-5)^1 + 6(2x)^2(-5)^2 + 4(2x)^1(-5)^3 + 1(-5)^4$

$= 16x^4 - 160x^3 + 600x^2 - 1000x + 625$

Index = 4 so look at the 5th row of Pascal's triangle.

This is the expansion of $(a + b)^4$ with $a = 2x$ and $b = -5$.

Be careful with the negative numbers.

Example 2 SKILLS PROBLEM-SOLVING

The coefficient of x^2 in the expansion of $(2 - cx)^3$ is 294.
Find the possible value(s) of the constant c.

Problem-solving

If there is an unknown in the original expression, you might be able to form an equation involving that unknown.

The coefficients are 1, 3, 3, 1:

The term in x^2 is $3 \times 2(-cx)^2 = 6c^2x^2$

So $6c^2 = 294$

$c^2 = 49$

$c = \pm 7$

Index = 3 so use the 4th row of Pascal's triangle.

From the expansion of $(a + b)^3$ the x^2 term is $3ab^2$ where $a = 2$ and $b = -cx$.

Form and solve an equation in c.

Exercise 4A SKILLS ANALYSIS

1 State the row of Pascal's triangle that would give the coefficients of each expansion:

a $(x + y)^3$ **b** $(3x - 7)^{15}$ **c** $\left(2x + \frac{1}{2}\right)^n$ **d** $(y - 2x)^{n+4}$

2 Write down the expansion of:

a $(x + y)^4$ **b** $(p + q)^5$ **c** $(a - b)^3$ **d** $(x + 4)^3$
e $(2x - 3)^4$ **f** $(a + 2)^5$ **g** $(3x - 4)^4$ **h** $(2x - 3y)^4$

3 Find the coefficient of x^3 in the expansion of:

a $(4 + x)^4$ **b** $(1 - x)^5$ **c** $(3 + 2x)^3$ **d** $(4 + 2x)^5$
e $(2 + x)^6$ **f** $\left(4 - \frac{1}{2}x\right)^4$ **g** $(x + 2)^5$ **h** $(3 - 2x)^4$

(P) 4 Fully expand the expression $(1 + 3x)(1 + 2x)^3$.

Problem-solving

Expand $(1 + 2x)^3$, then multiply each term by 1 and by $3x$.

(P) 5 Expand $(2 + y)^3$. Hence or otherwise, write down the expansion of $(2 + x - x^2)^3$ in ascending powers of x.

(P) 6 The coefficient of x^2 in the expansion of $(2 + ax)^3$ is 54. Find the possible values of the constant a.

(P) **7** The coefficient of x^2 in the expansion of $(2 - x)(3 + bx)^3$ is 45. Find possible values of the constant b.

(P) **8** Work out the coefficient of x^2 in the expansion of $(p - 2x)^3$. Give your answer in terms of p.

(P) **9** After 5 years, the value of an investment of \$500 at an interest rate of X% per annum is given by:

$$500\left(1 + \frac{X}{100}\right)^5$$

Find an approximation for this expression in the form $A + BX + CX^2$, where A, B and C are constants to be found. You can ignore higher powers of X.

Challenge

Find the constant term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^3$.

4.2 Factorial notation

You can use combinations and factorial notation to help you expand binomial expressions. For larger indices, it is quicker than using Pascal's triangle.

Notation You say 'n factorial'. By definition, $0! = 1$.

Using **factorial notation** $3 \times 2 \times 1 = 3!$

- **You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.**
  - **The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$:**

    $${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

  - **The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$**

Notation You can say 'n choose r' for ${}^nC_r$. It is sometimes written without superscripts and subscripts as nCr.

Example 3 SKILLS INTERPRETATION

Calculate:

a $5!$ **b** ${}^5C_2$ **c** the 6th entry in the 10th row of Pascal's triangle

a $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

b ${}^5C_2 = \frac{5!}{2!3!} = \frac{120}{12} = 10$

c ${}^9C_5 = 126$

Online Use the ${}^nC_r$ and ! functions on your calculator to answer this question.

You can calculate ${}^5C_2$ by using the ${}^nC_r$ function on your calculator.

$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{5!}{2!(5-2)!}$$

The rth entry in the nth row is ${}^{n-1}C_{r-1}$.

In the expansion of $(a + b)^9$ this would give the term $126a^4b^5$.

Exercise 4B SKILLS INTERPRETATION

1 Work out:

a $4!$ **b** $9!$ **c** $\frac{10!}{7!}$ **d** $\frac{15!}{13!}$

2 Without using a calculator, work out:

a $\binom{4}{2}$ **b** $\binom{6}{4}$ **c** $^6C_3$ **d** $\binom{5}{4}$ **e** $^{10}C_8$ **f** $\binom{9}{5}$

3 Use a calculator to work out:

a $\binom{15}{6}$ **b** $^{10}C_7$ **c** $\binom{20}{10}$ **d** $\binom{20}{17}$ **e** $^{14}C_9$ **f** $^{18}C_5$

4 Write each value a to d from Pascal's triangle using ^nC_r notation:

1

1 1

1 2 1

1 3 3 1

1 a 6 4 1

1 5 b 10 5 1

1 6 c d 15 6 1

5 Work out the 5th number on the 12th row from Pascal's triangle.

6 The 11th row of Pascal's triangle is shown below.

1 10 45 … …

a Find the next two values in the row.

b Hence find the coefficient of x^3 in the expansion of $(1 + 2x)^{10}$.

7 The 14th row of Pascal's triangle is shown below.

1 13 78 … …

a Find the next two values in the row.

b Hence find the coefficient of x^4 in the expansion of $(1 + 3x)^{13}$.

8 The probability of throwing exactly 10 heads when a fair coin is tossed 20 times is given by $\binom{20}{10}0.5^{20}$. Calculate the probability and describe the likelihood of this occurring.

(P) **9** Show that:

a $^nC_1 = n$ **b** $^nC_2 = \frac{n(n-1)}{2}$

(E) **10** Given that $\binom{50}{13} = \frac{50!}{13!a!}$, write down the value of a. **(1 mark)**

(E) **11** Given that $\binom{35}{p} = \frac{35!}{p!18!}$, write down the value of p. **(1 mark)**

Challenge

SKILLS CREATIVITY

a Work out ${}^{10}C_3$ and ${}^{10}C_7$

b Work out ${}^{14}C_5$ and ${}^{14}C_9$

c What do you notice about your answers to parts **a** and **b**?

d Prove that ${}^{n}C_r = {}^{n}C_{n-r}$

4.3 The binomial expansion

A binomial expression has two terms. The binomial expansion allows you to expand powers of binomial expressions. For example, in the expansion of $(a + b)^5 = (a + b)(a + b)(a + b)(a + b)(a + b)$ the term a^2b^3 occurs $\binom{5}{3}$ times. This is because you need to choose b 3 times from the 5 brackets. You can do this in $\binom{5}{3}$ ways so when the expansion is simplified, the term in a^2b^3 is $\binom{5}{3}a^2b^3$.

- The binomial expansion is:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^{n}C_r = \dfrac{n!}{r!(n-r)!}$

Notation $n \in \mathbb{N}$ means that n must be a member of the **natural numbers**. This is all the **positive integers**.

Example 4 SKILLS ANALYSIS

Use the binomial theorem to find the expansion of $(3 - 2x)^5$.

$$(3 - 2x)^5 = 3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \binom{5}{3}3^2(-2x)^3 + \binom{5}{4}3^1(-2x)^4 + (-2x)^5$$

$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

There will be 6 terms.
Each term has a total index of 5.
Use $(a + b)^n$ where $a = 3$, $b = -2x$ and $n = 5$.

There are $\binom{5}{2}$ ways of choosing two '$-2x$' terms from five brackets.

Online Work out each coefficient quickly using the ${}^{n}C_r$ and power functions on your calculator.

Example 5

Find the first four terms in the binomial expansion of:

a $(1 + 2x)^{10}$ **b** $\left(10 - \frac{1}{2}x\right)^6$

a $(1 + 2x)^{10}$

$= 1^{10} + \binom{10}{1}1^9(2x) + \binom{10}{2}1^8(2x)^2 + \binom{10}{3}1^7(2x)^3 + \ldots$

$= 1 + 20x + 180x^2 + 960x^3 + \ldots$

Notation This is sometimes called the expansion in **ascending powers of** x.

b $\left(10 - \frac{1}{2}x\right)^6$

$= 10^6 + \binom{6}{1}10^5\left(-\frac{1}{2}x\right) + \binom{6}{2}10^4\left(-\frac{1}{2}x\right)^2 + \binom{6}{3}10^3\left(-\frac{1}{2}x\right)^3 + \ldots$

$= 1\,000\,000 - 300\,000x + 37\,500x^2 - 2500x^3 + \ldots$

Write each coefficient in its simplest form.

Exercise 4C

SKILLS ANALYSIS

1 Write down the expansion of the following:

a $(1 + x)^4$ **b** $(3 + x)^4$ **c** $(4 - x)^4$ **d** $(x + 2)^6$ **e** $(1 + 2x)^4$ **f** $\left(1 - \frac{1}{2}x\right)^4$

2 Use the binomial theorem to find the first four terms in the expansion of:

a $(1 + x)^{10}$ **b** $(1 - 2x)^5$ **c** $(1 + 3x)^6$ **d** $(2 - x)^8$ **e** $\left(2 - \frac{1}{2}x\right)^{10}$ **f** $(3 - x)^7$

3 Use the binomial theorem to find the first four terms in the expansion of:

a $(2x + y)^6$ **b** $(2x + 3y)^5$ **c** $(p - q)^8$ **d** $(3x - y)^6$ **e** $(x + 2y)^8$ **f** $(2x - 3y)^9$

4 Use the binomial expansion to find the first four terms, in ascending powers of x, of:

a $(1 + x)^8$ **b** $(1 - 2x)^6$ **c** $\left(1 + \frac{x}{2}\right)^{10}$

d $(1 - 3x)^5$ **e** $(2 + x)^7$ **f** $(3 - 2x)^3$

g $(2 - 3x)^6$ **h** $(4 + x)^4$ **i** $(2 + 5x)^7$

Hint Your answers should be in the form $a + bx + cx^2 + dx^3$ where a, b, c and d are numbers.

E **5** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2 - x)^6$ and simplify each term. **(4 marks)**

E **6** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(3 - 2x)^5$ giving each term in its simplest form. **(4 marks)**

E/P **7** Find the binomial expansion of $\left(x + \frac{1}{x}\right)^5$ giving each term in its simplest form. **(4 marks)**

Challenge

SKILLS REASONING/ ARGUMENTATION

a Show that $(a + b)^4 - (a - b)^4 = 8ab(a^2 + b^2)$.

b Given that $82\,896 = 17^4 - 5^4$, write 82 896 as a product of its prime factors.

4.4 Solving binomial problems

You can use the general term of the binomial expansion to find individual coefficients in a binomial expansion.

- **In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.**

Example 6

a Find the coefficient of x^4 in the binomial expansion of $(2 + 3x)^{10}$.

b Find the coefficient of x^3 in the binomial expansion of $(2 + x)(3 - 2x)^7$.

a x^4 term $= \binom{10}{4}2^6(3x)^4$

$= 210 \times 64 \times 81x^4$

$= 1088640x^4$

The coefficient of x^4 in the binomial expansion of $(2 + 3x)^{10}$ is 1088640.

Use the general term. The power is 10, so $n = 10$, and you need to find the x^4 term so $r = 4$.

There are $\binom{10}{4}$ ways of choosing 4 '$3x$' terms from 10 brackets.

b $(3 - 2x)^7$

$= 3^7 + \binom{7}{1}3^6(-2x) + \binom{7}{2}3^5(-2x)^2 + \binom{7}{3}3^4(-2x)^3 + \ldots$

$= 2187 - 10206x + 20412x^2 - 22680x^3 + \ldots$

First find the first four terms of the binomial expansion of $(3 - 2x)^7$.

$(2 + x)(2187 - 10206x + 20412x^2 - 22680x^3 + \ldots)$

Now expand the brackets $(2 + x)(3 - 2x)^7$.

x^3 term $= 2 \times (-22680x^3) + x \times 20412x^2$

$= -24948x^3$

The coefficient of x^3 in the binomial expansion of $(2 + x)(3 - 2x)^7$ is -24948.

There are two ways of making the x^3 term: (constant term $\times$ x^3 term) and (x term $\times$ x^2 term).

Example 7

$g(x) = (1 + kx)^{10}$, where k is a constant.

Given that the coefficient of x^3 in the binomial expansion of $g(x)$ is 15, find the value of k.

x^3 term $= \binom{10}{3}1^7(kx)^3 = 15x^3$

$a = 1$, $b = kx$, $n = 10$ and $r = 3$.

$120k^3x^3 = 15x^3$

$k = \frac{1}{2}$

$k^3x^3 = \frac{15}{120}x^3$

$k^3x^3 = \frac{1}{8}x^3$

$k^3 = \frac{1}{8}$, $k = \sqrt[3]{\frac{1}{8}}$

Example 8 SKILLS PROBLEM-SOLVING

a Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 + qx)^8$, where q is a non-zero constant.

b Given that, in the expansion of $(1 + qx)^8$, the coefficient of x is $-r$ and the coefficient of x^2 is $7r$, find the value of q and the value of r.

a $(1 + qx)^8$

$= 1^8 + \binom{8}{1}1^7(qx)^1 + \binom{8}{2}1^6(qx)^2 + \ldots$

$= 1 + 8qx + 28q^2x^2 + \ldots$

b $8q = -r$ and $28q^2 = 7r$

$8q = -4q^2$

$4q^2 + 8q = 0$

$4q(q + 2) = 0$

$q = -2, r = 16$

Problem-solving

There are two unknowns in this expression. Your expansion will be in terms of q and x.

Using $28q^2 = 7r$, $r = 4q^2$ and $-r = -4q^2$.

q is non-zero so $q = -2$.

Exercise 4D SKILLS PROBLEM-SOLVING

1 Find the coefficient of x^3 in the binomial expansion of:

a $(3 + x)^5$ **b** $(1 + 2x)^5$ **c** $(1 - x)^6$ **d** $(3x + 2)^5$

e $(1 + x)^{10}$ **f** $(3 - 2x)^6$ **g** $(1 + x)^{20}$ **h** $(4 - 3x)^7$

i $\left(1 - \frac{1}{2}x\right)^6$ **j** $\left(3 + \frac{1}{2}x\right)^7$ **k** $\left(2 - \frac{1}{2}x\right)^8$ **l** $\left(5 + \frac{1}{4}x\right)^5$

(P) 2 The coefficient of x^2 in the expansion of $(2 + ax)^6$ is 60. Find two possible values of the constant a.

Problem-solving

$a = 2$, $b = ax$, $n = 6$. Use brackets when you substitute ax.

(P) 3 The coefficient of x^3 in the expansion of $(3 + bx)^5$ is -720. Find the value of the constant b.

(P) 4 The coefficient of x^3 in the expansion of $(2 + x)(3 - ax)^4$ is 30. Find the three possible values of the constant a.

(E/P) 5 When $(1 - 2x)^p$ is expanded, the coefficient of x^2 is 40. Given that $p > 0$, use this information to find:

a the value of the constant p **(6 marks)**

b the coefficient of x **(1 mark)**

c the coefficient of x^3 **(2 marks)**

Problem-solving

You will need to use the definition of $\binom{n}{r}$ to find an expression for $\binom{p}{2}$.

(E/P) 6 **a** Find the first three terms, in ascending powers of x, of the binomial expansion of $(5 + px)^{30}$, where p is a non-zero constant. **(2 marks)**

b Given that in this expansion the coefficient of x^2 is 29 times the coefficient of x work out the value of p. **(4 marks)**

E/P **7** **a** Find the first four terms, in ascending powers of x, of the binomial expansion of $(1 + qx)^{10}$, where q is a non-zero constant. **(2 marks)**

b Given that in the expansion of $(1 + qx)^{10}$ the coefficient of x^3 is 108 times the coefficient of x, work out the value of q. **(4 marks)**

E/P **8** **a** Find the first three terms, in ascending powers of x of the binomial expansion of $(1 + px)^{11}$, where p is a constant. **(2 marks)**

b The first 3 terms in the same expansion are 1, $77x$ and qx^2, where q is a constant. Find the value of p and the value of q. **(4 marks)**

E/P **9** **a** Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 + px)^{15}$, where p is a non-zero constant. **(2 marks)**

b Given that, in the expansion of $(1 + px)^{15}$, the coefficient of x is $(-q)$ and the coefficient of x^2 is $5q$, find the value of p and the value of q. **(4 marks)**

E/P **10** In the binomial expansion of $(1 + x)^{30}$, the coefficients of x^9 and x^{10} are p and q respectively. Find the value of $\frac{q}{p}$. **(4 marks)**

Challenge

Find the coefficient of x^4 in the binomial expansion of: **a** $(3 - 2x^2)^9$ **b** $\left(\frac{5}{x} + x^2\right)^8$

4.5 Binomial estimation

In engineering and science, it is often useful to find simple **approximations** for complicated functions. If the value of x is less than 1, then x^n gets smaller as n gets larger. If x is small you can sometimes **ignore large powers** of x to approximate a function or estimate a value.

Example 9 SKILLS ANALYSIS

a Find the first four terms of the binomial expansion, in ascending powers of x, of $\left(1 - \frac{x}{4}\right)^{10}$.

b Use your expansion to estimate the value of 0.975^{10}, giving your answer to 4 decimal places.

a $\left(1 - \frac{x}{4}\right)^{10}$

$= 1^{10} + \binom{10}{1}1^9\left(-\frac{x}{4}\right) + \binom{10}{2}1^8\left(-\frac{x}{4}\right)^2 + \binom{10}{3}1^7\left(-\frac{x}{4}\right)^3 + \ldots$

$= 1 - 2.5x + 2.8125x^2 - 1.875x^3 + \ldots$

b We want $\left(1 - \frac{x}{4}\right) = 0.975$

$\frac{x}{4} = 0.025$

$x = 0.1$

Calculate the value of x.

Substitute $x = 0.1$ into the expansion for $\left(1 - \frac{x}{4}\right)^{10}$ from part **a**:

$0.975^{10} \approx 1 - 0.25 + 0.028125 - 0.001875$

Substitute $x = 0.1$ into your expansion.

$= 0.77625$

$0.975^{10} \approx 0.7763$ to 4 d.p.

Using a calculator, $0.975^{10} = 0.776\,329\,62$.
So approximation is correct to 4 decimal places.

Online Use technology to find the values of x for which the first four terms of this expansion give a good approximation to the value of the function.

Exercise 4E

SKILLS ANALYSIS

1 a Find the first four terms of the binomial expansion, in ascending powers of x, of $\left(1 - \frac{x}{10}\right)^6$.

b By substituting an appropriate value for x, find an approximate value for 0.99^6.

2 a Write down the first four terms of the binomial expansion of $\left(2 + \frac{x}{5}\right)^{10}$.

b By substituting an appropriate value for x, find an approximate value for 2.1^{10}.

(P) **3** If x is so small that terms of x^3 and higher can be ignored, show that:

$$(2 + x)(1 - 3x)^5 \approx 2 - 29x + 165x^2$$

Hint Start by using the binomial expansion to expand $(1 - 3x)^5$. You can ignore terms of x^3 and higher so you only need to expand up to and including the x^2 term.

(P) **4** If x is so small that terms of x^3 and higher can be ignored, and

$$(2 - x)(3 + x)^4 \approx a + bx + cx^2$$

find the values of the constants a, b and c.

Problem-solving

Find the first 3 terms in the expansion of $(2 - x)(3 + x)^4$, compare with $a + bx + cx^2$ and write down the values of a, b and c.

5 a Write down the first four terms in the expansion of $(1 + 2x)^8$.

b By substituting an appropriate value of x (which should be stated), find an approximate value of 1.02^8.

6 $f(x) = (1 - 5x)^{30}$

a Find the first four terms, in ascending powers of x, in the binomial expansion of $f(x)$.

b Use your answer to part **a** to estimate the value of $(0.995)^{30}$, giving your answer to 6 decimal places.

c Use your calculator to evaluate 0.995^{30} and calculate the percentage error in your answer to part **b**.

(E/P) **7 a** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $\left(3 - \frac{x}{5}\right)^{10}$, giving each term in its simplest form. **(4 marks)**

b Explain how you would use your expansion to give an estimate for the value of 2.98^{10}. **(1 mark)**

E **8** **a** Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 - 3x)^5$. Give each term in its simplest form. **(4 marks)**

b If x is small, so that x^2 and higher powers can be ignored, show that $(1 + x)(1 - 3x)^5 \approx 1 - 14x$. **(2 marks)**

E/P **9** A microchip company models the probability of having no faulty chips on a single production run as:

$$\text{P(no fault)} = (1 - p)^n,\ p < 0.001$$

where p is the probability of a single chip being faulty, and n being the total number of chips produced.

a State why the model is restricted to small values of p. **(1 mark)**

b Given that $n = 200$, find an approximate expression for P(no fault) in the form $a + bp + cp^2$. **(2 marks)**

c The company wants to achieve a 92% likelihood of having no faulty chips on a production run of 200 chips. Use your answer to part **b** to suggest a maximum value of p for this to be the case. **(4 marks)**

Chapter review 4

P **1** The 16th row of Pascal's triangle is shown below.

1 15 105 … …

a Find the next two values in the row.

b Hence find the coefficient of x^3 in the expansion of $(1 + 2x)^{15}$.

E **2** Given that $\binom{45}{17} = \frac{45!}{17!a!}$, write down the value of a. **(1 mark)**

3 20 people play a game at a school fair.
The probability that exactly n people win a prize is modelled as $\binom{20}{n}p^n(1 - p)^{20-n}$, where p is the probability of any one person winning.

Calculate the probability of:

a 5 people winning when $p = \frac{1}{2}$

b nobody winning when $p = 0.7$

c 13 people winning when $p = 0.6$

Give your answers to 3 significant figures.

E/P **4** When $\left(1 - \frac{3}{2}x\right)^p$ is expanded in ascending powers of x, the coefficient of x is -24.

a Find the value of p. **(2 marks)**

b Find the coefficient of x^2 in the expansion. **(3 marks)**

c Find the coefficient of x^3 in the expansion. **(1 mark)**

E/P **5** Given that:

$$(2 - x)^{13} \equiv A + Bx + Cx^2 + \dots$$

find the values of the integers A, B and C. **(4 marks)**

E **6** **a** Expand $(1 - 2x)^{10}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient in the expansion. **(4 marks)**

b Use your expansion to find an approximation of 0.98^{10}, stating clearly the substitution which you have used for x. **(3 marks)**

E **7** **a** Use the binomial series to expand $(2 - 3x)^{10}$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an integer. **(4 marks)**

b Use your series expansion, with a suitable value for x, to obtain an estimate for 1.97^{10}, giving your answer to 2 decimal places. **(3 marks)**

E/P **8** **a** Expand $(3 + 2x)^4$ in ascending powers of x, giving each coefficient as an integer. **(4 marks)**

b Hence, or otherwise, write down the expansion of $(3 - 2x)^4$ in ascending powers of x. **(2 marks)**

c Hence by choosing a suitable value for x show that $(3 + 2\sqrt{2})^4 + (3 - 2\sqrt{2})^4$ is an integer and state its value. **(2 marks)**

E/P **9** The coefficient of x^2 in the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, where n is a positive integer, is 7.

a Find the value of n. **(2 marks)**

b Using the value of n found in part **a**, find the coefficient of x^4. **(4 marks)**

E **10** **a** Use the binomial theorem to expand $(3 + 10x)^4$ giving each coefficient as an integer. **(4 marks)**

b Use your expansion, with an appropriate value for x, to find the exact value of 1003^4. State the value of x which you have used. **(3 marks)**

E **11** **a** Expand $(1 + 2x)^{12}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient. **(4 marks)**

b By substituting a suitable value for x, which must be stated, into your answer to part **a**, calculate an approximate value of 1.02^{12}. **(3 marks)**

c Use your calculator, writing down all the digits in your display, to find a more exact value of 1.02^{12}. **(1 mark)**

d Calculate, to 3 significant figures, the percentage error of the approximation found in part **b**. **(1 mark)**

E/P **12** Expand $\left(x - \frac{1}{x}\right)^5$, simplifying the coefficients. **(4 marks)**

E/P **13** In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positive integer, the coefficient of x^2 is equal to the coefficient of x^3.

a Prove that $n = 6k + 2$. **(3 marks)**

b Given also that $k = \frac{2}{3}$, expand $(2k + x)^n$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an exact fraction in its simplest form. **(4 marks)**

E/P **14** **a** Expand $(2 + x)^6$ as a binomial series in ascending powers of x, giving each coefficient as an integer. **(4 marks)**

b By making suitable substitutions for x in your answer to part **a**, show that $(2 + \sqrt{3})^6 - (2 - \sqrt{3})^6$ can be simplified to the form $k\sqrt{3}$, stating the value of the integer k. **(3 marks)**

E/P **15** The coefficient of x^2 in the binomial expansion of $(2 + kx)^8$, where k is a positive constant, is 2800.

a Use algebra to calculate the value of k. **(2 marks)**

b Use your value of k to find the coefficient of x^3 in the expansion. **(4 marks)**

E/P **16** **a** Given that

$$(2 + x)^5 + (2 - x)^5 \equiv A + Bx^2 + Cx^4,$$

find the value of the constants A, B and C. **(4 marks)**

b Using the substitution $y = x^2$ and your answers to part **a**, solve

$$(2 + x)^5 + (2 - x)^5 = 349.$$ **(3 marks)**

E/P **17** In the binomial expansion of $(2 + px)^5$, where p is a constant, the coefficient of x^3 is 135. Calculate:

a the value of p, **(4 marks)**

b the value of the coefficient of x^4 in the expansion. **(2 marks)**

P **18** Find the constant term in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x}\right)^9$.

E/P **19** **a** Find the first three terms, in ascending powers of x of the binomial expansion of $(2 + px)^7$, where p is a constant. **(2 marks)**

The first 3 terms are 128, $2240x$ and qx^2, where q is a constant.

b Find the value of p and the value of q. **(4 marks)**

E/P **20** **a** Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 - px)^{12}$, where p is a non-zero constant. **(2 marks)**

b Given that, in the expansion of $(1 - px)^{12}$, the coefficient of x is q and the coefficient of x^2 is $6q$, find the value of p and the value of q. **(4 marks)**

E/P **21** **a** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $\left(2 + \frac{x}{2}\right)^7$, giving each term in its simplest form. **(4 marks)**

b Explain how you would use your expansion to give an estimate for the value of 2.05^7. **(1 mark)**

E/P **22** $g(x) = (4 + kx)^5$, where k is a constant.

Given that the coefficient of x^3 in the binomial expansion of $g(x)$ is 20, find the value of k. **(3 marks)**

Challenge

SKILLS CREATIVITY

1 $f(x) = (2 - px)(3 + x)^5$ where p is a constant.
There is no x^2 term in the expansion of $f(x)$.
Show that $p = \frac{4}{3}$

2 Find the coefficient of x^2 in the expansion of $(1 + 2x)^8(2 - 5x)^7$.

Summary of key points

1 Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.

2 The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.

3 $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$.

4 You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.

- The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$.

5 The binomial expansion is:

$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \ (n \in \mathbb{N})$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

6 In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

7 If x is small, the first few terms in the binomial expansion can be used to find an approximate value for a complicated expression.

1 Review exercise

E **1** The circle C has centre $(-3, 8)$ and passes through the point $(0, 9)$. Find an equation for C. **(4)**

← Pure 2 Section 2.2

E/P **2** **a** Show that $x^2 + y^2 - 6x + 2y - 10 = 0$ can be written in the form $(x - a)^2 + (y - b)^2 = r^2$, where a, b and r are numbers to be found. **(2)**

b Hence write down the centre and radius of the circle with equation $x^2 + y^2 - 6x + 2y - 10 = 0$. **(2)**

← Pure 2 Section 2.2

E/P **3** The line $3x + y = 14$ intersects the circle $(x - 2)^2 + (y - 3)^2 = 5$ at the points A and B.

a Find the coordinates of A and B. **(4)**

b Determine the length of the chord AB. **(2)**

← Pure 2 Section 2.3

E/P **4** The line with equation $y = 3x - 2$ does not intersect the circle with centre $(0, 0)$ and radius r. Find the range of possible values of r. **(8)**

← Pure 2 Section 2.3

E/P **5** The circle C has centre $(1, 5)$ and passes through the point $P(4, -2)$. Find:

a an equation for the circle C. **(4)**

b an equation for the tangent to the circle at P. **(3)**

← Pure 2 Section 2.4

E/P **6** The points $A(2, 1)$, $B(6, 5)$ and $C(8, 3)$ lie on a circle.

a Show that $\angle ABC = 90°$. **(2)**

b Deduce a geometrical property of the line segment AC. **(1)**

c Hence find the equation of the circle. **(4)**

← Pure 2 Section 2.5

E/P **7** $$\frac{2x^2 + 20x + 42}{224x + 4x^2 - 4x^3} = \frac{x + a}{bx(x + c)}$$

where a, b and c are constants. Work out the values of a, b and c. **(4)**

← Pure 2 Section 1.1

E **8** **a** Show that $(2x - 1)$ is a factor of $2x^3 - 7x^2 - 17x + 10$. **(2)**

b Factorise $2x^3 - 7x^2 - 17x + 10$ completely. **(4)**

c Hence, or otherwise, sketch the graph of $y = 2x^3 - 7x^2 - 17x + 10$, labelling any intersections with the coordinate axes clearly. **(2)**

← Pure 2 Section 1.3

E/P **9** $f(x) = 3x^3 + x^2 - 38x + c$

Given that $f(3) = 0$,

a find the value of c, **(2)**

b factorise $f(x)$ completely, **(4)**

← Pure 2 Section 1.3

E **10** $g(x) = x^3 - 13x + 12$

a Use the factor theorem to show that $(x - 3)$ is a factor of $g(x)$. **(2)**

b Factorise $g(x)$ completely. **(4)**

← Pure 2 Section 1.3

E/P **11** When $x^3 + ax^2 + bx + 8$ is divided by $(x - 3)$ the remainder is 2 and when it is divided by $(x + 1)$ the remainder is -2.

a Find the value of a and the value of b. **(4)**

b Hence find the remainder when this expression is divided by $(x - 2)$. **(2)**

← Pure 2 Section 1.4

E/P **12** $f(x) = 2x^3 + ax^2 + bx + 6$

When $f(x)$ is divided by $(x - 1)$ there is no remainder, and when $f(x)$ is divided by $(x + 1)$ the remainder is 10.

a Find the value of a and the value of b. **(4)**

b Hence solve the equation $f(x) = 0$. **(5)**

← Pure 2 Section 1.4

E/P **13** $f(x) = x^4 + 5x^3 + ax + b$, where a and b are constants.

The remainder when $f(x)$ is divided by $(x - 2)$ is equal to the remainder when $f(x)$ is divided by $(x + 1)$.

a Find the value of a. **(5)**

Given that $(x + 3)$ is a factor of $f(x)$,

b find the value of b. **(3)**

← Pure 2 Section 1.4

E/P **14** **a** It is claimed that the following inequality is true for all real numbers a and b. Use a counter-example to show that the claim is false:

$$a^2 + b^2 < (a + b)^2$$ **(2)**

b Specify conditions on a and b that make this inequality true. Prove your result. **(4)**

← Pure 2 Section 1.5

E/P **15** **a** Use proof by exhaustion to prove that for all prime numbers p, $3 < p < 20$, p^2 is one greater than a multiple of 24. **(2)**

b Find a counter-example that disproves the statement 'All numbers which are one greater than a multiple of 24 are the squares of prime numbers.' **(2)**

← Pure 2 Section 1.5

E/P **16** **a** Show that $x^2 + y^2 - 10x - 8y + 32 = 0$ can be written in the form $(x - a)^2 + (y - b)^2 = r^2$, where a, b and r are numbers to be found. **(2)**

b Circle C has equation $x^2 + y^2 - 10x - 8y + 32 = 0$ and circle D has equation $x^2 + y^2 = 9$. Calculate the distance between the centre of circle C and the centre of circle D. **(3)**

c Using your answer to part **b**, or otherwise, prove that circles C and D do not touch. **(2)**

← Pure 2 Sections 1.5, 2.4

E **17** **a** Find, to 3 significant figures, the value of x for which $5^x = 0.75$. **(2)**

b Solve the equation $2\log_5 x - \log_5 3x = 1$. **(3)**

← Pure 2 Sections 3.1, 3.4

E **18** **a** Solve $3^{2x-1} = 10$, giving your answer to 3 significant figures. **(3)**

b Solve $\log_2 x + \log_2(9 - 2x) = 2$. **(3)**

← Pure 2 Sections 3.1, 3.3, 3.4

E/P **19** **a** Express $\log_p 12 - (\frac{1}{2}\log_p 9 + \frac{1}{3}\log_p 8)$ as a single logarithm to base p. **(3)**

b Find the value of x in $\log_4 x = -1.5$. **(2)**

← Pure 2 Sections 3.3, 3.4

E/P **20** Solve the equation $\log_x 64 + 3\log_4 x - \log_x 4 = 5$. **(7)**

← Pure 2 Sections 3.3, 3.4, 3.5

E/P **21** Solve the equation $\log_2 x + 6\log_x 2 = 7$. **(6)**

← Pure 2 Sections 3.3, 3.4, 3.5

E/P **22** Solve the equation $\log_3 9t = \log_9\left(\frac{12}{t}\right)^2 + 2$.

Give your answer in the form $a\sqrt{b}$. **(8)**

← Pure 2 Sections 3.3, 3.4, 3.5

E **23** **a** Expand $(1 - 2x)^{10}$ in ascending powers of x up to and including the term in x^3. **(3)**

b Use your answer to part **a** to evaluate $(0.98)^{10}$ correct to 3 decimal places. **(1)**

← Pure 2 Section 4.5

E/P **24** If x is so small that terms of x^3 and higher can be ignored, $(2 - x)(1 + 2x)^5 \approx a + bx + cx^2$. Find the values of the constants a, b and c. **(5)**

← Pure 2 Section 4.4

E/P **25** The coefficient of x in the binomial expansion of $(2 - 4x)^q$, where q is a positive integer, is $-32q$. Find the value of q. **(4)**

← Pure 2 Section 4.4

E **26** **a** Find the binomial expansion of $(1 + 4x)^{\frac{3}{2}}$ in ascending powers of x up to and including the x^3 term, simplifying each term. **(4)**

b Show that, when $x = \frac{3}{100}$, the exact value of $(1 + 4x)^{\frac{3}{2}}$ is $\frac{112\sqrt{112}}{1000}$ **(2)**

c Substitute $x = \frac{3}{100}$ into the binomial expansion in part **a** and hence obtain an approximation to $\sqrt{112}$. Give your answer to 5 decimal places. **(3)**

d Calculate the percentage error in your estimate to 5 decimal places. **(2)**

← Pure 2 Section 4.1

Challenge

SKILLS CREATIVITY

1 $f(x) = 2x^4 + ax^3 - 23x^2 + bx + 24$, where a and b are real constants.

a Given that $x^2 + x - 6$ is a factor of $f(x)$, find the values of a and b.

b Hence, factorise $f(x)$ completely.

← Pure 2 Section 1.3

SKILLS INNOVATION

2 $f(x)$ is a polynomial.

Given that $f(-2) = -11$ and $f(1) = 4$, find the remainder when $f(x)$ is divided by $(x + 2)(x - 1)$.

← Pure 2 Section 1.4

Hint The remainder will be of the form $ax + b$.

3 Prove that the circle $(x + 4)^2 + (y - 5)^2 = 8^2$ lies completely inside the circle $x^2 + y^2 + 8y - 10y = 59$

← Pure 2 Section 2.2

4 Solve the simultaneous equations

$$8^{(2y+1)} = 4^{[2x-2]}$$
$$\log_2 y = 1 + \log_4 x$$

← Pure 2 Sections 3.1, 3.2, 3.3, 3.4, 3.5

SKILLS INNOVATION

5 Prove that for all positive integers n and k,

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

← Pure 2 Sections 1.4, 4.2

5 SEQUENCES AND SERIES

4.1
4.2
4.3
4.4

Learning objectives

After completing this chapter you should be able to:

- Find the nth term of an arithmetic sequence → pages 81–83
- Prove and use the formula for the sum of the first n terms of an arithmetic series → pages 84–87
- Find the nth term of a geometric sequence → pages 87–91
- Prove and use the formula for the sum of a finite geometric series → pages 91–94
- Prove and use the formula for the sum to infinity of a convergent geometric series → pages 94–97
- Use sigma notation to describe series → pages 97–99
- Generate sequences from recurrence relations → pages 100–104
- Model real-life situations with sequences and series → pages 104–107

Prior knowledge check

1 Write down the next three terms of each sequence.

a 2, 7, 12, 17 **b** 11, 8, 5, 2

c −15, −9, −3, 3 **d** 3, 6, 12, 24

e $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ **f** $-\frac{1}{16}, \frac{1}{4}, -1, 4$

← International GCSE Mathematics

2 Solve, giving your answers to 3 significant figures:

a $2^x = 50$

b $0.2^x = 0.0035$

c $4 \times 3^x = 78\,732$

← Pure 2 Section 3.2

Sequences and series can be found in nature, and can be used to model population growth or decline, or the spread of a virus.

5.1 Arithmetic sequences

- In an **arithmetic sequence**, the difference between consecutive terms is constant.

Notation An arithmetic sequence is sometimes called an **arithmetic progression**.

5, 7, 9, 11,
+2 +2 +2

This sequence is arithmetic. The difference between consecutive terms is +2. The sequence is **increasing**.

12.5, 10, 7.5, 5,
−2.5 −2.5 −2.5

This sequence is arithmetic. The difference between consecutive terms is −2.5. The sequence is **decreasing**.

4, 7, 12, 19,
+3 +5 +7

The difference is not constant so the sequence is not arithmetic.

- The formula for the ***n***th term of an arithmetic sequence is:

$$u_n = a + (n - 1)d$$

where a is the first term and d is the **common difference**.

Notation In questions on sequences and series:
- u_n is the nth term
- a is the first term
- d is the common difference

Example 1

The nth term of an arithmetic sequence is $u_n = 55 - 2n$.

a Write down the first 5 terms of the sequence.

b Find the 99th term in the sequence.

c Find the first term in the sequence that is negative.

Online Use the table function on your calculator to generate terms in the sequence for this function, or to check an nth term.

a $u_n = 55 - 2n$

$n = 1 \rightarrow u_1 = 55 - 2(1) = 53$

$n = 2 \rightarrow u_2 = 55 - 2(2) = 51$

$n = 3 \rightarrow u_3 = 55 - 2(3) = 49$

$n = 4 \rightarrow u_4 = 55 - 2(4) = 47$

$n = 5 \rightarrow u_5 = 55 - 2(5) = 45$

Remember, n is the position in the sequence, so for the first term substitute $n = 1$.

For the second term substitute $n = 2$.

b $u_{99} = 55 - 2(99) = -143$

For the 99th term substitute $n = 99$.

c $55 - 2n < 0$

$-2n < -55$

$n > 27.5$

$n = 28$

$u_{28} = 55 - 2(28) = -1$

Problem-solving To find the first negative term, set $u_n < 0$ and solve the inequality. n is the term number so it must be a positive integer.

Example 2

Find the nth term of each arithmetic sequence.

a 6, 20, 34, 48, 62

b 101, 94, 87, 80, 73

a $a = 6$, $d = 14$
$u_n = 6 + 14(n - 1)$
$u_n = 6 + 14n - 14$
$u_n = 14n - 8$

b $a = 101$, $d = -7$
$u_n = 101 - 7(n - 1)$
$u_n = 101 - 7n + 7$
$u_n = 108 - 7n$

Write down the values of a and d.

Substitute the values of a and d into the formula $a + (n - 1)d$ and simplify.

Watch out If the sequence is decreasing then d is negative.

Example 3

SKILLS PROBLEM-SOLVING

A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.
Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b.

Problem-solving You know two terms and there are two unknowns in the expression for the nth term. You can use this information to form two simultaneous equations. ← Year 1 Section 3.1

$u_3 = 5$, so $3a + b = 5$. (1)

Substitute $n = 3$ and $u_3 = 5$ in $u_n = an + b$.

$u_8 = 20$, so $8a + b = 20$. (2)

Substitute $n = 8$ and $u_8 = 20$ in $u_n = an + b$.

(2) − (1) gives:
$5a = 15$
$a = 3$
Substitute $a = 3$ in (1):
$9 + b = 5$
$b = -4$
Constants are $a = 3$ and $b = -4$.

Solve simultaneously.

Exercise 5A

SKILLS INTERPRETATION

1 For each sequence:
 i write down the first 4 terms of the sequence
 ii write down a and d.

 a $u_n = 5n + 2$ **b** $u_n = 9 - 2n$ **c** $u_n = 7 + 0.5n$ **d** $u_n = n - 10$

2 Find the nth terms and the 10th terms in the following arithmetic progressions:

a 5, 7, 9, 11, … **b** 5, 8, 11, 14, …
c 24, 21, 18, 15, … **d** −1, 3, 7, 11, …
e $x, 2x, 3x, 4x, \ldots$ **f** $a, a + d, a + 2d, a + 3d, \ldots$

P **3** Calculate the number of terms in each of the following arithmetic sequences.

a 3, 7, 11, …, 83, 87 **b** 5, 8, 11, …, 119, 122
c 90, 88, 86, …, 16, 14 **d** 4, 9, 14, …, 224, 229
e $x, 3x, 5x, \ldots, 35x$
f $a, a + d, a + 2d, \ldots, a + (n - 1)d$

Problem-solving

Find an expression for u_n and set it equal to the final term in the sequence. Solve the equation to find the value of n.

P **4** The first term of an arithmetic sequence is 14. The fourth term is 32. Find the common difference.

P **5** A sequence is generated by the formula $u_n = pn + q$ where p and q are constants to be found. Given that $u_6 = 9$ and $u_9 = 11$, find the constants p and q.

P **6** For an arithmetic sequence $u_3 = 30$ and $u_9 = 9$. Find the first negative term in the sequence.

P **7** The 20th term of an arithmetic sequence is 14. The 40th term is −6. Find the value of the 10th term.

P **8** The first three terms of an arithmetic sequence are $5p$, 20 and $3p$, where p is a constant. Find the 20th term in the sequence.

E/P **9** The first three terms in an arithmetic sequence are $-8, k^2, 17k$ …
Find two possible values of k. **(3 marks)**

E/P **10** An arithmetic sequence has first term k^2 and common difference k, where $k > 0$. The fifth term of the sequence is 41. Find the value of k, giving your answer in the form $p + q\sqrt{5}$, where p and q are integers to be found. **(4 marks)**

Problem-solving

You will need to make use of the condition $k > 0$ in your answer.

5.2 Arithmetic series

- An arithmetic series is the sum of the terms of an arithmetic sequence.

5, 7, 9, 11 is an arithmetic **sequence**.
5 + 7 + 9 + 11 is an arithmetic **series**.

Notation S_n is used for the sum of the first n terms of a series.

Example 4

Prove that the sum of the first 100 natural numbers is 5050.

$S_{100} = 1 + 2 + 3 + \ldots + 98 + 99 + 100 \quad (1)$
$S_{100} = 100 + 99 + 98 + \ldots + 3 + 2 + 1 \quad (2)$
Adding (1) and (2):
$2 \times S_{100} = 100 \times 101$
$S_{100} = \frac{100 \times 101}{2}$
$= 5050$

The **natural numbers** are the positive integers: 1, 2, 3, 4, …

Problem-solving
Write out the sum longhand, then write it out in reverse. You can pair up the numbers so that each pair has a sum of 101. There are 100 pairs in total.

- The sum of the first n terms of an arithmetic series is given by the formula

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

where a is the first term and d is the common difference.

You can also write this formula as

$$S_n = \frac{n}{2}(a + l)$$

where l is the last term.

Example 5

Prove that the sum of the first n terms of an arithmetic series is $\frac{n}{2}(2a + (n-1)d)$.

$S_n = a + (a + d) + (a + 2d) + \ldots$
$+ (a + (n-2)d) + (a + (n-1)d) \quad (1)$

Write out the terms of the sum.

$S_n = (a + (n-1)d) + (a + (n-2)d) + \ldots$
$+ (a + 2d) + (a + d) + a \quad (2)$

This is the sum reversed.

Adding (1) and (2):

Adding together the two sums.

$2 \times S_n = n(2a + (n-1)d)$
$S_n = \frac{n}{2}(2a + (n-1)d)$

Problem-solving
You need to learn this proof for your exam.

Example 6

Find the sum of the first 50 terms of the arithmetic series 32 + 27 + 22 + 17 + 12 + …

$a = 32$, $d = -5$ — Write down a and d.

$S_{50} = \frac{50}{2}(2(32) + (50 - 1)(-5))$ — Substitute into the formula.

$S_{50} = -4525$ — Simplify.

Example 7

SKILLS PROBLEM-SOLVING

Find the least number of terms required for the sum of 4 + 9 + 14 + 19 + … to exceed 2000.

$4 + 9 + 14 + 19 + \ldots > 2000$ — Always establish what you are given in a question. As you are adding on positive terms, it is easier to solve the equality $S_n = 2000$.

Using $S_n = \frac{n}{2}(2a + (n - 1)d)$ — Knowing $a = 4$, $d = 5$ and $S_n = 2000$, you need to find n. Substitute into $S_n = \frac{n}{2}(2a + (n - 1)d)$.

$2000 = \frac{n}{2}(2 \times 4 + (n - 1)5)$

$4000 = n(8 + 5n - 5)$

$4000 = n(5n + 3)$

$4000 = 5n^2 + 3n$

$0 = 5n^2 + 3n - 4000$

$n = \frac{-3 \pm \sqrt{9 + 80\,000}}{10}$ — Solve using the quadratic formula.

$n = 27.99$ or -28.59 (2 d.p.)

28 terms are needed. — n is the number of terms, so must be a positive integer.

Exercise 5B

SKILLS PROBLEM-SOLVING

1 Find the sums of the following series.

a $3 + 7 + 11 + 14 + \ldots$ (20 terms) **b** $2 + 6 + 10 + 14 + \ldots$ (15 terms)

c $30 + 27 + 24 + 21 + \ldots$ (40 terms) **d** $5 + 1 + -3 + -7 + \ldots$ (14 terms)

e $5 + 7 + 9 + \ldots + 75$

f $4 + 7 + 10 + \ldots + 91$

g $34 + 29 + 24 + 19 + \ldots + -111$

h $(x + 1) + (2x + 1) + (3x + 1) + \ldots + (21x + 1)$

Hint For parts **e** to **h**, start by using the last term to work out the number of terms in the series.

2 Find how many terms of the following series are needed to make the given sums.

a $5 + 8 + 11 + 14 + \ldots = 670$

b $3 + 8 + 13 + 18 + \ldots = 1575$

c $64 + 62 + 60 + \ldots = 0$

d $34 + 30 + 26 + 22 + \ldots = 112$

Hint Set the expression for S_n equal to the total and solve the resulting equation to find n.

(P) **3** Find the sum of the first 50 even numbers.

(P) **4** Find the least number of terms for the sum of $7 + 12 + 17 + 22 + 27 + \ldots$ to exceed 1000.

(P) **5** The first term of an arithmetic series is 4. The sum to 20 terms is -15. Find, in any order, the common difference and the 20th term.

(P) **6** The sum of the first three terms of an arithmetic series is 12. If the 20th term is -32, find the first term and the common difference.

(P) **7** Prove that the sum of the first 50 natural numbers is 1275.

Problem-solving

Use the same method as Example 4.

(P) **8** Show that the sum of the first $2n$ natural numbers is $n(2n + 1)$.

(P) **9** Prove that the sum of the first n odd numbers is n^2.

(E/P) **10** The fifth term of an arithmetic series is 33. The tenth term is 68. The sum of the first n terms is 2225.

a Show that $7n^2 + 3n - 4450 = 0$. **(4 marks)**

b Hence find the value of n. **(1 mark)**

(E/P) **11** An arithmetic series is given by $(k + 1) + (2k + 3) + (3k + 5) + \ldots + 303$

a Find the number of terms in the series in terms of k. **(1 mark)**

b Show that the sum of the series is given by $\dfrac{152k + 46208}{k + 2}$ **(3 marks)**

c Given that $S_n = 2568$, find the value of k. **(1 mark)**

(E/P) **12** **a** Calculate the sum of all the multiples of 3 from 3 to 99 inclusive,

$$3 + 6 + 9 + \ldots + 99$$ **(3 marks)**

b In the arithmetic series

$$4p + 8p + 12p + \ldots + 400$$

where p is a positive integer and a factor of 100,

i find, in terms of p, an expression for the number of terms in this series.

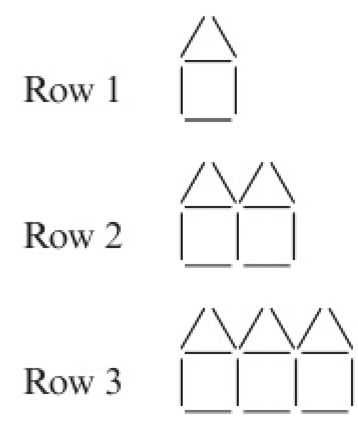
ii Show that the sum of this series is $200 + \dfrac{20\,000}{p}$ **(4 marks)**

c Find, in terms of p, the 80th term of the arithmetic sequence

$$(3p + 2), (5p + 3), (7p + 4), \ldots,$$

giving your answer in its simplest form. **(2 marks)**

E/P **13** Mia has some sticks that are all of the same length.
She arranges them in shapes as shown opposite and has made the following 3 rows of patterns.



She notices that 6 sticks are required to make the single pentagon in the first row, 11 sticks in the second row and for the third row she needs 16 sticks.

a Find an expression, in terms of n, for the number of sticks required to make a similar arrangement of n pentagons in the nth row. **(3 marks)**

Mia continues to make pentagons following the same pattern. She continues until she has completed 10 rows.

b Find the total number of sticks Mia uses in making these 10 rows. **(3 marks)**

Mia started with 1029 sticks. Given that Mia continues the pattern to complete k rows but does not have enough sticks to complete the $(k + 1)$th row:

c show that k satisfies $(5k - 98)(k + 21) \leqslant 0$ **(4 marks)**

d find the value of k. **(2 marks)**

Challenge

The sum S_n of the first n terms of an arithmetic sequence is given by $S_n = 3n(n + 2)$
The nth term of the series is T_n
Given that $4T_{(n+4)} + 12 = S_{(n-4)}$, find the value of n.

5.3 Geometric sequences

■ A **geometric sequence** has a **common ratio** between consecutive terms.

Notation A geometric sequence is sometimes called a **geometric progression**.

To get from one term to the next you **multiply** by the common ratio.

2, 4, 8, 16
×2 ×2 ×2

This is a geometric sequence with common ratio 2. This sequence is increasing.

$\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}$
$\times\frac{1}{3}$ $\times\frac{1}{3}$ $\times\frac{1}{3}$

This is a geometric sequence with common ratio $\frac{1}{3}$. This sequence is decreasing but will never get to zero.

Notation A geometric sequence with a common ratio $|r| < 1$ converges. This means it tends to a certain value. You call the value the **limit** of the sequence.

5, −10, 20, −40, 80
×(−2) ×(−2) ×(−2) ×(−2)

Here the common ratio is −2. The sequence alternates between positive and negative terms.

Notation An **alternating sequence** is a sequence in which terms are alternately positive and negative.

■ The formula for the nth term of a geometric sequence is:

$$u_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

Example 8

Find the **i** 10th and **ii** nth terms in the following geometric sequences:

a 3, 6, 12, 24, …

b 40, −20, 10, −5, …

a 3, 6, 12, 24, …

For this sequence $a = 3$ and $r = \frac{6}{3} = 2$.

i 10th term $= 3 \times 2^9$
$= 3 \times 512$
$= 1536$

For the 10th term use ar^{n-1} with $a = 3$, $r = 2$ and $n = 10$.

ii nth term $= 3 \times 2^{n-1}$

For the nth term use ar^{n-1} with $a = 3$ and $r = 2$.

b 40, −20, 10, −5, …

For this sequence $a = 40$ and $r = -\frac{20}{40} = -\frac{1}{2}$

i 10th term $= 40 \times \left(-\frac{1}{2}\right)^9$
$= 40 \times -\frac{1}{512}$
$= -\frac{5}{64}$

Use ar^{n-1} with $a = 40$, $r = -\frac{1}{2}$ and $n = 10$.

ii nth term $= 40 \times \left(-\frac{1}{2}\right)^{n-1}$

Use ar^{n-1} with $a = 40$, $r = -\frac{1}{2}$ and $n = n$.

$= 5 \times 8 \times \left(-\frac{1}{2}\right)^{n-1}$
$= 5 \times 2^3 \times \left(-\frac{1}{2}\right)^{n-1}$
$= (-1)^{n-1} \times \frac{5}{2^{n-4}}$

Use laws of indices $\frac{x^m}{x^n} = x^{(m-n)} = \frac{1}{x^{n-m}}$

So $2^3 \times \frac{1}{2^{n-1}} = \frac{1}{2^{n-1-3}}$

Example 9

SKILLS ANALYSIS

The 2nd term of a geometric sequence is 4 and the 4th term is 8. Given that the common ratio is positive, find the exact value of the 11th term in the sequence.

nth term $= ar^{n-1}$, so the 2nd term is ar, and the 4th term is ar^3

$ar = 4$ (1)
$ar^3 = 8$ (2)

Dividing equation (2) by equation (1):

$\frac{ar^3}{ar} = \frac{8}{4}$

$r^2 = 2$

$r = \sqrt{2}$

You are told in the question that $r > 0$ so use the positive square root.

Problem-solving

You can use the general term of a geometric sequence to write two equations. Solve these simultaneously to find a and r, then find the 11th term in the sequence.

Substituting back into equation (1):

$a\sqrt{2} = 4$

$a = \frac{4}{\sqrt{2}}$

$a = 2\sqrt{2}$ — Rationalise the denominator.

nth term $= ar^{n-1}$, so

11th term $= (2\sqrt{2})(\sqrt{2})^{10}$

$= 64\sqrt{2}$ — Simplify your answer as much as possible.

Example 10

SKILLS EXECUTIVE FUNCTION

The numbers 3, x and $(x + 6)$ form the first three terms of a geometric sequence with all positive terms. Find:

a the possible values of x, **b** the 10th term of the sequence.

a

$\frac{u_2}{u_1} = \frac{u_3}{u_2}$

$\frac{x}{3} = \frac{x+6}{x}$

$x^2 = 3(x + 6)$

$x^2 = 3x + 18$

$x^2 - 3x - 18 = 0$

$(x - 6)(x + 3) = 0$ — Factorise.

$x = 6$ or -3

So x is either 6 or −3, but there are no negative terms so $x = 6$. — If there are no negative terms then −3 cannot be an answer.

b 10th term $= ar^9$

$= 3 \times 2^9$ — Use the formula nth term $= ar^{n-1}$ with $n = 10$, $a = 3$ and $r = \frac{x}{3} = \frac{6}{3} = 2$.

$= 3 \times 512$

$= 1536$

The 10th term is 1536.

Problem-solving

In a geometric sequence the ratio between consecutive terms is the same, so $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

Simplify the algebraic fraction to form a quadratic equation. ← Pure 1 Section 3.2

Example 11

What is the first term in the geometric progression 3, 6, 12, 24, … to exceed 1 million?

nth term $= ar^{n-1}$

$= 3 \times 2^{n-1}$ — Sequence has $a = 3$ and $r = 2$.

We want nth term > 1000000

Problem-solving

Determine a and r, then write an inequality using the formula for the general term of a geometric sequence.

So $3 \times 2^{n-1} > 1\,000\,000$

$2^{n-1} > \frac{1\,000\,000}{3}$

$\log 2^{n-1} > \log\left(\frac{1\,000\,000}{3}\right)$

$(n-1)\log 2 > \log\left(\frac{1\,000\,000}{3}\right)$

$n - 1 > \frac{\log\left(\frac{1\,000\,000}{3}\right)}{\log(2)}$

$n - 1 > 18.35$ (2 d.p.)

$n > 19.35$

$n \geqslant 20$

The 20th term is the first to exceed 1 000 000.

Divide by 3.

To solve this inequality take logs of both sides.

$\log a^n = n \log a$ ← Pure 2 Section 3.4

Divide by log 2.

n has to be an integer.

Online Use your calculator to check your answer.

Exercise 5C

SKILLS ANALYSIS

1 Which of the following are geometric sequences? For the ones that are, give the value of the common ratio, r.

a 1, 2, 4, 8, 16, 32, …
b 2, 5, 8, 11, 14, …
c 40, 36, 32, 28, …
d 2, 6, 18, 54, 162, …
e 10, 5, 2.5, 1.25, …
f 5, −5, 5, −5, 5, …
g 3, 3, 3, 3, 3, 3, 3, …
h 4, −1, 0.25, −0.0625, …

2 Continue the following geometric sequences for three more terms.

a 5, 15, 45, …
b 4, −8, 16, …
c 60, 30, 15, …
d $1, \frac{1}{4}, \frac{1}{16}, \ldots$
e $1, p, p^2, \ldots$
f $x, -2x^2, 4x^3, \ldots$

(P) 3 If 3, x and 9 are the first three terms of a geometric sequence, find:

a the exact value of x,
b the exact value of the 4th term.

Problem-solving

In a geometric sequence the common ratio can be calculated by $\frac{u_2}{u_1}$ or $\frac{u_3}{u_2}$

4 Find the sixth and nth terms of the following geometric sequences.

a 2, 6, 18, 54, …
b 100, 50, 25, 12.5, …
c 1, −2, 4, −8, …
d 1, 1.1, 1.21, 1.331, …

5 The nth term of a geometric sequence is 2×5^n. Find the first and 5th terms.

6 The sixth term of a geometric sequence is 32 and the 3rd term is 4. Find the first term and the common ratio.

7 A geometric sequence has first term 4 and third term 1. Find the two possible values of the 6th term.

E/P 8 The first three terms of a geometric sequence are given by $8 - x$, $2x$, and x^2 respectively where $x > 0$.

a Show that $x^3 - 4x^2 = 0$. **(2 marks)**

b Find the value of the 20th term. **(3 marks)**

c State, with a reason, whether 4096 is a term in the sequence. **(1 mark)**

E/P 9 A geometric sequence has first term 200 and a common ratio p where $p > 0$. The 6th term of the sequence is 40.

a Show that p satisfies the equation $5 \log p + \log 5 = 0$. **(3 marks)**

b Hence or otherwise, find the value of p correct to 3 significant figures. **(1 mark)**

P 10 A geometric sequence has first term 4 and fourth term 108. Find the smallest value of k for which the kth term in this sequence exceeds 500 000.

P 11 The first three terms of a geometric sequence are 9, 36, 144. State, with a reason, whether 383 616 is a term in the sequence.

Problem-solving

Determine the values of a and r and find the general term of the sequence. Set the number given equal to the general term and solve to find n. If n is an integer, then the number is in the sequence.

P 12 The first three terms of a geometric sequence are 3, −12, 48. State, with a reason, whether 49 152 is a term in the sequence.

P 13 Find which term in the geometric progression 3, 12, 48, …is the first to exceed 1 000 000.

5.4 Geometric series

A geometric **series** is the sum of the terms of a geometric **sequence**. 3, 6, 12, 24, … is a geometric sequence. 3 + 6 + 12 + 24 + … is a geometric series.

- **The sum of the first n terms of a geometric series is given by the formula**

$$S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$$

or $$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

where a is the first term and r is the common ratio.

Hint These two formulae are equivalent. It is often easier to use the first one if $r < 1$ and the second one if $r > 1$.

Example 12

A geometric series has first term a and common difference r. Prove that the sum of the first n terms of this series is given by $S_n = \dfrac{a(1-r^n)}{1-r}$

Let $S_n = a + ar + ar^2 + ar^3 + \dots ar^{n-2} + ar^{n-1}$ (1)

$rS_n = ar + ar^2 + ar^3 + \dots ar^{n-1} + ar^n$ (2)

Multiply by r.

(1) − (2) gives $S_n - rS_n = a - ar^n$

Subtract rS_n from S_n.

$S_n(1-r) = a(1-r^n)$

Take out the common factor.

$S_n = \dfrac{a(1-r^n)}{1-r}$

Divide by $(1-r)$.

Problem-solving

You need to learn this proof for your exam.

Example 13

Find the sums of the following geometric series.

a $2 + 6 + 18 + 54 + \dots$ (for 10 terms)

b $1024 - 512 + 256 - 128 + \dots + 1$

a Series is

$2 + 6 + 18 + 54 + \dots$ (for 10 terms)

As in all questions, write down what is given.

So $a = 2$, $r = \frac{6}{2} = 3$ and $n = 10$

So $S_{10} = \dfrac{2(3^{10}-1)}{3-1} = 59\,048$

When $r > 1$ it is easier to use the formula $S_n = \dfrac{a(r^n-1)}{r-1}$

b Series is

$1024 - 512 + 256 - 128 + \dots + 1$

So $a = 1024$, $r = -\frac{512}{1024} = -\frac{1}{2}$

and the nth term $= 1$

$1024\left(-\frac{1}{2}\right)^{n-1} = 1$

First solve $ar^{n-1} = 1$ to find n.

$(-2)^{n-1} = 1024$

$2^{n-1} = 1024$

$(-2)^{n-1} = (-1)^{n-1}(2^{n-1}) = 1024$, so $(-1)^{n-1}$ must be positive and $2^{n-1} = 1024$.

$n - 1 = \dfrac{\log 1024}{\log 2}$

$n - 1 = 10$

$1024 = 2^{10}$

$n = 11$

So $S_n = \dfrac{1024\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)}$

When $r < 1$, it is easier to use the formula $S_n = \dfrac{a(1-r^n)}{1-r}$

$= \dfrac{1024\left(1+\frac{1}{2048}\right)}{1+\frac{1}{2}}$

$= \dfrac{1024.5}{\frac{3}{2}} = 683$

Example 14 SKILLS INTERPRETATION

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \ldots$ to n terms exceeds 2 000 000.

Sum to n terms is $S_n = \frac{1(2^n - 1)}{2 - 1}$
$= 2^n - 1$

If this is to exceed 2 000 000 then

$S_n > 2\,000\,000$
$2^n - 1 > 2\,000\,000$
$2^n > 2\,000\,001$ — Add 1.
$n \log 2 > \log(2\,000\,001)$ — Use laws of logs: $\log a^n = n \log a$.
$n > \frac{\log(2\,000\,001)}{\log(2)}$
$n > 20.9$

It needs 21 terms to exceed 2 000 000. — Round n up to the nearest integer.

Problem-solving

Determine the values of a and r, then use the formula for the sum of a geometric series to form an inequality.

Exercise 5D SKILLS INTERPRETATION

1 Find the sum of the following geometric series (to 3 decimal places if necessary).

a $1 + 2 + 4 + 8 + \ldots$ (8 terms)

b $32 + 16 + 8 + \ldots$ (10 terms)

c $\frac{2}{3} + \frac{4}{15} + \frac{8}{75} + \ldots + \frac{256}{234\,375}$

d $4 - 12 + 36 - 108 + \ldots$ (6 terms)

e $729 - 243 + 81 - \ldots - \frac{1}{3}$

f $-\frac{5}{2} + \frac{5}{4} - \frac{5}{8} \ldots - \frac{5}{32\,768}$

2 A geometric series has first three terms $3 + 1.2 + 0.48\ldots$ **Evaluate** S_{10} giving your answer to 4 decimal places.

3 A geometric series has first term 5 and common ratio $\frac{2}{3}$. Find the value of S_8.

(P) 4 The sum of the first three terms of a geometric series is 30.5. If the first term is 8, find possible values of r.

(P) 5 Find the least value of n such that the sum $3 + 6 + 12 + 24 + \ldots$ to n terms exceeds 1.5 million.

(P) 6 Find the least value of n such that the sum $5 + 4.5 + 4.05 + \ldots$ to n terms exceeds 45.

(E) 7 A geometric series has first term 25 and common ratio $\frac{3}{5}$
Given that the sum to k terms of the series is greater than 61,

a show that $k > \frac{\log(0.024)}{\log(0.6)}$ **(4 marks)**

b find the smallest possible value of k. **(1 mark)**

E/P **8** A geometric series has first term a and common ratio r. The sum of the first two terms of the series is 4.48. The sum of the first four terms is 5.1968. Find the two possible values of r. **(4 marks)**

Problem-solving One value will be positive and one value will be negative.

E/P **9** The first term of a geometric series is a and the common ratio is $\sqrt{3}$. Show that $S_{10} = 121a(\sqrt{3} + 1)$. **(4 marks)**

E/P **10** A geometric series has first term a and common ratio 2. A different geometric series has first term b and common ratio 3. Given that the sum of the first 4 terms of both series is the same, show that $a = \frac{8}{3}b$. **(4 marks)**

E/P **11** The first three terms of a geometric series are $(k - 6)$, k, $(2k + 5)$, where k is a positive constant.

a Show that $k^2 - 7k - 30 = 0$. **(4 marks)**

b Hence find the value of k. **(2 marks)**

c Find the common ratio of this series. **(1 mark)**

d Find the sum of the first 10 terms of this series, giving your answer to the nearest whole number. **(2 marks)**

5.5 Sum to infinity

You can work out the sum of the first n terms of a geometric series. As n tends to infinity, the sum of the series is called the **sum to infinity**.

Notation You can write the sum to infinity of a geometric series as S_∞

Consider the sum of the first n terms of the geometric series 2 + 4 + 8 + 16 +…

The terms of this series are getting larger, so as n tends to infinity, S_n also tends to infinity. This is called a **divergent** series.

Now consider the sum of the first n terms of the geometric series $1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \ldots$

The terms of this series are getting smaller. As n tends to infinity, S_n gets closer and closer to a finite value, S_∞. This is called a **convergent** series.

- **A geometric series is convergent if and only if $|r| < 1$, where r is the common ratio.**

Hint You can also write this condition as $-1 < r < 1$.

The sum of the first n terms of a geometric series is given by $S_n = \frac{a(1 - r^n)}{1 - r}$

When $|r| < 1$, $\lim_{n\to\infty}\left(\frac{a(1 - r^n)}{1 - r}\right) = \frac{a}{1 - r}$

This is because $r^n \to 0$ as $n \to \infty$.

Notation $\lim_{n\to\infty}$ means 'the limit as n tends to ∞'. You can't evaluate the expression when n is ∞, but as n gets larger the expression gets closer to a fixed (or **limiting**) value.

- **The sum to infinity of a convergent geometric series is given by $S_\infty = \frac{a}{1 - r}$**

Watch out You can only use this formula for a convergent series, i.e. when $|r| < 1$.

Example 15

The fourth term of a geometric series is 1.08 and the seventh term is 0.233 28.

a Show that this series is convergent.

b Find the sum to infinity of the series.

a $ar^3 = 1.08$ (1)

$ar^6 = 0.233\,28$ (2)

Dividing (2) by (1):

$\frac{ar^6}{ar^3} = \frac{0.233\,28}{1.08}$

$r^3 = 0.216$

$r = 0.6$

The series is convergent as $|r| = 0.6 < 1$.

b Substituting the value of r^3 into equation (1) to find a

$0.216a = 1.08$

$a = \frac{1.08}{0.216}$

$a = 5$

Substituting into S_∞ formula:

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{5}{1-0.6}$

$S_\infty = 12.5$

Use the nth term of a geometric sequence ar^{n-1} to write down 2 simultaneous equations.

Divide equation (2) by equation (1) to eliminate a.

Problem-solving

To show that a series is convergent you need to find r, then state that the series is convergent if $|r| < 1$.

Example 16

SKILLS EXECUTIVE FUNCTION

For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a Find the possible values of r.

b Given that all the terms in the series are positive, find the value of a.

a $\frac{a(1-r^4)}{1-r} = 15$ (1)

$\frac{a}{1-r} = 16$ (2)

$16(1-r^4) = 15$

$1 - r^4 = \frac{15}{16}$

$r^4 = \frac{1}{16}$

$r = \pm\frac{1}{2}$

$S_4 = 15$ so use the formula $S_n = \frac{a(1-r^n)}{1-r}$ with $n = 4$.

$S_\infty = 16$ so use the formula $S_\infty = \frac{a}{1-r}$ with $S_\infty = 16$.

Solve equations simultaneously.

Replace $\frac{a}{1-r}$ by 16 in equation (1).

Take the 4th root of $\frac{1}{16}$

b As all terms are positive, $r = +\frac{1}{2}$

$\frac{a}{1 - \frac{1}{2}} = 16$

$16(1 - \frac{1}{2}) = a$

$a = 8$

The first term in the series is 8.

Substitute $r = \frac{1}{2}$ into equation (2) to fnd a.

Exercise 5E

SKILLS ANALYSIS

1 For each of the following series:

i state, with a reason, whether the series is convergent.

ii If the series is convergent, find the sum to infinity.

a $1 + 0.1 + 0.01 + 0.001 + \ldots$ **b** $1 + 2 + 4 + 8 + 16 + \ldots$ **c** $10 - 5 + 2.5 - 1.25 + \ldots$

d $2 + 6 + 10 + 14 + \ldots$ **e** $1 + 1 + 1 + 1 + 1 + \ldots$ **f** $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

g $0.4 + 0.8 + 1.2 + 1.6 + \ldots$ **h** $9 + 8.1 + 7.29 + 6.561 + \ldots$

2 A geometric series has first term 10 and sum to infinity 30. Find the common ratio.

3 A geometric series has first term −5 and sum to infinity −3. Find the common ratio.

4 A geometric series has sum to infinity 60 and common ratio $\frac{2}{3}$. Find the first term.

5 A geometric series has common ratio $-\frac{1}{3}$ and $S_\infty = 10$. Find the first term.

P **6** Find the fraction equal to the recurring decimal $0.\dot{2}\dot{3}$.

Hint $0.23 = \frac{23}{100} + \frac{23}{10\,000} + \frac{23}{1\,000\,000} + \ldots$

7 For a geometric series $a + ar + ar^2 + \ldots$, $S_3 = 9$ and $S_\infty = 8$, find the values of a and r.

E/P **8** Given that the geometric series $1 - 2x + 4x^2 - 8x^3 + \ldots$ is convergent,

a find the range of possible values of x **(3 marks)**

b find an expression for S_∞ in terms of x. **(1 mark)**

E/P **9** In a convergent geometric series the common ratio is r and the first term is 2.
Given that $S_\infty = 16 \times S_3$,

a find the value of the common ratio, giving your answer to 4 significant figures **(3 marks)**

b find the value of the fourth term. **(2 marks)**

E/P **10** The first term of a geometric series is 30. The sum to infinity of the series is 240.

a Show that the common ratio, r, is $\frac{7}{8}$ **(2 marks)**

b Find to 3 significant figures, the difference between the 4th and 5th terms. **(2 marks)**

c Calculate the sum of the first 4 terms, giving your answer to 3 significant figures. **(2 marks)**

The sum of the first n terms of the series is greater than 180.

d Calculate the smallest possible value of n. **(4 marks)**

E/P **11** A geometric series has first term a and common ratio r. The second term of the series is $\frac{15}{8}$ and the sum to infinity of the series is 8.

a Show that $64r^2 - 64r + 15 = 0$. **(4 marks)**

b Find the two possible values of r. **(2 marks)**

c Find the corresponding two possible values of a. **(2 marks)**

Given that r takes the smaller of its two possible values,

d find the smallest value of n for which S_n exceeds 7.99. **(2 marks)**

Challenge

The sum to infinity of a geometric series is 7. A second series is formed by squaring every term in the first geometric series.

a Show that the second series is also geometric.

b Given that the sum to infinity of the second series is 35, show that the common ratio of the original series is $\frac{1}{6}$

5.6 Sigma notation

- **The Greek capital letter 'sigma' is used to signify a sum. You write it as $\sum$. You write limits on the top and bottom to show which terms you are summing.**

This tells you that are summing the expression in brackets with $r = 1$, $r = 2$, … up to $r = 5$.

$$\sum_{r=1}^{5}(2r - 3) = -1 + 1 + 3 + 5 + 7$$

Substitute $r = 1$, $r = 2$, $r = 3$, $r = 4$, $r = 5$ to find the five terms in this arithmetic series.

Look at the limits carefully: they don't have to start at 1.

$$\sum_{r=3}^{7}(5 \times 2^r) = 40 + 80 + 160 + 320 + 640$$

To find the terms in this geometric series, you substitute $r = 3$, $r = 4$, $r = 5$, $r = 6$, $r = 7$.

You can write some results that you already know using sigma notation:

- $\sum_{r=1}^{n} 1 = n$
- $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Hint $\sum_{r=1}^{n} 1 = \underbrace{1 + 1 + 1 + \ldots + 1}_{n \text{ times}}$

Example 17

Calculate $\sum_{r=1}^{20}(4r + 1)$

$\sum_{r=1}^{20}(4r + 1) = 5 + 9 + 13 + \ldots + 81$

$a = 5$, $d = 4$ and $n = 20$

Problem-solving

Substitute $r = 1, 2$, etc. to find the terms in the series.

$S = \frac{n}{2}(2a + (n - 1)d)$

$= \frac{20}{2}(2 \times 5 + (20 - 1)4)$

$= 10(10 + 19 \times 4)$

$= 10 \times 86$

$= 860$

Use the formula for the sum to n terms of an arithmetic series.
Substitute $a = 5$, $d = 4$ and $n = 20$ into $S = \frac{n}{2}(2a + (n - 1)d)$.

Online Check your answer by using your calculator to calculate the sum of the series.

Example 18 SKILLS INTERPRETATION

Find the values of:

a $\sum_{k=1}^{12} 5 \times 3^{k-1}$ **b** $\sum_{k=5}^{12} 5 \times 3^{k-1}$

a $\sum_{k=1}^{12} 5 \times 3^{k-1}$

$= 5 + 15 + 45 + \ldots$

Substitute $k = 1$, $k = 2$ and so on to write out the first few terms of the series. This will help you determine the correct values for a, r and n.

$a = 5$, $r = 3$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{12} = \frac{5(3^{12} - 1)}{3 - 1}$

$S_{12} = 1328600$

Since $r > 1$ use the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ and substitute in $a = 5$, $r = 3$ and $n = 12$.

b $\sum_{k=5}^{12} 5 \times 3^{k-1} = \sum_{k=1}^{12} 5 \times 3^{k-1} - \sum_{k=1}^{4} 5 \times 3^{k-1}$

$S_{12} = 1328600$

$S_4 = \frac{5(3^4 - 1)}{3 - 1} = 200$

$\sum_{k=5}^{12} 5 \times 3^{k-1} = 1328600 - 200 = 1328400$

Problem-solving

When we are summing series from k to n, we can consider the sum of the terms from 1 to n and subtract the terms from 1 to $k - 1$.

Exercise 5F SKILLS INTERPRETATION

1 For each series:
i write out every term in the series
ii hence find the value of the sum.

a $\sum_{r=1}^{5}(3r + 1)$ **b** $\sum_{r=1}^{6} 3r^2$ **c** $\sum_{r=1}^{5}\sin(90r^\circ)$ **d** $\sum_{r=5}^{8} 2\left(-\frac{1}{3}\right)^r$

2 For each series:
i write the series using sigma notation
ii evaluate the sum.

a $2 + 4 + 6 + 8$ **b** $2 + 6 + 18 + 54 + 162$ **c** $6 + 4.5 + 3 + 1.5 + 0 - 1.5$

3 For each series:
i find the number of terms in the series
ii write the series using sigma notation.

a $7 + 13 + 19 + \ldots + 157$ **b** $\frac{1}{3} + \frac{2}{15} + \frac{4}{75} + \ldots + \frac{64}{46875}$ **c** $8 - 1 - 10 - 19 \ldots - 127$

4 Evaluate:

a $\sum_{r=1}^{20}(7 - 2r)$ **b** $\sum_{r=1}^{10} 3 \times 4^r$ **c** $\sum_{r=1}^{100}(2r - 8)$ **d** $\sum_{r=1}^{\infty} 7\left(-\frac{1}{3}\right)^r$

(P) **5** Evaluate:

a $\sum_{r=9}^{30}\left(5r - \frac{1}{2}\right)$ **b** $\sum_{r=100}^{200}(3r + 4)$ **c** $\sum_{r=5}^{100} 3 \times 0.5^r$ **d** $\sum_{i=5}^{100} 1$

Problem-solving

$\sum_{r=k}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{k-1} u_r$

(P) **6** Show that $\sum_{r=1}^{n} 2r = n + n^2$.

(P) **7** Show that $\sum_{r=1}^{n} 2r - \sum_{r=1}^{n}(2r - 1) = n$.

8 Find in terms of k:

a $\sum_{r=1}^{k} 4(-2)^r$ **b** $\sum_{r=1}^{k}(100 - 2r)$ **c** $\sum_{r=10}^{k}(7 - 2r)$

(P) **9** Find the value of $\sum_{r=10}^{\infty} 200 \times \left(\frac{1}{4}\right)^r$

(E/P) **10** Given that $\sum_{r=1}^{k}(8 + 3r) = 377$,

a show that $(3k + 58)(k - 13) = 0$ **(3 marks)**

b hence find the value of k. **(1 mark)**

(E/P) **11** Given that $\sum_{r=1}^{k} 2 \times 3^r = 59\,046$,

a show that $k = \frac{\log 19\,683}{\log 3}$ **(4 marks)**

b For this value of k, calculate $\sum_{r=k+1}^{13} 2 \times 3^r$. **(3 marks)**

(E/P) **12** A geometric series is given by $1 + 3x + 9x^2 + \ldots$
The series is convergent.

a Write down the range of possible values of x. **(3 marks)**

Given that $\sum_{r=1}^{\infty}(3x)^{r-1} = 2$

b calculate the value of x. **(3 marks)**

Challenge

Given that $\sum_{r=1}^{10}(a + (r - 1)d) = \sum_{r=11}^{14}(a + (r - 1)d)$, show that $d = 6a$.

5.7 Recurrence relations

If you know the rule to get from one term to the next in a sequence you can write a recurrence relation.

- **A recurrence relation of the form $u_{n+1} = f(u_n)$ defines each term of a sequence as a function of the previous term.**

For example, the recurrence relation $u_{n+1} = 2u_n + 3$, $u_1 = 6$ produces the following sequence:

6, 15, 33, 69, … $u_2 = 2u_1 + 3 = 2(6) + 3 = 15$

Watch out In order to generate a sequence from a recurrence relation like this, you need to know the **first term** of the sequence.

Example 19

Find the first four terms of the following sequences.

a $u_{n+1} = u_n + 4$, $u_1 = 7$ **b** $u_{n+1} = u_n + 4$, $u_1 = 5$

a $u_{n+1} = u_n + 4$, $u_1 = 7$
Substituting $n = 1$, $u_2 = u_1 + 4 = 7 + 4 = 11$.
Substituting $n = 2$, $u_3 = u_2 + 4 = 11 + 4 = 15$.
Substituting $n = 3$, $u_4 = u_3 + 4 = 15 + 4 = 19$.
Sequence is 7, 11, 15, 19, …

Substitute $n = 1$, 2 and 3. Use u_1 to find u_2, and then u_2 to find u_3.

b $u_{n+1} = u_n + 4$, $u_1 = 5$
Substituting $n = 1$, $u_2 = u_1 + 4 = 5 + 4 = 9$.
Substituting $n = 2$, $u_3 = u_2 + 4 = 9 + 4 = 13$.
Substituting $n = 3$, $u_4 = u_3 + 4 = 13 + 4 = 17$.
Sequence is 5, 9, 13, 17, …

This is the same recurrence formula. It produces a different sequence because u_1 is different.

Example 20

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = p$$
$$a_{n+1} = (a_n)^2 - 1, n \geqslant 1$$

where $p < 0$.

a Show that $a_3 = p^4 - 2p^2$.

b Given that $a_2 = 0$, find the value of p.

c Find $\sum_{r=1}^{200} a_r$

d Write down the value of a_{199}

a $a_1 = p$
$a_2 = (a_1)^2 - 1 = p^2 - 1$
$a_3 = (a_2)^2 - 1$
$= (p^2 - 1)^2 - 1$
$= p^4 - 2p^2 + 1 - 1$
$= p^4 - 2p^2$

Use $a_2 = (a_1)^2 - 1$ and substitute $a_1 = p$.

Now substitute the expression for a_2 to find a_3.

b $p^2 - 1 = 0$
$p^2 = 1$
$p = \pm 1$ but since $p < 0$ is given, $p = -1$

Set the expression for a_2 equal to zero and solve.

c $a_1 = -1$, $a_2 = 0$, $a_3 = -1$ series alternates between -1 and 0

In 200 terms, there will be one hundred -1s and one hundred 0s.

$$\sum_{r=1}^{200} a_r = -100$$

Since this is a recurrence relation, we can see that the sequence is going to alternate between –1 and 0. The first 200 terms will have one hundred –1s and one hundred 0s.

d $a_{199} = -1$ as 199 is odd

Problem-solving

For an alternating series, consider the sums of the odd and even terms separately. Write the first few terms of the series. The odd terms are –1 and the even terms are 0. Only the odd terms contribute to the sum.

Exercise 5G

1 Find the first four terms of the following recurrence relationships.

a $u_{n+1} = u_n + 3$, $u_1 = 1$ **b** $u_{n+1} = u_n - 5$, $u_1 = 9$

c $u_{n+1} = 2u_n$, $u_1 = 3$ **d** $u_{n+1} = 2u_n + 1$, $u_1 = 2$

e $u_{n+1} = \frac{u_n}{2}$, $u_1 = 10$ **f** $u_{n+1} = (u_n)^2 - 1$, $u_1 = 2$

2 Suggest possible recurrence relationships for the following sequences. (Remember to state the first term.)

a 3, 5, 7, 9, … **b** 20, 17, 14, 11, … **c** 1, 2, 4, 8, …

d 100, 25, 6.25, 1.5625, … **e** 1, –1, 1, –1, 1, … **f** 3, 7, 15, 31, …

g 0, 1, 2, 5, 26, … **h** 26, 14, 8, 5, 3.5, …

3 By writing down the first four terms or otherwise, find the recurrence formula that defines the following sequences:

a $u_n = 2n - 1$ **b** $u_n = 3n + 2$ **c** $u_n = n + 2$

d $u_n = \frac{n+1}{2}$ **e** $u_n = n^2$ **f** $u_n = 3^n - 1$

(P) **4** A sequence of terms is defined for $n \geqslant 1$ by the recurrence relation $u_{n+1} = ku_n + 2$, where k is a constant. Given that $u_1 = 3$,

a find an expression in terms of k for u_2

b hence find an expression for u_3

Given that $u_3 = 42$:

c find the possible values of k.

(E/P) **5** A sequence is defined for $n \geqslant 1$ by the recurrence relation

$$u_{n+1} = pu_n + q,\ u_1 = 2$$

Given that $u_2 = -1$ and $u_3 = 11$, find the values of p and q. **(4 marks)**

E/P **6** A sequence is given by

$$x_1 = 2$$

$$x_{n+1} = x_n(p - 3x_n)$$

where p is an integer.

a Show that $x_3 = -10p^2 + 132p - 432$. **(2 marks)**

b Given that $x_3 = -288$ find the value of p. **(1 mark)**

c Hence find the value of x_4. **(1 mark)**

E/P **7** A sequence $a_1, a_2, a_3, \dots$ is defined by

$$a_1 = k$$

$$a_{n+1} = 4a_n + 5$$

a Find a_3 in terms of k. **(2 marks)**

b Show that $\sum_{r=1}^{4} a_r$ is a multiple of 5. **(3 marks)**

- **A sequence is increasing if $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.**
- **A sequence is decreasing if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.**
- **A sequence is periodic if the terms repeat in a cycle. For a periodic sequence there is an integer k such that $u_{n+k} = u_n$ for all $n \in \mathbb{N}$. The value k is called the order of the sequence.**

Notation The order of a periodic sequence is sometimes called its **period**.

- 2, 3, 4, 5… is an increasing sequence.
- −3, −6, −12, −24… is a decreasing sequence.
- −2, 1, −2, 1, −2, 1 is a periodic sequence with a period of 2.
- 1, −2, 3, −4, 5, −6… is not increasing, decreasing or periodic.

Example 21 SKILLS ANALYSIS

For each sequence:

i state whether the sequence is increasing, decreasing, or periodic

ii if the sequence is periodic, write down its order.

a $u_{n+1} = u_n + 3, u_1 = 7$ **b** $u_{n+1} = (u_n)^2, u_1 = \frac{1}{2}$ **c** $u_n = \sin(90n°)$

a 7, 10, 13, 16, …

$u_{n+1} > u_n$ for all n, so the sequence is increasing.

Write out the first few terms of the sequence.

State the condition for an increasing sequence. You could also write that $k + 3 > k$ for all numbers k.

b $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$

$u_{n+1} < u_n$ for all n, so the sequence is decreasing.

The starting value in the sequence makes a big difference. Because $u_1 < 1$ the numbers get smaller every time you square them.

c $u_1 = \sin(90°) = 1$

To find u_1 substitute $n = 1$ into $\sin(90n°)$.

$u_2 = \sin(180°) = 0$
$u_3 = \sin(270°) = -1$
$u_4 = \sin(360°) = 0$
$u_5 = \sin(450°) = 1$
$u_6 = \sin(540°) = 0$
$u_7 = \sin(630°) = -1$

Watch out Although every even term of the sequence is 0, the period is not 2 because the odd terms alternate between 1 and –1.

The sequence is periodic, with order 4.

The graph of $y = \sin x$ repeats with period 360°. So $\sin(x + 360°) = \sin x$. ← **Pure 1 Section 6.5**

Exercise 5H SKILLS ANALYSIS

1 For each sequence:

i state whether the sequence is increasing, decreasing, or periodic

ii if the sequence is periodic, write down its order.

a 2, 5, 8, 11, 14 **b** $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ **c** 5, 9, 15, 23, 33 **d** 3, –3, 3, –3, 3

2 For each sequence:

i write down the first 5 terms of the sequence

ii state whether the sequence is increasing, decreasing, or periodic

iii if the sequence is periodic, write down its order.

a $u_n = 20 - 3n$ **b** $u_n = 2^{n-1}$ **c** $u_n = \cos(180n°)$

d $u_n = (-1)^n$ **e** $u_{n+1} = u_n - 5,\ u_1 = 20$ **f** $u_{n+1} = 5 - u_n,\ u_1 = 20$

g $u_{n+1} = \frac{2}{3}u_n,\ u_1 = k$

3 The sequence of numbers $u_1, u_2, u_3, \ldots$ is given by $u_{n+1} = ku_n,\ u_1 = 5$.
Find the range of values of k for which the sequence is strictly decreasing.

E/P **4** The sequence with recurrence relation $u_{k+1} = pu_k + q,\ u_1 = 5$, where p is a constant and $q = 13$, is periodic with order 2.
Find the value of p. **(5 marks)**

E/P **5** A sequence has nth term $a_n = \cos(90n°),\ n \geqslant 1$.

a Find the order of the sequence. **(1 mark)**

b Find $\sum_{r=1}^{444} a_r$ **(2 marks)**

Challenge

SKILLS CREATIVITY

The sequence of numbers $u_1, u_2, u_3, \ldots$ is given by $u_{n+2} = \frac{1 + u_{n+1}}{u_n}$, $u_1 = a$, $u_2 = b$, where a and b are positive integers.

a Show that the sequence is periodic for all positive a and b.

b State the order of the sequence.

Hint Each term in this sequence is defined in terms of the **previous two** terms.

5.8 Modelling with series

You can model real-life situations with series. For example if a person's salary increases by the same percentage every year, their salaries each year would form a **geometric sequence** and the amount they had been paid in total over n years would be modelled by the corresponding **geometric series**.

Example 22

Ahmed starts a new company. In year 1 his profits will be \$20 000. He predicts his profits to increase by \$5000 each year, so that his profits in year 2 are modelled to be \$25 000, in year 3, \$30 000 and so on. He predicts this will continue until he reaches annual profits of \$100 000. He then models his annual profits to remain at \$100 000.

a Calculate the profits for Ahmed's business in the first 20 years.

b State one reason why this may not be a suitable model.

c Ahmed's financial advisor says the yearly profits are likely to increase by 5% per annum. Using this model, calculate the profits for Ahmed's business in the first 20 years.

a Year 1 $P = 20\,000$, Year 2 $P = 25\,000$, Year 3 $P = 30\,000$

This is an arithmetic sequence as the difference is constant.

$a = 20\,000$, $d = 5000$

Write down the values of a and d.

$u_n = a + (n - 1)d$

$100\,000 = 20\,000 + (n - 1)(5000)$

Use the nth term of an arithmetic sequence to work out n when profits will reach £100 000.

$100\,000 = 20\,000 + 5000n - 5000$

$85\,000 = 5000n$

$n = \frac{85\,000}{5000} = 17$

Solve to find n.

$S_{17} = \frac{17}{2}(2(20\,000) + (17 - 1)(5000))$

$= 1\,020\,000$

You want to know how much he made overall in the 17 years, so find the sum of the arithmetic series.

$S_{20} = 1\,020\,000 + 3(100\,000)$

$= 1\,320\,000$

In the 18th, 19th and 20th year he makes \$100 000 each year, so add on 3 × \$100 000 to the sum of the first 17 years.

So Ahmed's total profit after 20 years is \$1 320 000.

b It is unlikely that Ahmed's profits will increase by exactly the same amount each year.

c $a = \$20\,000$, $r = 1.05$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{20} = \frac{20\,000(1.05^{20} - 1)}{1.05 - 1}$$

$$S_{20} = 661\,319.08$$

So Ahmed's total profit after 20 years is \$661 319.08.

This is a geometric series, as to get the next term you multiply the current term by 1.05.

Use the formula for the sum of the first n terms of a geometric series $S_n = \frac{a(r^n - 1)}{r - 1}$

Example 23

SKILLS PROBLEM-SOLVING

A piece of A4 paper is folded in half repeatedly. The thickness of the A4 paper is 0.5 mm.

a Work out the thickness of the paper after four folds.

b Work out the thickness of the paper after 20 folds.

c State one reason why this might be an unrealistic model.

a $a = 0.5$ mm, $r = 2$

After 4 folds:

$u_5 = 0.5 \times 2^4 = 8$ mm

b After 20 folds

$u_{21} = 0.5 \times 2^{20} = 524\,288$ mm

c It is impossible to fold the paper that many times so the model is unrealistic.

This is a geometric sequence, as each time we fold the paper the thickness doubles.

Since u_1 is the first term (after 0 folds), u_2 is after 1 fold, so u_5 is after 4 folds.

Problem-solving

If you have to comment on the validity of a model, always refer to the context given in the question.

Exercise 5I

SKILLS PROBLEM-SOLVING

1 An investor puts \$4000 in an account. Every month thereafter she deposits another \$200. How much money in total will she have invested at the start of **a** the 10th month and **b** the mth month?

Hint At the start of the 6th month she will have only made 5 deposits of \$200.

(P) **2** Nour starts a new job on a salary of €20 000. She is given an annual wage rise of €500 at the end of every year until she reaches her maximum salary of €5 000. Find the total amount she earns (assuming no other rises), **a** in the first 10 years, **b** over 15 years and **c** state one reason why this may be an unsuitable model.

Problem-solving

This is an arithmetic series with $a = 20\,000$ and $d = 500$. First find how many years it will take her to reach her maximum salary.

P **3** James decides to save some money during the six-week holiday. He saves 1 cent on the first day, 2c on the second, 3c on the third and so on.

a How much will he have at the end of the holiday (42 days)?

b If he carried on, how long would it be before he has saved \$100?

P **4** A population of ants is growing at a rate of 10% a year. If there were 200 ants in the initial population, write down the number of ants after:

a 1 year **b** 2 years **c** 3 years **d** 10 years.

Problem-solving

This is a geometric sequence. $a = 200$ and $r = 1.1$

P **5** A motorcycle has four gears. The maximum speed in bottom gear is 40 km h^{-1} and the maximum speed in top gear is 120 km h^{-1}. Given that the maximum speeds in each successive gear form a geometric progression, calculate, in km h^{-1} to one decimal place, the maximum speeds in the two intermediate gears.

P **6** A car becomes less valuable by 15% a year. After 3 years it is worth €11 054.25.

a What was the car's initial price?

b When will the car's value first be less than €5000?

Problem-solving

Use your answer to part **a** to write an inequality, then solve it using **logarithms**.

E **7** A salesman is paid commission of \$10 per week for each life insurance policy that he has sold. Each week he sells one new policy so that he is paid \$10 commission in the first week, \$20 commission in the second week, \$30 commission in the third week and so on.

a Find his total commission in the first year of 52 weeks. **(2 marks)**

b In the second year the commission increases to \$11 per week on new policies sold, although it remains at \$10 per week for policies sold in the first year. He continues to sell one policy per week. Show that he is paid \$542 in the second week of his second year. **(3 marks)**

c Find the total commission paid to him in the second year. **(2 marks)**

E **8** Prospectors are drilling for oil. The cost of drilling to a depth of 50 m is \$500. To drill a further 50 m costs \$640 and, hence, the total cost of drilling to a depth of 100 m is £1140. Each subsequent extra depth of 50 m costs \$140 more to drill than the previous 50 m.

a Show that the cost of drilling to a depth of 500 m is \$11 300. **(3 marks)**

b The total sum of money available for drilling is \$76 000. Find, to the nearest 50 m, the greatest depth that can be drilled. **(3 marks)**

E **9** Each year, for 40 years, Sara will pay money into a savings scheme. In the first year she pays in €500. Her payments then increase by €50 each year, so that she pays in €550 in the second year, €600 in the third year, and so on.

a Find the amount that Sara will pay in the 40th year. **(2 marks)**

b Find the total amount that Sara will pay in over the 40 years. **(3 marks)**

c Over the same 40 years, Max will also pay money into the savings scheme. In the first year he pays in €890 and his payments then increase by €d each year. Given that Max and Sara will pay in exactly the same amount over the 40 years, find the value of d. **(4 marks)**

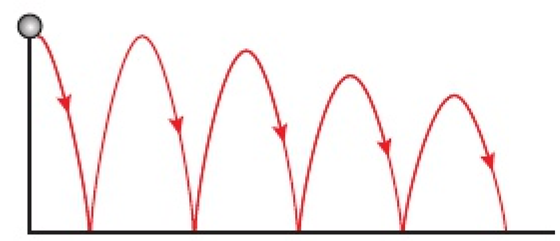
(P) **10** A virus is spreading such that the number of people infected increases by 4% a day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected?

(P) **11** I invest £A in the bank at a rate of interest of 3.5% per annum. How long will it be before I double my money?

(P) **12** The fish in a particular area of the North Sea are being reduced by 6% each year due to overfishing. How long will it be before the fish stocks are halved?

(P) **13** The man who invented the game of chess was asked to name his reward. He asked for 1 grain of corn to be placed on the first square of his chessboard, 2 on the second, 4 on the third and so on until all 64 squares were covered. He then said he would like as many grains of corn as the chessboard carried. How many grains of corn did he claim as his prize?

(P) **14** A ball is dropped from a height of 10 m. It bounces to a height of 7 m and continues to bounce. Subsequent heights to which it bounces follow a geometric sequence. Find out:



a how high it will bounce after the fourth bounce

b the total vertical distance travelled up to the point when the ball hits the ground for the sixth time.

(P) **15** Rafi is doing a sponsored cycle. He plans to cycle 1000 miles over a number of days. He plans to cycle 10 miles on day 1 and increase the distance by 10% a day.

a How long will it take Rafi to complete the challenge?

b What will be his greatest number of miles completed in a day?

(P) **16** A savings scheme is offering a rate of interest of 3.5% per annum for the lifetime of the plan. Liu Wei wants to save up €20 000. He works out that he can afford to save €500 every year, which he will deposit on 1 January. If interest is paid on 31 December, how many years will it be before he has saved up his €20 000?

Chapter review 5

(E/P) **1** A geometric series has third term 27 and sixth term 8.

a Show that the common ratio of the series is $\frac{2}{3}$ **(2 marks)**

b Find the first term of the series. **(2 marks)**

c Find the sum to infinity of the series. **(2 marks)**

d Find the difference between the sum of the first 10 terms of the series and the sum to infinity. Give your answer to 3 significant figures. **(2 marks)**

(E/P) **2** The second term of a geometric series is 80 and the fifth term of the series is 5.12.

a Show that the common ratio of the series is 0.4. **(2 marks)**

Calculate:

b the first term of the series **(2 marks)**

c the sum to infinity of the series, giving your answer as an exact fraction **(1 mark)**

d the difference between the sum to infinity of the series and the sum of the first 14 terms of the series, giving your answer in the form $a \times 10^n$, where $1 \leqslant a < 10$ and n is an integer. **(2 marks)**

E/P **3** The nth term of a sequence is u_n, where $u_n = 95\left(\frac{4}{5}\right)^n$, $n = 1, 2, 3, \ldots$

a Find the values of u_1 and u_2. **(2 marks)**

Giving your answers to 3 significant figures, calculate:

b the value of u_{21} **(1 mark)**

c $\sum_{n=1}^{15} u_n$ **(2 marks)**

d the sum to infinity of the series whose first term is u_1 and whose nth term is u_n. **(1 mark)**

E/P **4** A sequence of numbers $u_1, u_2, \ldots, u_n, \ldots$ is given by the formula $u_n = 3\left(\frac{2}{3}\right)^n - 1$ where n is a positive integer.

a Find the values of u_1, u_2 and u_3. **(2 marks)**

b Show that $\sum_{n=1}^{15} u_n = -9.014$ to 4 significant figures. **(2 marks)**

c Prove that $u_{n+1} = \dfrac{2u_n - 1}{3}$ **(2 marks)**

E/P **5** The third and fourth terms of a geometric series are 6.4 and 5.12 respectively. Find:

a the common ratio of the series **(2 marks)**

b the first term of the series **(2 marks)**

c the sum to infinity of the series. **(2 marks)**

d Calculate the difference between the sum to infinity of the series and the sum of the first 25 terms of the series. **(2 marks)**

E/P **6** The price of a car depreciates by 15% per annum. Its price when new is \$20 000.

a Find the value of the car after 5 years. **(2 marks)**

b Find when the value will be less than \$4000. **(3 marks)**

E/P **7** The first three terms of a geometric series are $p(3q + 1)$, $p(2q + 2)$ and $p(2q - 1)$, where p and q are non-zero constants.

a Show that one possible value of q is 5 and find the other possible value. **(2 marks)**

b Given that $q = 5$, and the sum to infinity of the series is 896, find the sum of the first 12 terms of the series. Give your answer to 2 decimal places. **(4 marks)**

E/P **8** **a** Prove that the sum of the first n terms in an arithmetic series is

$$S = \frac{n}{2}(2a + (n - 1)d)$$

where a = first term and d = common difference. **(3 marks)**

b Use this to find the sum of the first 100 natural numbers. **(2 marks)**

E/P **9** Find the least value of n for which $\sum_{r=1}^{n}(4r - 3) > 2000$. **(2 marks)**

E/P **10** The sum of the first two terms of an arithmetic series is 47.
The thirtieth term of this series is −62. Find:

a the first term of the series and the common difference **(3 marks)**

b the sum of the first 60 terms of the series. **(2 marks)**

E/P **11** **a** Find the sum of the integers which are divisible by 3 and lie between 1 and 400. **(3 marks)**

b Hence, or otherwise, find the sum of the integers, from 1 to 400 inclusive, which are **not** divisible by 3. **(2 marks)**

E/P **12** A polygon has 10 sides. The lengths of the sides, starting with the shortest, form an arithmetic series. The perimeter of the polygon is 675 cm and the length of the longest side is twice that of the shortest side. Find the length of the shortest side of the polygon. **(4 marks)**

E/P **13** Prove that the sum of the first $2n$ multiples of 4 is $4n(2n + 1)$. **(4 marks)**

E/P **14** A sequence of numbers is defined, for $n \geqslant 1$, by the recurrence relation $u_{n+1} = ku_n - 4$, where k is a constant. Given that $u_1 = 2$:

a find expressions, in terms of k, for u_2 and u_3. **(2 marks)**

b Given also that $u_3 = 26$, use algebra to find the possible values of k. **(2 marks)**

E/P **15** The fifth term of an arithmetic series is 14 and the sum of the first three terms of the series is −3.

a Use algebra to show that the first term of the series is −6 and calculate the common difference of the series. **(3 marks)**

b Given that the nth term of the series is greater than 282, find the least possible value of n. **(3 marks)**

E/P **16** The fourth term of an arithmetic series is $3k$, where k is a constant, and the sum of the first six terms of the series is $7k + 9$.

a Show that the first term of the series is $9 - 8k$. **(3 marks)**

b Find an expression for the common difference of the series in terms of k. **(2 marks)**

Given that the seventh term of the series is 12, calculate:

c the value of k **(2 marks)**

d the sum of the first 20 terms of the series. **(2 marks)**

E/P **17** A sequence is defined by the recurrence relation

$$a_{n+1} = \frac{1}{a_n}, \; a_1 = p$$

a Show that the sequence is periodic and state its order. **(2 marks)**

b Find $\sum_{r=1}^{1000} a_n$ in terms of p. **(2 marks)**

E/P **18** A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$
$$a_{n+1} = 2a_n + 6, \; n \geqslant 1$$

where k is an integer.

a Given that the sequence is increasing for the first 3 terms, show that $k > p$, where p is an integer to be found. **(2 marks)**

b Find a_4 in terms of k. **(2 marks)**

c Show that $\sum_{r=1}^{4} a_r$ is divisible by 3. **(3 marks)**

E/P **19** The first term of a geometric series is 130. The sum to infinity of the series is 650.

a Show that the common ratio, r, is $\frac{4}{5}$ **(3 marks)**

b Find, to 2 decimal places, the difference between the 7th and 8th terms. **(2 marks)**

c Calculate the sum of the first 7 terms. **(2 marks)**

The sum of the first n terms of the series is greater than 600.

d Show that $n > \frac{-\log 13}{\log 0.8}$ **(4 marks)**

E/P **20** The adult population of a town is 25 000 at the beginning of 2018.
A model predicts that the adult population of the town will increase by 2% each year, forming a geometric sequence.

a Show that the predicted population at the beginning of 2020 is 26 010. **(1 mark)**

The model predicts that after n years, the population will first exceed 50 000.

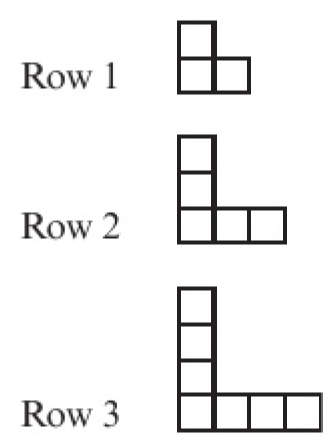
b Show that $n > \frac{\log 2}{\log 1.02}$ **(3 marks)**

c Find the year in which the population first exceeds 50 000. **(2 marks)**

d Every member of the adult population is modelled to visit the doctor once per year. Calculate the number of appointments the doctor has from the beginning of 2018 to the end of 2025. **(4 marks)**

e Give a reason why this model for doctors' appointments may not be appropriate. **(1 mark)**

E/P **21** Leo is making some patterns out of squares. He has made 3 rows so far.



a Find an expression, in terms of n, for the number of squares required to make a similar arrangement in the nth row. **(3 marks)**

b Leo counts the number of squares used to make the pattern in the kth row. He counts 301 squares. Write down the value of k. **(1 mark)**

c In the first q rows, Leo uses a total of p squares.

i Show that $q^2 + 2q - p = 0$. **(3 marks)**

ii Given that $p > 1520$, find the minimum number of rows that Leo makes. **(3 marks)**

E/P **22** A convergent geometric series has first term a and common ratio r. The second term of the series is -3 and the sum to infinity of the series is 6.75.

a Show that $27r^2 - 27r - 12 = 0$. **(4 marks)**

b Given that the series is convergent, find the value of r. **(2 marks)**

c Find the sum of the first 5 terms of the series, giving your answer to 2 decimal places. **(3 marks)**

Challenge

SKILLS INNOVATION

A sequence is defined by the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$.

a Prove that any sequence of the form $u_n = p \times 3^n + q \times 2^n$, where p and q are constants, satisfies this recurrence relation.

Given that $u_1 = 5$ and $u_2 = 12$,

b find an expression for u_n in terms of n only.

c Hence determine the number of digits in u_{100}.

Summary of key points

1 In an **arithmetic sequence**, the difference between consecutive terms is constant.

2 The formula for the nth term of an arithmetic sequence is $\boldsymbol{u_n = a + (n-1)d}$, where a is the first term and d is the common difference.

3 An arithmetic series is the sum of the terms of an arithmetic sequence.

The sum of the first n terms of an arithmetic series is given by $\boldsymbol{S_n = \frac{n}{2}(2a + (n-1)d)}$, where where a is the first term and d is the common difference.

You can also write this formula as $\boldsymbol{S_n = \frac{n}{2}(a + l)}$, where l is the last term.

4 A **geometric sequence** has a **common ratio** between consecutive terms.

5 The formula for the nth term of a geometric sequence is $\boldsymbol{u_n = ar^{n-1}}$, where a is the first term and r is the common ratio.

6 The sum of the first n terms of a geometric series is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1 \qquad \text{or} \qquad S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

where a is the first term and r is the common ratio.

7 A geometric series is convergent if and only if $|r| < 1$, where r is the common ratio.

The **sum to infinity** of a convergent geometric series is given by $\boldsymbol{S_\infty = \frac{a}{1 - r}}$

8 The Greek capital letter 'sigma' is used to signify a sum. You write it as $\sum$. You write limits on the top and bottom to show which terms you are summing.

9 A recurrence relation of the form $u_{n+1} = \mathrm{f}(u_n)$ defines each term of a sequence as a function of the previous term.

10 A sequence is **increasing** if $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.

A sequence is **decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

A sequence is **periodic** if the terms repeat in a cycle. For a periodic sequence there is an integer k such that $u_{n+k} = u_n$ for all $n \in \mathbb{N}$. The value k is called the **order** of the sequence.

6 TRIGONOMETRIC IDENTITIES AND EQUATIONS

6.1
6.2

Learning objectives

After completing this chapter you should be able to:

- Calculate the sine, cosine and tangent of any angle → pages 113–118
- Know the exact trigonometric ratios for 30°, 45° and 60° → pages 119–120
- Know and use the relationships $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$ → pages 120–123
- Solve simple trigonometric equations of the forms $\sin\theta = k$, $\cos\theta = k$ and $\tan\theta = k$ → pages 124–127
- Solve more complicated trigonometric equations of the forms $\sin n\theta = k$ and $\sin(\theta \pm \alpha) = k$ and equivalent equations involving cos and tan → pages 128–130
- Solve trigonometric equations that produce quadratics → pages 130–133

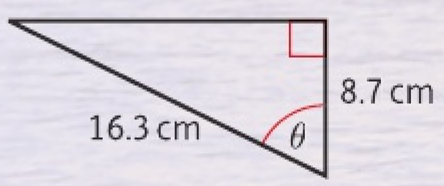
Prior knowledge check

1 **a** Sketch the graph of $y = \sin x$ for $0 \leqslant x \leqslant 540°$.
b How many solutions are there to the equation $\sin x = 0.6$ in the range $0 \leqslant x \leqslant 540°$?
c Given that $\sin^{-1}(0.6) = 36.9°$ (to 3 significant figures), write down three other solutions to the equation $\sin x = 0.6$.

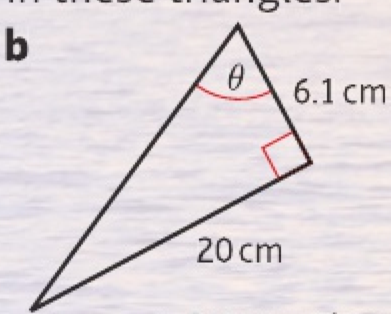
← Pure 1 Section 6.5

2 Work out the marked angles in these triangles.

a



b



← International GCSE Mathematics

3 Solve the following equations.
a $2x - 7 = 15$ **b** $3x + 5 = 7x - 4$
c $\sin x = -0.7$

← International GCSE Mathematics

4 Solve the following equations.
a $x^2 - 4x + 3 = 0$ **b** $x^2 + 8x - 9 = 0$
c $2x^2 - 3x - 7 = 0$

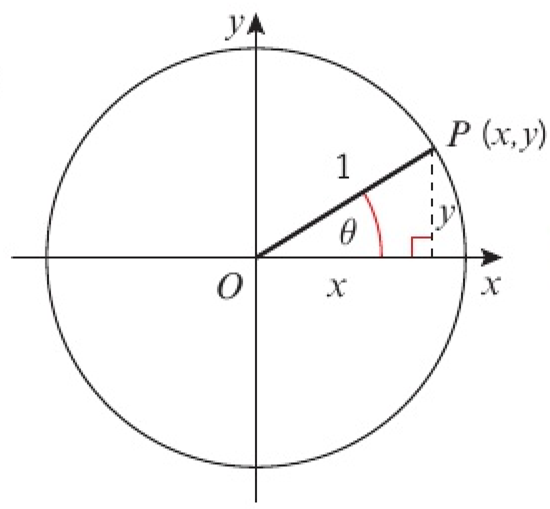
← Pure 1 Section 2.1

Trigonometric equations can be used to model many real-life situations such as the rise and fall of the tides or the angle of elevation of the sun at different times of the day.

6.1 Angles in all four quadrants

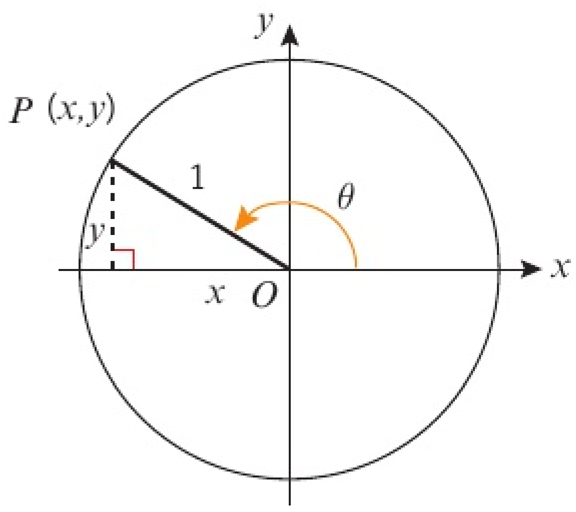
You can use a **unit circle** with its centre at the origin to help you understand the trigonometric ratios.

- For a point $P(x, y)$ on a unit circle such that OP makes an angle θ with the positive x-axis:
  - $\cos\theta = x = x$-coordinate of P
  - $\sin\theta = y = y$-coordinate of P
  - $\tan\theta = \frac{y}{x} =$ gradient of OP



Notation A **unit circle** is a circle with a radius of 1 unit.

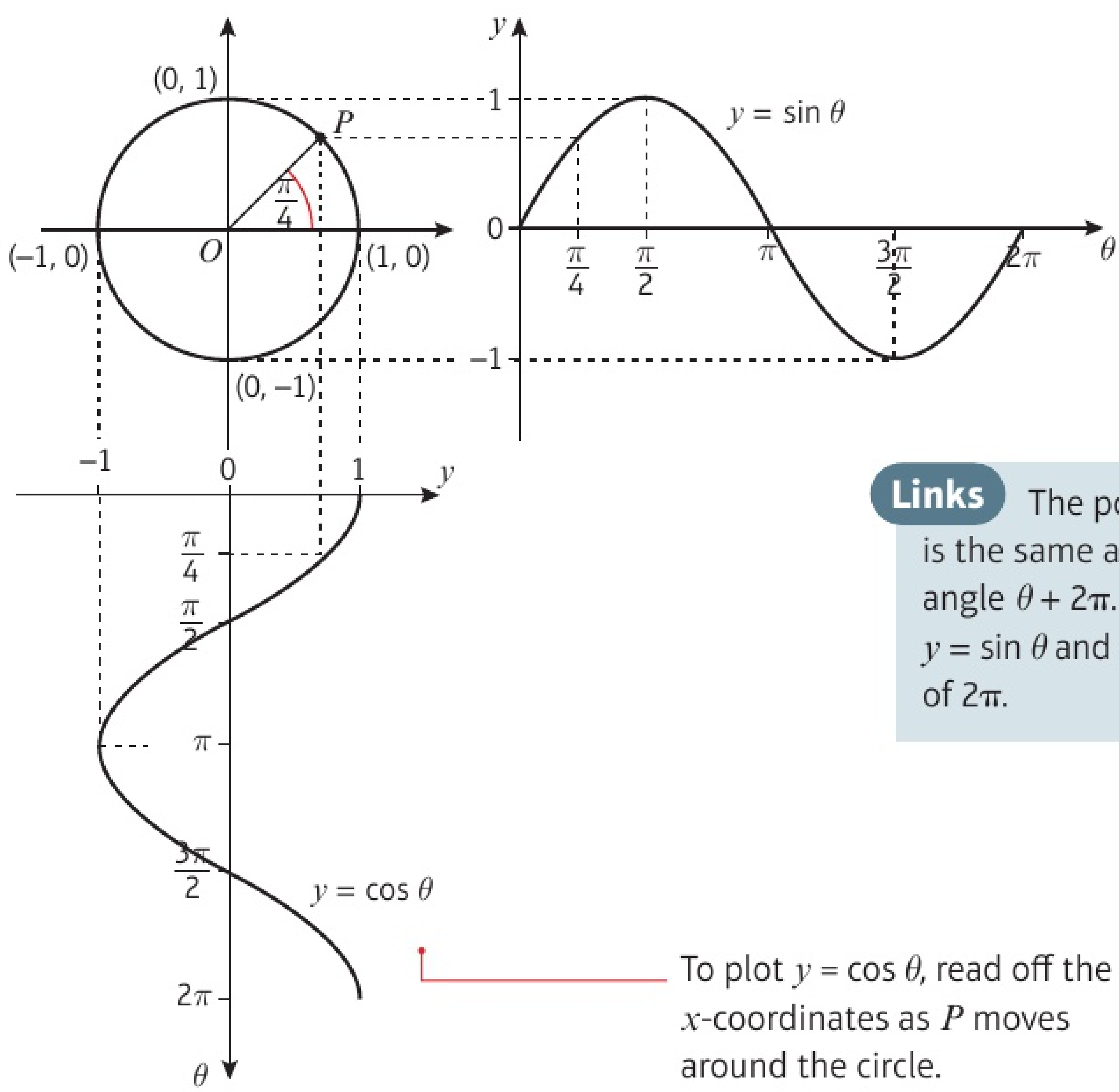
You can use these definitions to find the values of sine, cosine and tangent for any angle θ. You always measure positive angles θ **anticlockwise** from the **positive x-axis**.



When θ is **obtuse**, $\cos\theta$ is negative because the x-coordinate of P is negative.

Online Use GeoGebra to explore the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for any angle θ in a unit circle.

You can also use these definitions to generate the graphs of $y = \sin\theta$ and $y = \cos\theta$.



Links The point P corresponding to an angle is the same as the point P corresponding to an angle $\theta + 2\pi$. This shows you that the graphs of $y = \sin\theta$ and $y = \cos\theta$ are periodic with period of 2π. ← Pure 1 Section 6.5

To plot $y = \cos\theta$, read off the x-coordinates as P moves around the circle.

Example 1

Write down the values of:

a $\sin 90°$ **b** $\sin 180°$ **c** $\sin \frac{3\pi}{2}$ **d** $\sin\left(-\frac{\pi}{2}\right)$ **e** $\cos 180°$ **f** $\cos(-90°)$

g $\cos 3\pi$ **h** $\cos 5\pi$

a $\sin 90° = 1$

b $\sin 180° = 0$

c $\sin \frac{3\pi}{2} = -1$

d $\sin\left(-\frac{\pi}{2}\right) = -1$

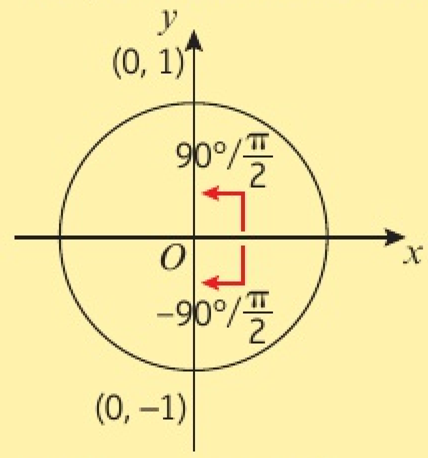
e $\cos 180° = -1$

f $\cos(-90°) = 0$

g $\cos 3\pi = -1$

h $\cos 5\pi = -1$

The y-coordinate is 1 when $\theta = 90°$ or $\frac{\pi}{2}$



$90°/\frac{\pi}{2}$ and $-90°/\frac{\pi}{2}$

If θ is negative, then measure **clockwise** from the positive x-axis.

An angle of −90° is equivalent to a positive angle of 270°. The x-coordinate is 0 when $\theta = -90°$ or 270°.

An angle of $-\frac{\pi}{2}$ is equivalent to a positive angle of $\frac{3\pi}{2}$. The x-coordinate is 0 when $\theta = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Example 2

SKILLS INTERPRETATION

Write down the values of:

a $\tan 45°$ **b** $\tan 135°$ **c** $\tan 225°$

d $\tan\left(-\frac{\pi}{4}\right)$ **e** $\tan \pi$ **f** $\tan \frac{\pi}{2}$

a $\tan 45° = 1$

b $\tan 135° = 0$

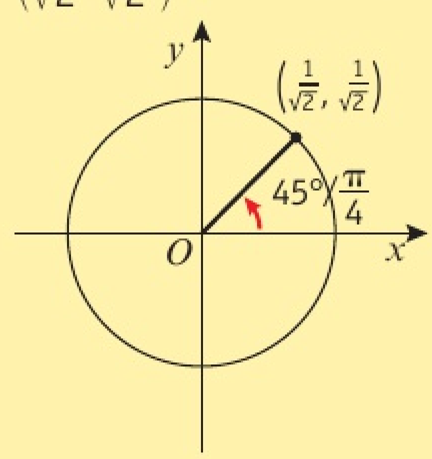
c $\tan 225° = 1$

d $\tan\left(-\frac{\pi}{4}\right) = \tan \frac{7\pi}{4} = -1$

e $\tan \pi = 0$

f $\tan \frac{\pi}{2} =$ undefined

When $\theta = 45°$ or $\frac{\pi}{4}$, the coordinates of OP are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ so the gradient of OP is 1.



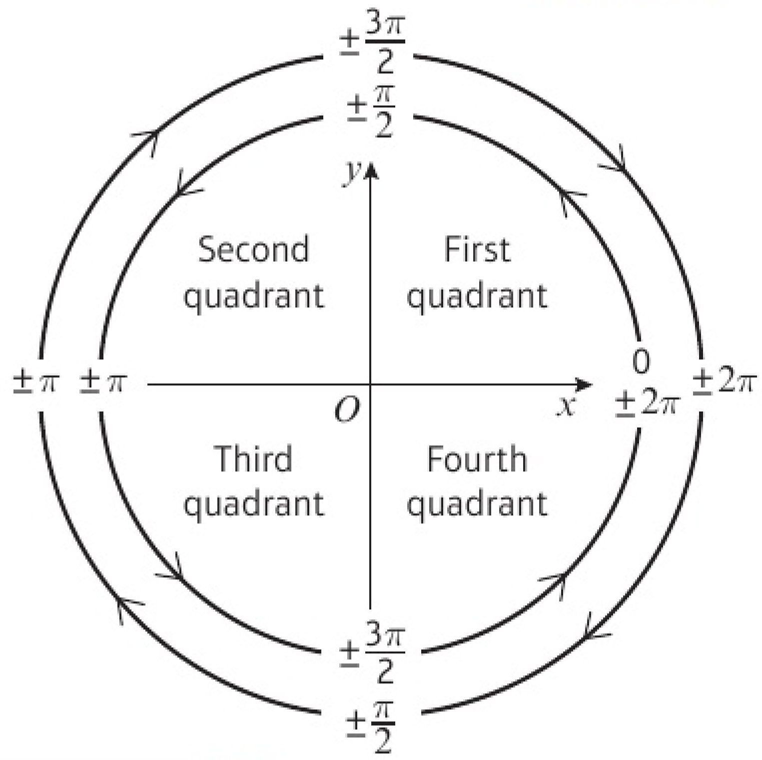
When $\theta = -\frac{\pi}{4}$ the gradient of OP is −1.

When $\theta = \pi$, P has coordinates (−1, 0) so the gradient of $OP = \frac{0}{1} = 0$.

When $\theta = \frac{\pi}{2}$, P has coordinates (0, 1) so the gradient of $OP = \frac{1}{0}$. This is undefined since you cannot divide by zero.

Links $\tan\theta$ is undefined when $\theta = \frac{\pi}{2}$ or any other odd multiple of $\frac{\pi}{2}$. These values of θ correspond to the asymptotes on the graph of $y = \tan\theta$. ← Pure 1 Section 6.5

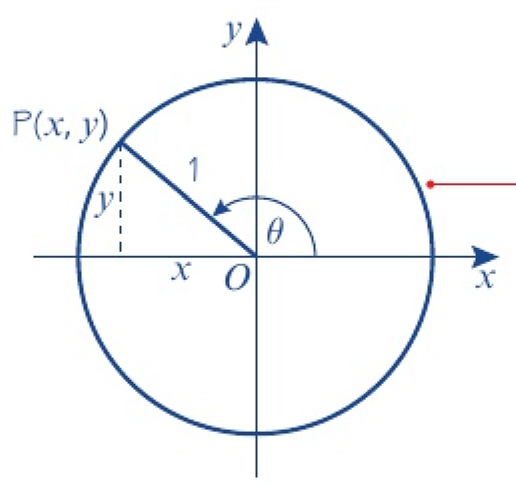
The x-y plane is divided into **quadrants**:



Angles may lie outside the range 0–360°, but they will always lie in one of the four quadrants.
For example, an angle of 600° would be equivalent to 600° – 360° = 240°, so it would lie in the third quadrant.

Example 3

Find the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in the second quadrant.



Draw a circle, centre O and radius 1, with $P(x, y)$ on the circle in the second quadrant.

In the second quadrant, θ is obtuse, or $90° < \theta < 180°$ in degrees or $\frac{\pi}{2} < \theta < \pi$ in radians.

As x is −ve and y is +ve in this quadrant

$\sin\theta = +\text{ve}$

$\cos\theta = -\text{ve}$

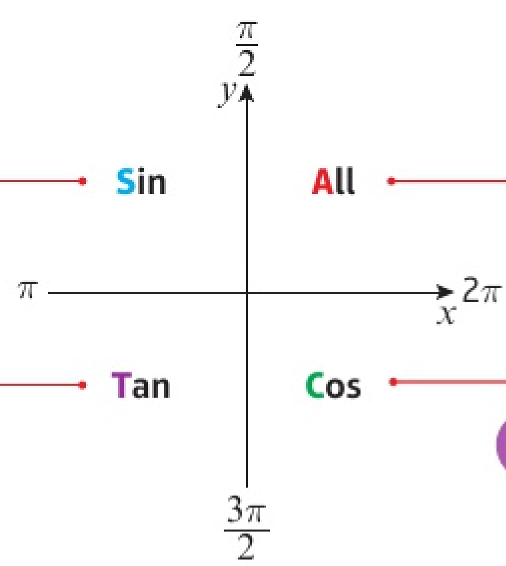
$\tan\theta = \frac{+\text{ve}}{-\text{ve}} = -\text{ve}$

So only $\sin\theta$ is positive.

- **You can use the quadrants to determine whether each of the trigonometric ratios is positive or negative.**

For an angle θ in the first quadrant, $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all positive.

For an angle θ in the second quadrant, only $\sin\theta$ is positive.



For an angle θ in the fourth quadrant, only $\cos\theta$ is positive.

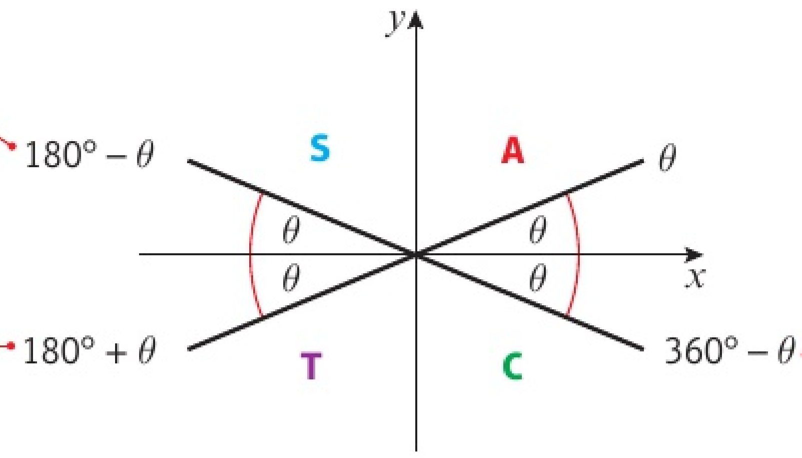
For an angle θ in the third quadrant, only $\tan\theta$ is positive.

Notation This diagram is often referred to as a CAST diagram since the word is spelled out from the bottom right going anti-clockwise.

- You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding **acute** angle made with the x-axis, θ.

$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

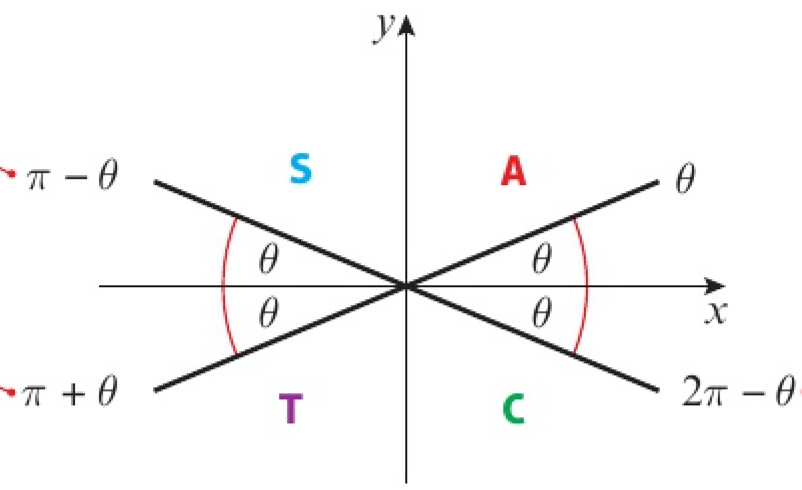
$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$



$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$

$\sin(\pi - \theta) = \sin\theta$
$\cos(\pi - \theta) = -\cos\theta$
$\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$
$\tan(\pi + \theta) = \tan\theta$



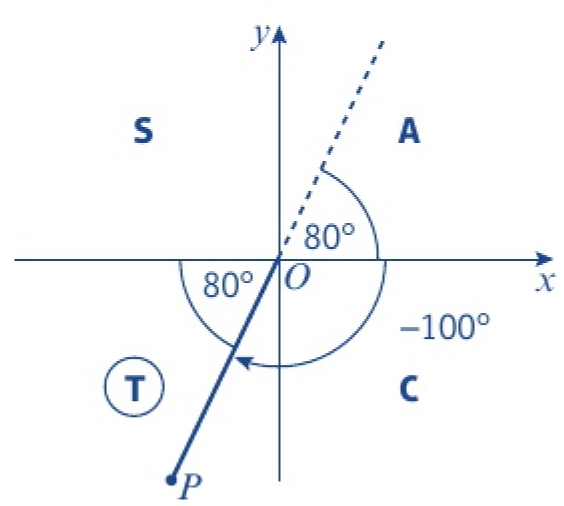
$\sin(2\pi - \theta) = -\sin\theta$
$\cos(2\pi - \theta) = \cos\theta$
$\tan(2\pi - \theta) = -\tan\theta$

Example 4 SKILLS INTERPRETATION

Express in terms of trigonometric ratios of acute angles:

a $\sin(-100°)$ **b** $\cos 330°$ **c** $\tan 500°$ **d** $\sin\frac{2\pi}{3}$ **e** $\cos\left(-\frac{5\pi}{6}\right)$ **f** $\tan\frac{11}{6}\pi$

a



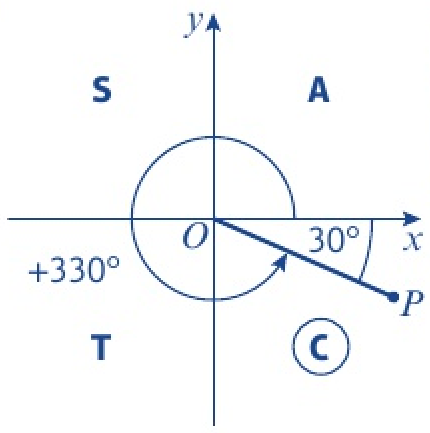
For each part, draw diagrams showing the position of OP for the given angle and insert the acute angle that OP makes with the x-axis.

The acute angle made with the x-axis is 80°.

In the third quadrant only tan is +ve, so sin is −ve.

So $\sin(-100)° = -\sin 80°$

b

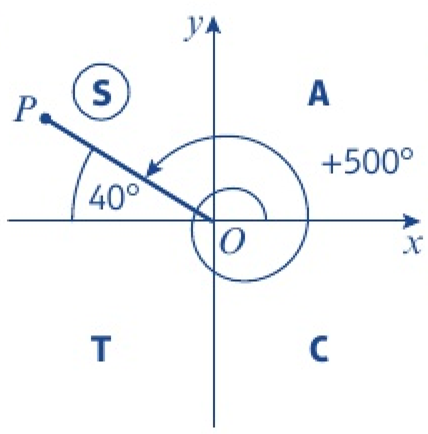


The acute angle made with the x-axis is 30°.

In the fourth quadrant only cos is +ve.

So $\cos 330° = +\cos 30°$

c

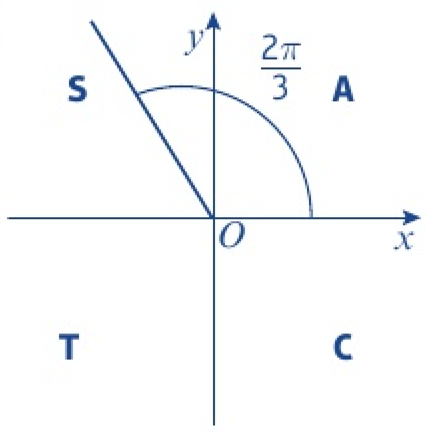


The acute angle made with the x-axis is 40°.

In the second quadrant only sin is +ve.

So $\tan 500° = -\tan 40°$

d

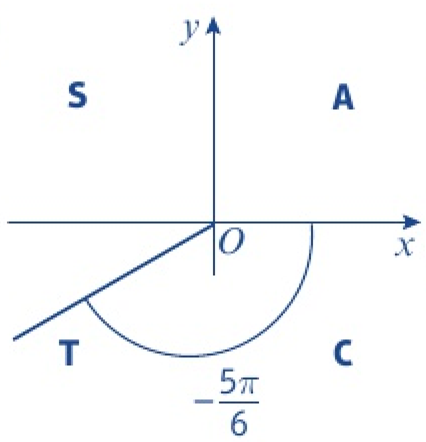


The acute angle made with the x-axis is $\frac{\pi}{3}$

In the second quadrant sine is +ve.

So $\sin \frac{2\pi}{3} = \sin \frac{\pi}{3}$

e

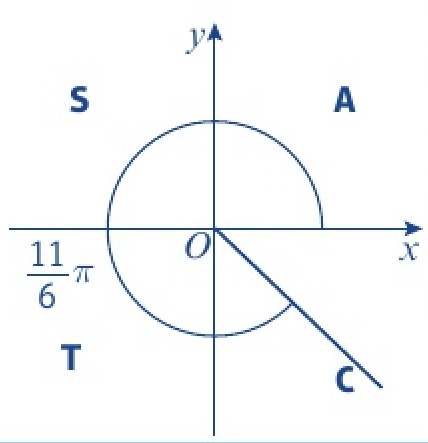


The acute angle made with the x-axis is $\frac{\pi}{6}$

In the third quadrant cosine is −ve.

So $\cos\left(-\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)$

f



The acute angle made with the x-axis is $\frac{\pi}{6}$

In the fourth quadrant tangent is −ve.

So $\tan\left(\frac{11\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right)$

Exercise 6A

SKILLS INTERPRETATION

1 Draw diagrams to show the following angles. Mark in the acute angle that *OP* makes with the *x*-axis.

a $-80°$ **b** $100°$ **c** $200°$ **d** $165°$ **e** $-145°$

f $\frac{5\pi}{4}$ **g** $\frac{14\pi}{9}$ **h** $\frac{11\pi}{6}$ **i** $-\frac{8\pi}{9}$ **j** $-\frac{14\pi}{9}$

2 State the quadrant that *OP* lies in when the angle that *OP* makes with the positive *x*-axis is:

a $400°$ **b** $115°$ **c** $-210°$ **d** $\frac{5\pi}{4}$ **e** $\frac{5\pi}{9}$

3 Without using a calculator, write down the values of:

a $\sin(-90°)$ **b** $\sin 450°$ **c** $\sin 540°$ **d** $\sin(-450°)$ **e** $\cos(-180°)$

f $\cos\left(-\frac{3\pi}{2}\right)$ **g** $\cos\left(\frac{3\pi}{2}\right)$ **h** $\cos\left(\frac{9\pi}{2}\right)$ **i** $\tan(2\pi)$ **j** $\tan(-\pi)$

4 Express the following in terms of trigonometric ratios of acute angles:

a $\sin 240°$ **b** $\sin(-80°)$ **c** $\sin\left(-\frac{10\pi}{9}\right)$ **d** $\sin\left(\frac{5\pi}{3}\right)$ **e** $\cos 110°$

f $\cos 260°$ **g** $\cos\left(-\frac{10\pi}{9}\right)$ **h** $\cos\left(\frac{109\pi}{36}\right)$ **i** $\tan 100°$ **j** $\tan 325°$

k $\tan\left(-\frac{\pi}{6}\right)$ **l** $\tan\frac{10\pi}{3}$

5 Given that θ is an acute angle, express in terms of $\sin\theta$:

a $\sin(-\theta)$ **b** $\sin(\pi + \theta)$ **c** $\sin(360° - \theta)$

d $\sin(-(180° + \theta))$ **e** $\sin(-\pi + \theta)$ **f** $\sin(-360° + \theta)$

g $\sin(3\pi + \theta)$ **h** $\sin(720° - \theta)$ **i** $\sin(\theta + 4\pi)$

Hint The results obtained in questions **5** and **6** are true for all values of θ.

6 Given that θ is an acute angle, express in terms of $\cos\theta$ or $\tan\theta$:

a $\cos(\pi - \theta)$ **b** $\cos(180° + \theta)$ **c** $\cos(-\theta)$ **d** $\cos(-(180° - \theta))$

e $\cos(\theta - 2\pi)$ **f** $\cos(\theta - 540°)$ **g** $\tan(-\theta)$ **h** $\tan(\pi - \theta)$

i $\tan(180° + \theta)$ **j** $\tan(-\pi + \theta)$ **k** $\tan(3\pi - \theta)$ **l** $\tan(\theta - 2\pi)$

Challenge

a Prove that $\sin(180° - \theta) = \sin\theta$

b Prove that $\cos(-\theta) = \cos\theta$

c Prove that $\tan(\pi - \theta) = -\tan\theta$

Problem-solving

Draw a diagram showing the positions of θ and $180° - \theta$ or $\pi - \theta$ on the unit circle.

6.2 Exact values of trigonometrical ratios

You can find sin, cos and tan of $30°/\frac{\pi}{6}$, $45°/\frac{\pi}{4}$ and $60°/\frac{\pi}{3}$ exactly using triangles.

Consider an **equilateral** triangle ABC of side 2 units.

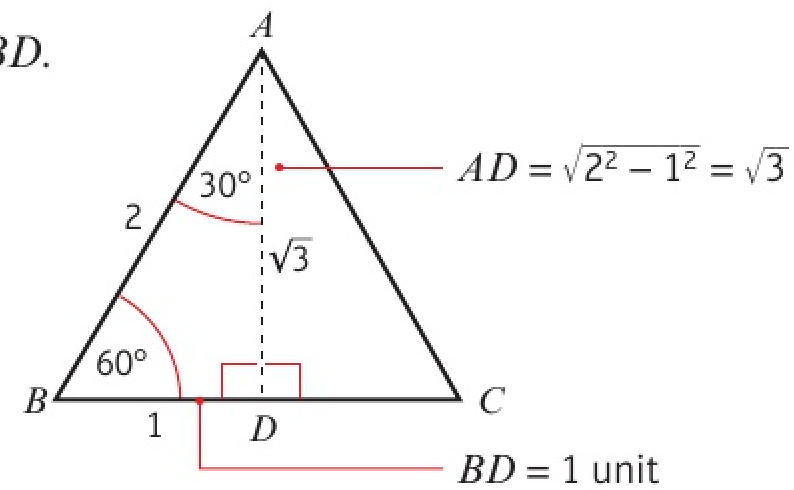
Draw a perpendicular from A to meet BC at D.

Apply the trigonometric ratios in the right-angled triangle ABD.

- $\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$ $\cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

  $\tan 30° = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

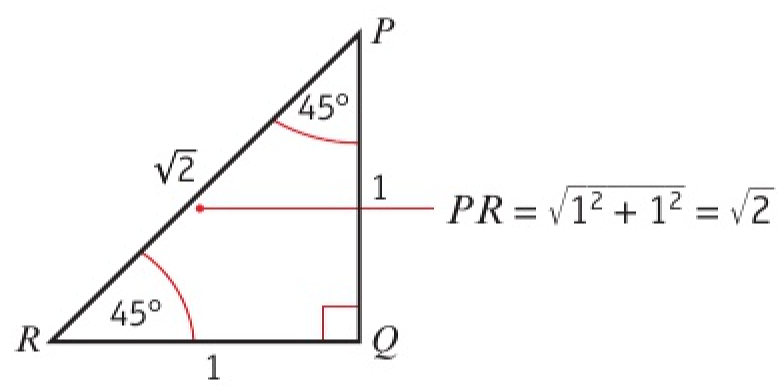
  $\cos 60° = \cos\frac{\pi}{3} = \frac{1}{2}$ $\tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$



Consider an **isosceles right-angled** triangle PQR with $PQ = RQ = 1$ unit.

- $\sin 45° = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\cos 45° = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

  $\tan 45° = \tan\frac{\pi}{4} = 1$

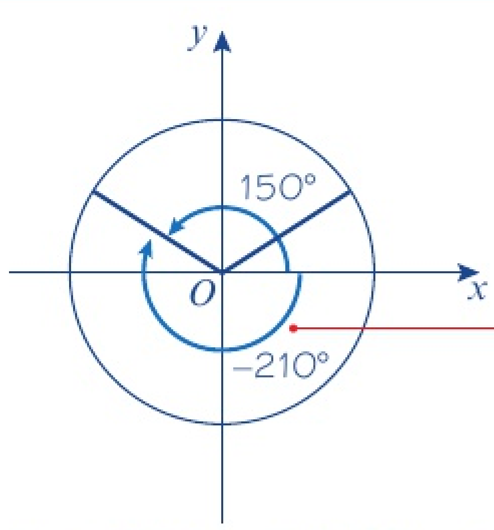


Example 5 SKILLS ANALYSIS

a Find the exact value of $\sin(-210°)$.

b Find the exact value of $\tan\left(\frac{7\pi}{4}\right)$

a

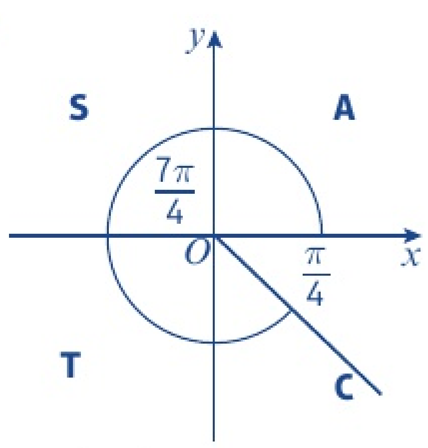


$\sin(-210°) = \sin(150°) = \sin(30°) = \frac{1}{2}$

$\sin(-210°) = \sin(150°)$

Use $\sin(180° - \theta) = \sin\theta$

b



$\tan\left(\frac{7\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$

Exercise 6B

SKILLS ANALYSIS

1 Express the following as trigonometric ratios of either 30°, 45° or 60°, and hence find their exact values.

a $\sin 135°$ **b** $\sin(-60°)$ **c** $\sin\left(\frac{11\pi}{6}\right)$ **d** $\sin\left(\frac{7\pi}{3}\right)$ **e** $\sin(-300°)$

f $\cos 120°$ **g** $\cos\left(\frac{5\pi}{3}\right)$ **h** $\cos\left(\frac{5\pi}{4}\right)$ **i** $\cos\left(\frac{-7\pi}{6}\right)$ **j** $\cos 495°$

k $\tan 135°$ **l** $\tan(-225°)$ **m** $\tan\left(\frac{7\pi}{6}\right)$ **n** $\tan 300°$ **o** $\tan\left(-\frac{2\pi}{3}\right)$

Challenge

The diagram shows an isosceles right-angled triangle ABC. $AE = DE = 1$ unit. Angle $ACD = 30°$.

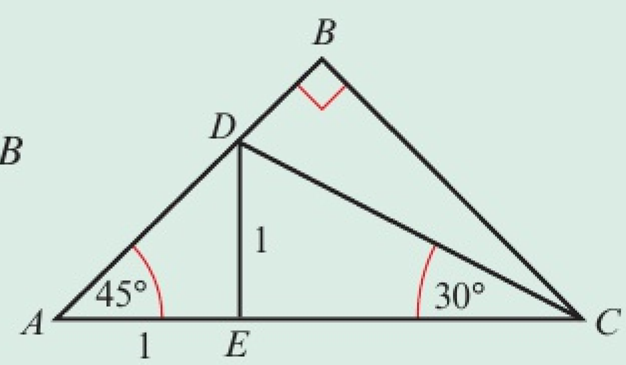
a Calculate the exact lengths of

i CE **ii** DC **iii** BC **iv** DB

b State the size of angle BCD.

c Hence find exact values for

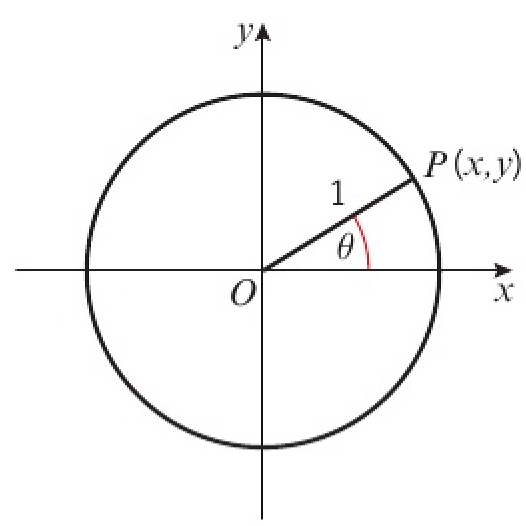
i sin 15° **ii** cos 15°



6.3 Trigonometric identities

You can use the definitions of sin, cos and tan, together with Pythagoras' theorem, to find two useful identities.

The unit circle has equation $x^2 + y^2 = 1$.



Links The equation of a circle with radius r and centre at the origin is $x^2 + y^2 = r^2$.

← Pure 2 Section 6.2

Since $\cos\theta = x$ and $\sin\theta = y$, it follows that $\cos^2\theta + \sin^2\theta = 1$.

- **For all values of θ, $\sin^2\theta + \cos^2\theta \equiv 1$.**

Since $\tan\theta = \frac{y}{x}$ it follows that $\tan\theta = \frac{\sin\theta}{\cos\theta}$

- **For all values of θ such that $\cos\theta \neq 0$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$**

You can use these two **identities** to simplify trigonometrical expressions and complete proofs.

Notation These results are called trigonometric identities. You use the $\equiv$ symbol instead of = to show that they are always true for all values of θ (subject to any conditions given).

Watch out $\tan\theta$ is undefined when the denominator = 0. This occurs when $\cos\theta = 0$, so when $\theta = \ldots -90°, 90°, 270°, 450°, \ldots$

Example 6

SKILLS PROBLEM-SOLVING

Simplify the following expressions:

a $\sin^2 3\theta + \cos^2 3\theta$ **b** $5 - 5\sin^2\theta$ **c** $\dfrac{\sin 2\theta}{\sqrt{1-\sin^2 2\theta}}$

a $\sin^2 3\theta + \cos^2 3\theta = 1$

$\sin^2\theta + \cos^2\theta = 1$, with θ replaced by 3θ.

b $5 - 5\sin^2\theta = 5(1-\sin^2\theta)$

Always look for factors.

$= 5\cos^2\theta$

$\sin^2\theta + \cos^2\theta = 1$, so $1 - \sin^2\theta = \cos^2\theta$.

c $\dfrac{\sin 2\theta}{\sqrt{1-\sin^2 2\theta}} = \dfrac{\sin 2\theta}{\sqrt{\cos^2 2\theta}}$

$\sin^2 2\theta + \cos^2 2\theta = 1$, so $1 - \sin^2 2\theta = \cos^2 2\theta$.

$= \dfrac{\sin 2\theta}{\cos 2\theta}$

$= \tan 2\theta$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, so $\dfrac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$.

Example 7

Prove that $\dfrac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$

Problem-solving

When you have to prove an identity like this you may quote the basic identities like $\sin^2 A + \cos^2 A \equiv 1$.

$\text{LHS} \equiv \dfrac{\cos^4\theta - \sin^4\theta}{\cos^2\theta}$

To prove an identity, start from the left-hand side, and manipulate the expression until it matches the right-hand side. ← Pure 2 Sections 1.5, 1.6

$\equiv \dfrac{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$

The numerator can be factorised as the 'difference of two squares'.

$\sin^2\theta + \cos^2\theta \equiv 1$.

$\equiv \dfrac{(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$

$\equiv \dfrac{\cos^2\theta}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta}$

$\equiv 1 - \tan^2\theta = \text{RHS}$

Divide through by $\cos^2\theta$ and note that $\dfrac{\sin^2\theta}{\cos^2\theta} \equiv \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 \equiv \tan^2\theta$.

Example 8

a Given that $\cos\theta = -\dfrac{3}{5}$ and that θ is **reflex**, find the value of $\sin\theta$.

b Given that $\sin\alpha = \dfrac{2}{5}$ and that α is obtuse, find the exact value of $\cos\alpha$.

a Since $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$

$$= 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

So $\sin\theta = -\frac{4}{5}$

b Using $\sin^2\alpha + \cos^2\alpha \equiv 1$,

$$\cos^2\alpha = 1 - \frac{4}{25} = \frac{21}{25}$$

As α is obtuse, $\cos\alpha$ is negative

so $\cos\alpha = -\frac{\sqrt{21}}{5}$

Watch out If you use your calculator to find $\cos^{-1}(-\frac{3}{5})$, then the sine of the result, you will get an incorrect answer. This is because the $\cos^{-1}$ function on your calculator gives results between 0 and 180°.

'θ is reflex' means θ is in the 3rd or 4th quadrants, but as $\cos\theta$ is negative, θ must be in the 3rd quadrant. $\sin\theta = \pm\frac{4}{5}$ but in the third quadrant $\sin\theta$ is negative.

Obtuse angles lie in the second quadrant, and have a negative cosine.

The question asks for the exact value so leave your answer as a surd.

Example 9

SKILLS EXECUTIVE FUNCTION

Given that $p = 3\cos\theta$, and that $q = 2\sin\theta$, show that $4p^2 + 9q^2 = 36$.

As $p = 3\cos\theta$, and $q = 2\sin\theta$,

$$\cos\theta = \frac{p}{3} \text{ and } \sin\theta = \frac{q}{2}$$

Using $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2 = 1$$

so $$\frac{q^2}{4} + \frac{p^2}{9} = 1$$

$\therefore$ $$4p^2 + 9q^2 = 36$$

Problem-solving

You need to eliminate θ from the equations. As you can find $\sin\theta$ and $\cos\theta$ in terms of p and q, use the identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Multiply both sides by 36.

Exercise 6C

SKILLS EXECUTIVE FUNCTION

1 Simplify each of the following expressions:

a $1 - \cos^2\frac{1}{2}\theta$ **b** $5\sin^2 3\theta + 5\cos^2 3\theta$ **c** $\sin^2 A - 1$

d $\frac{\sin\theta}{\tan\theta}$ **e** $\frac{\sqrt{1-\cos^2 x}}{\cos x}$ **f** $\frac{\sqrt{1-\cos^2 3A}}{\sqrt{1-\sin^2 3A}}$

g $(1 + \sin x)^2 + (1 - \sin x)^2 + 2\cos^2 x$ **h** $\sin^4\theta + \sin^2\theta\cos^2\theta$

i $\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta$

2 Given that $2\sin\theta = 3\cos\theta$, find the value of $\tan\theta$.

3 Given that $\sin x \cos y = 3\cos x \sin y$, express $\tan x$ in terms of $\tan y$.

4 Express in terms of $\sin\theta$ only:

a $\cos^2\theta$ **b** $\tan^2\theta$ **c** $\cos\theta\tan\theta$

d $\dfrac{\cos\theta}{\tan\theta}$ **e** $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

(P) **5** Using the identities $\sin^2 A + \cos^2 A \equiv 1$ and/or $\tan A = \dfrac{\sin A}{\cos A}$ $(\cos A \neq 0)$, prove that:

a $(\sin\theta + \cos\theta)^2 \equiv 1 + 2\sin\theta\cos\theta$ **b** $\dfrac{1}{\cos\theta} - \cos\theta \equiv \sin\theta\tan\theta$

c $\tan x + \dfrac{1}{\tan x} \equiv \dfrac{1}{\sin x\cos x}$ **d** $\cos^2 A - \sin^2 A \equiv 1 - 2\sin^2 A$

e $(2\sin\theta - \cos\theta)^2 + (\sin\theta + 2\cos\theta)^2 \equiv 5$ **f** $2 - (\sin\theta - \cos\theta)^2 \equiv (\sin\theta + \cos\theta)^2$

g $\sin^2 x\cos^2 y - \cos^2 x\sin^2 y \equiv \sin^2 x - \sin^2 y$

6 Find, without using your calculator, the values of:

a $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{5}{12}$ and θ is acute.

b $\sin\theta$ and $\tan\theta$, given that $\cos\theta = -\frac{3}{5}$ and θ is obtuse.

c $\cos\theta$ and $\tan\theta$, given that $\sin\theta = -\frac{7}{25}$ and $\frac{3\pi}{2} < \theta < 2\pi$.

7 Given that $\sin\theta = \frac{2}{3}$ and that θ is obtuse, find the exact value of: **a** $\cos\theta$ **b** $\tan\theta$

8 Given that $\tan\theta = -\sqrt{3}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$ **b** $\cos\theta$

9 Given that $\cos\theta = \frac{3}{4}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$ **b** $\tan\theta$

(P) **10** In each of the following, eliminate θ to give an equation relating x and y:

a $x = \sin\theta,\ y = \cos\theta$ **b** $x = \sin\theta,\ y = 2\cos\theta$

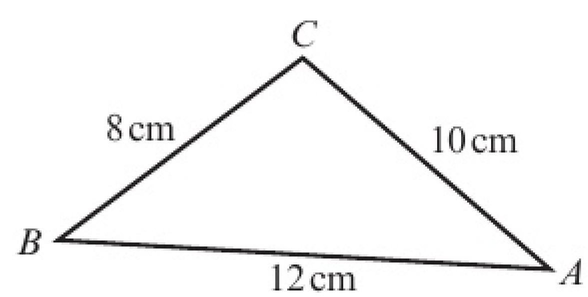
c $x = \sin\theta,\ y = \cos^2\theta$ **d** $x = \sin\theta,\ y = \tan\theta$

e $x = \sin\theta + \cos\theta,\ y = \cos\theta - \sin\theta$

Problem-solving

In part **e** find expressions for $x + y$ and $x - y$.

(E/P) **11** The diagram shows the triangle ABC with $AB = 12$ cm, $BC = 8$ cm and $AC = 10$ cm.

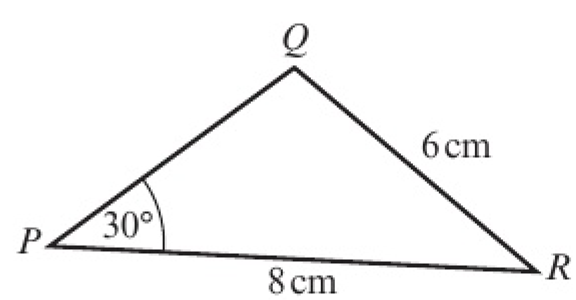


a Show that $\cos B = \dfrac{9}{16}$ **(3 marks)**

b Hence find the exact value of $\sin B$. **(2 marks)**

Hint Use the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ ← Pure 1 Section 6.1

(E/P) **12** The diagram shows triangle PQR with $PR = 8$ cm, $QR = 6$ cm and angle $QPR = 30°$.



a Show that $\sin Q = \dfrac{2}{3}$ **(3 marks)**

b Given that Q is obtuse, find the exact value of $\cos Q$ **(2 marks)**

6.4 Solve simple trigonometric equations

You need to be able to solve simple trigonometric equations of the form $\sin\theta = k$ and $\cos\theta = k$ (where $-1 \leqslant k \leqslant 1$) and $\tan\theta = p$ (where $p \in \mathbb{R}$) for given intervals of θ.

- **Solutions to $\sin\theta = k$ and $\cos\theta = k$ only exist when $-1 \leqslant k \leqslant 1$.**
- **Solutions to $\tan\theta = p$ exist for all values of p.**

Links The graphs of $y = \sin\theta$ and $y = \cos\theta$ have a maximum value of 1 and a minimum value of –1.
The graph of $y = \tan\theta$ has no maximum or minimum value. ← Pure 1 Section 6.5

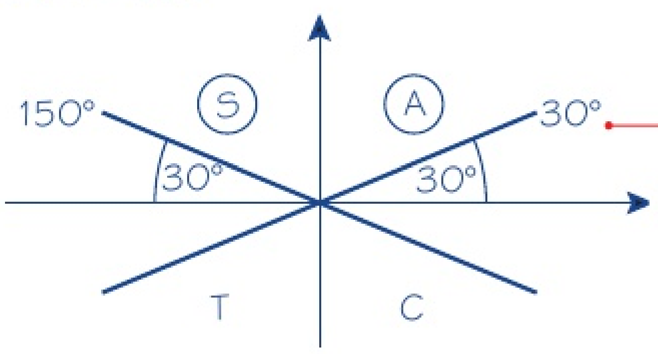
Example 10

Find the solutions of the equation $\sin\theta = \frac{1}{2}$ in the interval $0 \leqslant \theta \leqslant 360°$.

Method 1

$\sin\theta = \frac{1}{2}$

So $\theta = 30°$

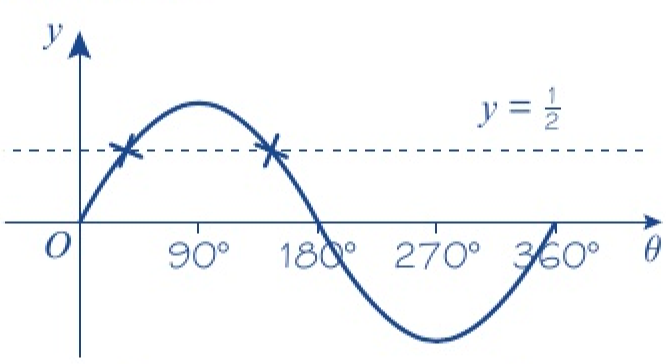


Putting 30° in the four positions shown gives the angles 30°, 150°, 210° and 330° but sine is only positive in the 1st and 2nd quadrants.

So $x = 30°$
or $x = 180° - 30° = 150°$

You can check this by putting sin 150° in your calculator.

Method 2



Draw the graph of $y = \sin\theta$ for the given interval.

$\sin\theta = \frac{1}{2}$ where the line $y = \frac{1}{2}$ cuts the curve.
Hence $\theta = 30°$ or $150°$

Use the **symmetry** properties of the $y = \sin\theta$ graph. ← Pure 1 Section 6.5

- **When you use the inverse trigonometric functions on your calculator, the angle you get is called the principal value.**

Your calculator will give principal values in the following ranges:

$\cos^{-1}$ in the range $0 \leqslant \theta \leqslant 180°$

$\sin^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$

$\tan^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$

Notation The inverse trigonometric functions are also called **arccos**, **arcsin** and **arctan**.

Example 11

SKILLS ANALYSIS

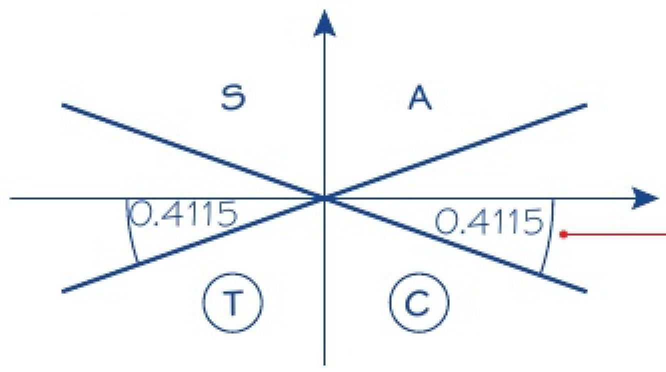
Solve, in radians to 3 significant figures, $5 \sin x = -2$ in the interval $0 \leqslant x \leqslant 2\pi$.

Method 1

$5 \sin x = -2$

$\sin x = -\frac{2}{5}$

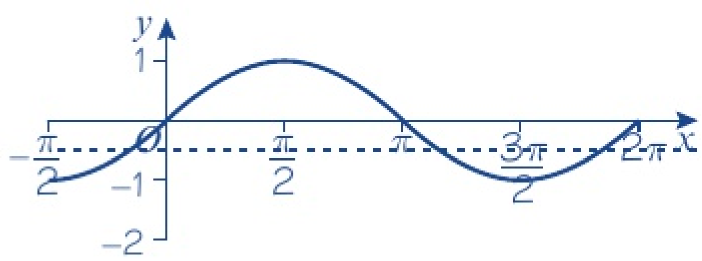
Principal value is $x = -0.4115$ (4 s.f.)



$x = \pi + 0.4115 = 3.5531... \Rightarrow 3.55$ (3 s.f.)

$x = 2\pi - 0.4115 = 5.8717... \Rightarrow 5.87$ (3 s.f.)

Method 2



$\sin^{-1}(-0.4) = -0.4115...$

$x = \pi + 0.4115 = 3.5531... \Rightarrow 3.55$ (3 s.f.)

$x = 2\pi - 0.4115 = 5.8717... \Rightarrow 5.87$ (3 s.f.)

First rewrite in the form $\sin x = ...$

Watch out The principal value will not always be a solution to the equation.

Sine is negative so you need to look in the 3rd and 4th quadrants for your solutions.

You can now find the solutions in the given interval.
Note that in this case, if $\alpha = \sin^{-1}(-0.4)$, the solutions are $\pi - \alpha$ and $2\pi + \alpha$.

Draw the graph of $y = \sin x$ starting from $-\pi$ since the principal solution given by $\sin^{-1}(-0.4)$ is negative.

Use the symmetry properties of the $y = \sin \theta$ graph.

Example 12

Solve, in the interval $0 < x \leqslant 360°$, $\cos x = \frac{\sqrt{3}}{2}$

A student writes down the following working:

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$

So $x = 30°$ or $x = 180° - 30° = 150°$

a Identify the error made by the student.

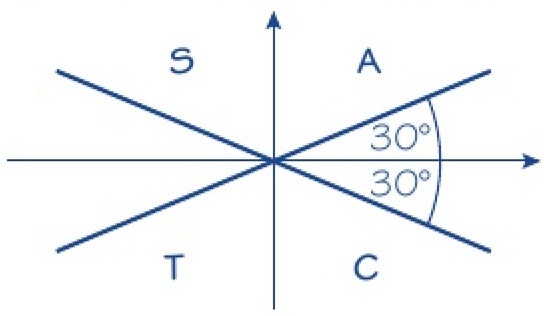
b Write down the correct answer.

a The principal solution is correct but the student has found a second solution in the second quadrant where cos is negative.

Problem-solving

In your exam you might have to analyse student working and identify errors. One strategy is to solve the problem yourself, then compare your working with the incorrect working that has been given.

b $x = 30°$ from the calculator



$x = 30°$ or $330°$

$\cos x$ is positive so you need to look in the 1st and 4th quadrants.

Find the solutions, in $0 < x \leqslant 360°$, from your diagram.

Note that these results are α and $360° - \alpha$ where $\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

You can use the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ to solve equations.

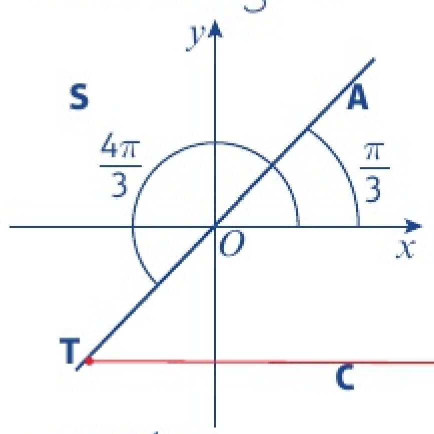
Example 13

Find the values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$ that satisfy the equation $\sin\theta = \sqrt{3}\cos\theta$.

$\sin\theta = \sqrt{3}\cos\theta$

So $\tan\theta = \sqrt{3}$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$



$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

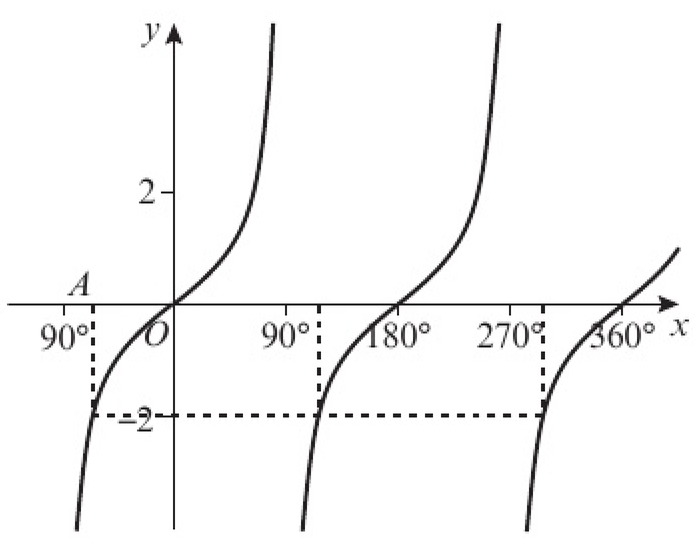
Since $\cos\theta = 0$ does not satisfy the equation, divide both sides by $\cos\theta$ and use the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

This is the principal solution.

Tangent is positive in the 1st and 3rd quadrants, so insert the angle in the correct positions.

Exercise 6D SKILLS ANALYSIS

1 The diagram shows a sketch of $y = \tan x$.

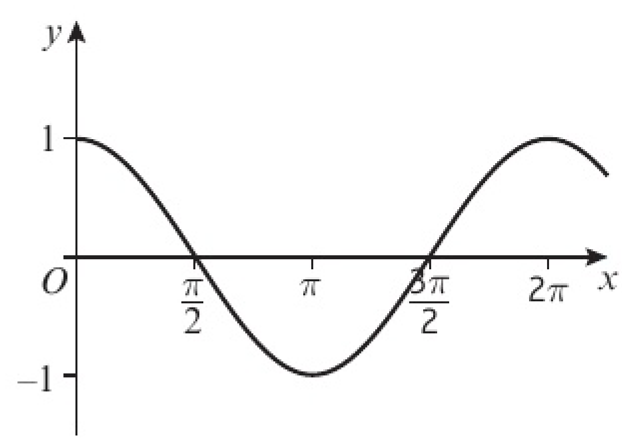


a Use your calculator to find the principal solution to the equation $\tan x = -2$.

Hint The principal solution is marked A on the diagram.

b Use the graph and your answer to part **a** to find solutions to the equation $\tan x = -2$ in the range $0 \leqslant x \leqslant 360°$.

2 The diagram shows a sketch of $y = \cos x$.



a Use your calculator to find the principal solution to the equation $\cos x = 0.4$.

b Use the graph and your answer to part **a** to find solutions to the equation $\cos x = \pm 0.4$ in the range $0 \leqslant x \leqslant 2\pi$.

3 Solve the following equations for θ, in the interval $0 < \theta \leqslant 360°$:

a $\sin\theta = -1$ **b** $\tan\theta = \sqrt{3}$ **c** $\cos\theta = \frac{1}{2}$

d $\sin\theta = \sin 15°$ **e** $\cos\theta = -\cos 40°$ **f** $\tan\theta = -1$

g $\cos\theta = 0$ **h** $\sin\theta = -0.766$

Hint Give your answers exactly where possible, or round to 3 significant figures.

4 Solve the following equations for θ, in the interval $0 < \theta \leqslant 2\pi$:

a $7\sin\theta = 5$ **b** $2\cos\theta = -\sqrt{2}$ **c** $3\cos\theta = -2$ **d** $4\sin\theta = -3$

5 Solve the following equations for θ, in the interval $0 < \theta \leqslant 360°$:

a $7\tan\theta = 1$ **b** $8\tan\theta = 15$ **c** $3\tan\theta = -11$ **d** $3\cos\theta = \sqrt{5}$

6 Solve the following equations for θ, in the interval $0 < \theta \leqslant 2\pi$:

a $\sqrt{3}\sin\theta = \cos\theta$ **b** $\sin\theta + \cos\theta = 0$ **c** $3\sin\theta = 4\cos\theta$

7 Solve the following equations for θ, in the range $0° \leqslant x \leqslant 360°$

a $2\sin\theta - 3\cos\theta = 0$ **b** $\sqrt{2}\sin\theta = 2\cos\theta$ **c** $\sqrt{5}\sin\theta + \sqrt{2}\cos\theta = 0$

8 Solve the following equations for x, giving your answers to 3 significant figures where appropriate, in the intervals indicated.

a $\sin x = -\frac{\sqrt{3}}{2}$, $-180° \leqslant x \leqslant 540°$ **b** $2\sin x = -0.3$, $-\pi \leqslant x \leqslant \pi$

c $\cos x = -0.809$, $-180° \leqslant x \leqslant 180°$ **d** $\cos x = 0.84$, $-2\pi < x < 0$

e $\tan x = -\frac{\sqrt{3}}{3}$, $0 \leqslant x \leqslant 4\pi$ **f** $\tan x = 2.90$, $80° \leqslant x \leqslant 440°$

E/P **9** A teacher asks two students to solve the equation $2\cos x = 3\sin x$ for $-180° \leqslant x \leqslant 180°$. The attempts are shown:

Student A:

$\tan x = \frac{3}{2}$

$x = 56.3°$ or $x = -123.7°$

Student B:

$4\cos^2 x = 9\sin^2 x$

$4(1 - \sin^2 x) = 9\sin^2 x$

$4 = 13\sin^2 x$

$\sin x = \pm\sqrt{\frac{4}{13}}$, $x = \pm 33.7°$ or $x = \pm 146.3°$

a Identify the mistake made by Student A. **(1 mark)**

b Identify the mistake made by Student B and explain the effect it has on their solution. **(2 marks)**

c Write down the correct answers to the question. **(1 mark)**

10 **a** Sketch the graphs of $y = 2\sin x$ and $y = \cos x$ on the same set of axes $(0 \leqslant x \leqslant 360°)$.

b Write down how many solutions there are in the given range for the equation $2\sin x = \cos x$.

c Solve the equation $2\sin x = \cos x$ algebraically, giving your answers to 1 d.p.

E/P **11** Find all the values of θ, to 1 decimal place, in the interval $0 < \theta < 360°$ for which $\tan^2\theta = 9$. **(5 marks)**

Problem-solving When you take square roots of both sides of an equation you need to consider both the positive and the negative square roots.

E/P **12** **a** Show that $4\sin^2 x - 3\cos^2 x = 2$ can be written as $7\sin^2 x = 5$.

b Hence solve, for $0 \leqslant x \leqslant 2\pi$, the equation $4\sin^2 x - 3\cos^2 x = 2$.

Give your answers to one decimal place.

E/P **13** **a** Show that the equation $2\sin^2 x + 5\cos^2 x = 1$ can be written as $3\sin^2 x = 4$. **(2 marks)**

b Use your result in part **a** to explain why the equation $2\sin^2 x + 5\cos^2 x = 1$ has no solutions. **(1 mark)**

6.5 Harder trigonometric equations

You need to be able to solve equations of the form $\sin n\theta = k$, $\cos n\theta = k$ and $\tan n\theta = p$.

Example 14

a Solve the equation $\cos 3\theta = 0.766$, in the interval $0 \leqslant \theta \leqslant 2\pi$.

b Solve the equation $2\sin 2\theta = \cos 2\theta$, in the interval $0 \leqslant \theta \leqslant 360°$.

a Let $X = 3\theta$

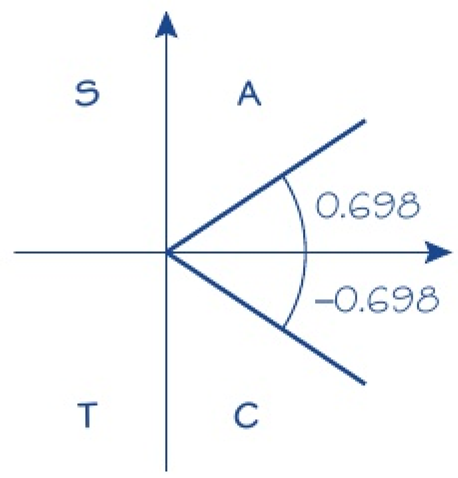
Replace 3θ by X and solve.

$\cos X = 0.766$

As $X = 3\theta$,
then as $0 \leqslant \theta \leqslant 2\pi$

So, $3 \times 0 \leqslant X \leqslant 3 \times 2\pi$

So the interval for X is
$0 \leqslant X \leqslant 6\pi$



Watch out If the range of values for θ is $0 \leqslant \theta \leqslant 2\pi$, then the range of values for 3θ is $0 \leqslant 3\theta \leqslant 6\pi$.

$X = 0.698, 5.585, 6.981, 11.868, 13.264, 18.151$

The value of X from your calculator is 0.698. You need to list **ALL** values in the 1st and 4th quadrants for three complete revolutions.

i.e. $3\theta = 0.698, 5.585, 6.981, 11.88, 13.264, 18.151$

So $\theta = 0.233, 1.86, 2.33, 3.96, 4, 42, 6.05$

Remember $X = 3\theta$.

b $\dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{1}{2}$, so $\tan 2\theta = \dfrac{1}{2}$

Use the identity for tan to rearrange the equation.

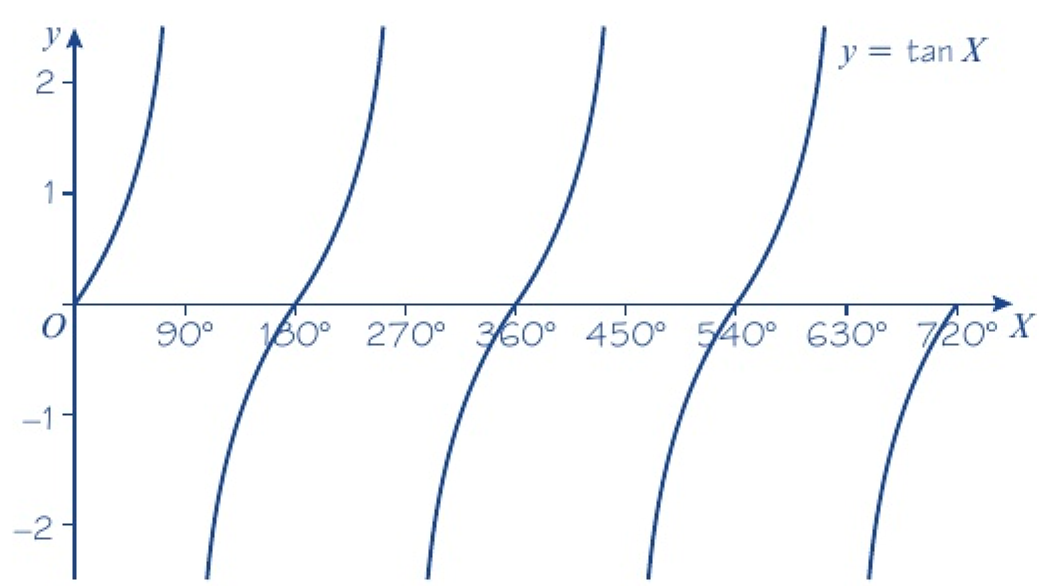
Let $X = 2\theta$

So $\tan X = \dfrac{1}{2}$

As $X = 2\theta$, then as $0 \leqslant \theta \leqslant 360°$

The interval for X is $0 \leqslant X \leqslant 720°$

Let $X = 2\theta$, and double both values to find the interval for X.



Draw a graph of $\tan X$ for this interval.
Alternatively, you could use a CAST diagram as in part **a**.

The principal solution for X is 26.565…°

Add multiples of 180°:

$X = 26.565…°, 206.565…°, 386.565…°, 566.565…°$

$\theta = 13.3°, 103°, 193°, 283°$

Convert your values of X back into values of θ.
Round each answer to a sensible degree of accuracy at the end.

You need to be able to solve equations of the form $\sin(\theta + \alpha) = k$, $\cos(\theta + \alpha) = k$ and $\tan(\theta + \alpha) = p$.

Example 15 SKILLS ANALYSIS

Solve the equation $\sin(x + 60°) = 0.3$ in the interval $0 \leqslant x \leqslant 360°$.

Let $X = x + 60°$

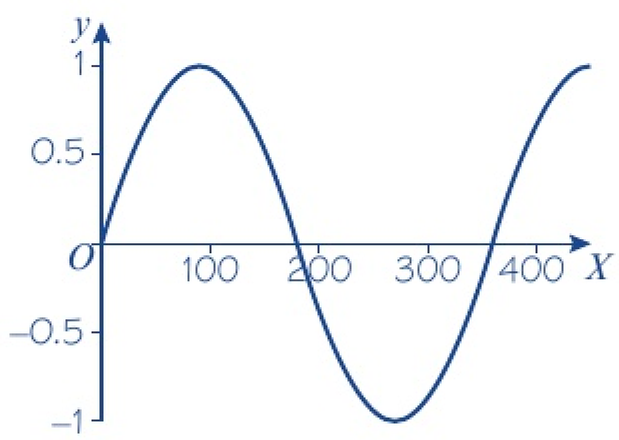
So $\sin X = 0.3$

The interval for X is

$0° + 60° \leqslant X \leqslant 360° + 60°$

So $60° \leqslant X \leqslant 420°$

Adjust the interval by adding 60° to both values.



Draw a sketch of the sin graph for the given interval.

The principal value for X is 17.45…°

$X = 162.54…°, 377.45…°$

This is not in the given interval so it does not correspond to a solution of the equation. Use the symmetry of the sin graph to find other solutions.

Subtract 60° from each value:

$x = 102.54…°, 317.45…°$

Hence $x = 102.5°$ or $317.5°$

You could also use a CAST diagram to solve this problem.

Exercise 6E SKILLS ANALYSIS

1 Find the values of θ, in the interval $0 \leqslant \theta \leqslant 360°$, for which:

a $\sin 4\theta = 0$ **b** $\cos 3\theta = -1$ **c** $\tan 2\theta = 1$

2 Find the values of θ, in the interval $0 \leqslant \theta \leqslant 2\pi$, for which:

a $\cos 2\theta = \frac{1}{2}$ **b** $\tan \frac{\theta}{2} = -\frac{1}{\sqrt{3}}$ **c** $\sin(-\theta) = \frac{1}{\sqrt{2}}$

3 Solve the following equations in the interval given:

a $\tan(45° - \theta) = -1, 0 \leqslant \theta \leqslant 360°$ **b** $2\sin\left(\theta - \frac{\pi}{9}\right) = 1, 0 \leqslant \theta \leqslant 2\pi$

c $\tan(\theta + 75°) = \sqrt{3}, 0 \leqslant \theta \leqslant 360°$ **d** $\sin\left(\theta - \frac{\pi}{18}\right) = -\frac{\sqrt{3}}{2}, 0 \leqslant \theta \leqslant 2\pi$

e $\cos(70° - x) = 0.6, 0 \leqslant \theta \leqslant 180°$

4 Solve the following equations in the interval given:

a $3\sin 3\theta = 2\cos 3\theta, 0 \leqslant \theta \leqslant 180°$ **b** $4\sin\left(\theta + \frac{\pi}{4}\right) = 5\cos\left(\theta + \frac{\pi}{4}\right), 0 \leqslant \theta \leqslant \frac{5\pi}{2}$

c $2\sin 2x - 7\cos 2x = 0, 0 \leqslant x \leqslant 180°$ **d** $\sqrt{3}\sin\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{\pi}{4}\right) = 0, 0 \leqslant x \leqslant \pi$

(E) **5** Solve for $0 \leqslant x \leqslant 180°$ the equations:

a $\sin(x + 20°) = \frac{1}{2}$ **(4 marks)**

b $\cos 2x = -0.8$, giving your answers to 1 decimal place. **(4 marks)**

(E) **6** **a** Sketch for $0 \leqslant x \leqslant 2\pi$ the graph of $y = \sin\left(x + \frac{\pi}{3}\right)$ **(2 marks)**

b Write down the exact coordinates of the points where the graph meets the coordinate axes. **(3 marks)**

c Solve, for $0 \leqslant x \leqslant 2\pi$, the equation $\sin\left(x + \frac{\pi}{3}\right) = 0.55$, giving your answers to 3 significant figures. **(5 marks)**

(E) **7** **a** Given that $4\sin x = 3\cos x$, write down the value of $\tan x$. **(1 mark)**

b Solve, for $0 \leqslant \theta \leqslant 360°$, $4\sin 2\theta = 3\cos 2\theta$ giving your answers to 1 decimal place. **(5 marks)**

(E/P) **8** The equation $\tan kx = -\frac{1}{\sqrt{3}}$, where k is a constant and $k > 0$, has a solution at $x = \frac{\pi}{3}$

a Find a possible value of k. **(3 marks)**

b State, with justification, whether this is the only such possible value of k. **(1 mark)**

Challenge

Solve the equation $\sin(3x - 45°) = \frac{1}{2}$ in the interval $0 \leqslant x \leqslant 180°$.

6.6 Equations and identities

You need to be able to solve quadratic equations in $\sin\theta$, $\cos\theta$ or $\tan\theta$. This may give rise to two sets of solutions.

$5\sin^2 x + 3\sin x - 2 = 0$ — This is a quadratic equation in the form $5A^2 + 3A - 2 = 0$ where $A = \sin x$.

$(5\sin x - 2)(\sin x + 1) = 0$ — Factorise

$5\sin x - 2 = 0$ $\quad$ $\sin x + 1 = 0$ — Setting each factor equal to zero produces two linear equations in $\sin x$.

Example 16 SKILLS EXECUTIVE FUNCTION

Solve for θ, in the interval $0 \leqslant x \leqslant 360°$, the equations

a $2\cos^2\theta - \cos\theta - 1 = 0$ $\quad$ **b** $\sin^2(\theta - 30°) = \frac{1}{2}$

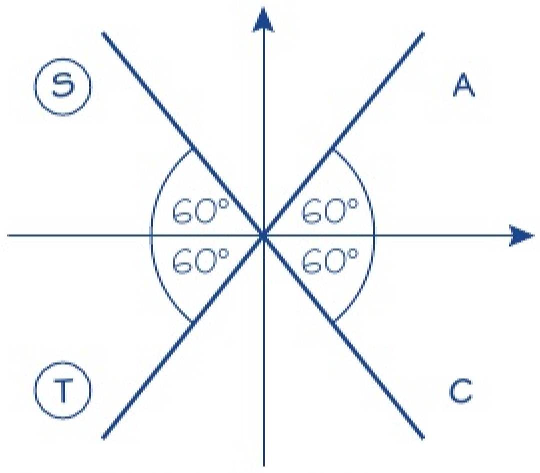
a $2\cos^2\theta - \cos\theta - 1 = 0$

So $(2\cos\theta + 1)(\cos\theta - 1) = 0$ — Compare with $2x^2 - x - 1 = (2x + 1)(x - 1)$

So $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

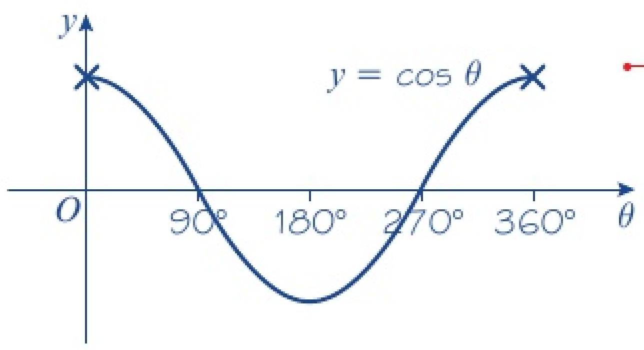
Set each factor equal to 0 to find two sets of solutions.

$\cos\theta = -\frac{1}{2}$ so $\theta = 120°$



120° makes an angle of 60° with the horizontal. But cosine is negative in the 2nd and 3rd quadrants so $\theta = 120°$ or $\theta = 240°$.

$\theta = 120°$ or $\theta = 240°$



Sketch the graph of $y = \cos\theta$.

Or $\cos\theta = 1$ so $\theta = 0$ or $360°$

So the solutions are

$\theta = 0°, 120°, 240°, 360°$

There are four solutions within the given interval.

b $\sin^2(\theta - 30°) = \frac{1}{2}$

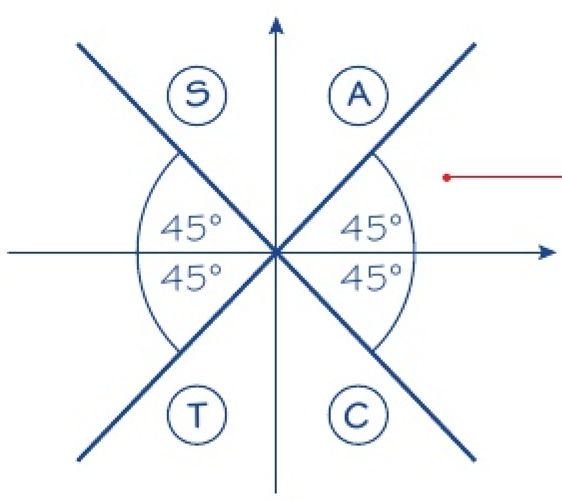
$\sin(\theta - 30°) = \frac{1}{\sqrt{2}}$

or $\sin(\theta - 30°) = -\frac{1}{\sqrt{2}}$

The solutions of $x^2 = k$ are $x = \pm\sqrt{k}$.

So $\theta - 30° = 45°$ or $\theta - 30° = -45°$

Use your calculator to find one solution for each equation.



Draw a diagram to find the quadrants where sine is positive and the quadrants where sine is negative.

So from $\sin(\theta - 30°) = \frac{1}{\sqrt{2}}$

$\theta - 30° = 45°, 135°$

and from $\sin(\theta - 30°) = -\frac{1}{\sqrt{2}}$

$\theta - 30° = 225°, 315°$

So the solutions are: $\theta = 75°, 165°, 255°, 345°$

In some equations you may need to use the identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Example 17

Find the values of x, in the interval $-\pi \leqslant x \leqslant \pi$, satisfying the equation $2\cos^2 x + 9\sin x = 3\sin^2 x$.

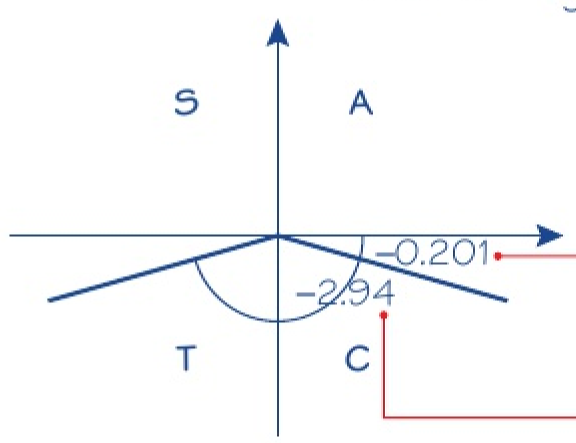
$2\cos^2 x + 9\sin x = 3\sin^2 x$

$2(1 - \sin^2 x) + 9\sin x = 3\sin^2 x$

$\Rightarrow 5\sin^2 x - 9\sin x - 2 = 0$

So $(5\sin x + 1)(\sin x - 2) = 0$

$\Rightarrow \sin x = -\frac{1}{5} \Rightarrow x = -0.20135$



The solutions are −2.94 and −0.201 (3 s.f.)

$\sin x$ in the equation informs you that you must change $\cos^2 x$ into $\sin^2 x$ using the identity $\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$ and so form a quadratic equation in $\sin x$.

Watch out The factor $(\sin x - 2)$ does not produce any solutions, because $\sin x = 2$ has no solutions.

Your calculator value is −0.201. Insert into CAST diagram. There are no values between 0 and π.

Exercise 6F

SKILLS EXECUTIVE FUNCTION

1 Solve for θ, in the interval $0° \leqslant \theta \leqslant 360°$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $4\cos^2\theta = 1$ **b** $2\sin^2\theta - 1 = 0$ **c** $3\sin^2\theta + \sin\theta = 0$

d $\tan^2\theta - 2\tan\theta - 10 = 0$ **e** $2\cos^2\theta - 5\cos\theta + 2 = 0$ **f** $\sin^2\theta - 2\sin\theta - 1 = 0$

g $\tan^2 2\theta = 3$

Hint In part **e**, only one factor leads to valid solutions.

2 Solve for θ, in the interval $-\pi \leqslant \theta \leqslant \pi$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $\sin^2 2\theta = 1$ **b** $\tan^2\theta = 2\tan\theta$

c $\cos\theta(\cos\theta - 2) = 1$ **d** $4\sin\theta = \tan\theta$

Warning! Do not cancel through by $\tan\theta$ in **b** or $\sin\theta$ in **d**. You will lose solutions.

3 Solve for θ, in the interval $0 \leqslant \theta \leqslant \pi$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $4(\sin^2\theta - \cos\theta) = 3 - 2\cos\theta$ **b** $2\sin^2\theta = 3(1 - \cos\theta)$ **c** $4\cos^2\theta - 5\sin\theta - 5 = 0$

4 Solve for θ, in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $5\sin^2\theta = 4\cos^2\theta$ **b** $\tan\theta = \cos\theta$

E **5** Find all the solutions, in the interval $0 \leqslant x \leqslant 360°$, to the equation $8\sin^2 x + 6\cos x - 9 = 0$ giving each solution to one decimal place. **(6 marks)**

E **6** Find, for $0 \leqslant x \leqslant 2\pi$, all the solutions of $1 + \sin^2 = \frac{7}{2}\cos^2 x$
Give each solution to 3 significant figures. **(6 marks)**

E/P **7** Show that the equation $2\cos^2 x + \cos x - 6 = 0$ has no solutions. **(3 marks)**

E/P **8** **a** Show that the equation $\cos^2 x = 2 - \sin x$ can be written as $\sin^2 x - \sin x + 1 = 0$. **(2 marks)**

b Hence show that the equation $\cos^2 x = 2 - \sin x$ has no solutions. **(3 marks)**

Problem-solving

If you have to answer a question involving the number of solutions to a quadratic equation, see if you can make use of the discriminant.

E/P **9** $\tan^2 x - 2\tan x - 4 = 0$

a Show that $\tan x = p \pm \sqrt{q}$ where p and q are numbers to be found. **(3 marks)**

b Hence solve the equation $\tan^2 x - 2\tan x - 4 = 0$ in the interval $0 \leqslant x \leqslant 540°$. **(5 marks)**

Challenge

SKILLS INNOVATION

1 Solve the equation $\cos^2 3\theta - \cos 3\theta = 2$ in the interval $-180° \leqslant \theta \leqslant 180°$.

2 Solve the equation $\tan^2(\theta - 45°) = 1$ in the interval $0 \leqslant \theta \leqslant 360°$.

Chapter review 6

1 Write each of the following as a trigonometric ratio of an acute angle:

a $\cos 237°$ **b** $\sin 312°$ **c** $\tan 190°$

d $\cos\left(\frac{11\pi}{6}\right)$ **e** $\sin\left(\frac{7\pi}{6}\right)$ **f** $\tan\left(\frac{11\pi}{4}\right)$

2 Without using your calculator, work out the values of:

a $\cos 270°$ **b** $\sin 225°$ **c** $\tan 240°$

d $\cos \pi$ **e** $\tan\left(\frac{5\pi}{4}\right)$ **f** $\sin\left(\frac{3\pi}{2}\right)$

P **3** Given that angle A is obtuse and $\cos A = -\sqrt{\frac{7}{11}}$, show that $\tan A = \frac{-2\sqrt{7}}{7}$

P **4** Given that angle B is reflex and $\tan B = +\frac{\sqrt{21}}{2}$, find the exact value of: **a** $\sin B$ **b** $\cos B$

5 Simplify the following expressions:

a $\cos^4\theta - \sin^4\theta$ **b** $\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$

c $\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta$

6 **a** Given that $2(\sin x + 2\cos x) = \sin x + 5\cos x$, find the exact value of $\tan x$.

b Given that $\sin x\cos y + 3\cos x\sin y = 2\sin x\sin y - 4\cos x\cos y$, express $\tan y$ in terms of $\tan x$.

(P) **7** Prove that, for all values of θ:

a $(1 + \sin\theta)^2 + \cos^2\theta \equiv 2(1 + \sin\theta)$ **b** $\cos^4\theta + \sin^2\theta \equiv \sin^4\theta + \cos^2\theta$

(P) **8** Without attempting to solve them, state how many solutions the following equations have in the interval $0 \leqslant \theta \leqslant 360°$. Give a brief reason for your answer.

a $2\sin\theta = 3$ **b** $\sin\theta = -\cos\theta$

c $2\sin\theta + 3\cos\theta + 6 = 0$ **d** $\tan\theta + \dfrac{1}{\tan\theta} = 0$

(E) **9** **a** Factorise $4xy - y^2 + 4x - y$. **(2 marks)**

b Solve the equation $4\sin\theta\cos\theta - \cos^2\theta + 4\sin\theta - \cos\theta = 0$, in the interval $0 \leqslant \theta \leqslant 2\pi$. **(5 marks)**

(E) **10** **a** Express $4\cos 3\theta - \sin(90° - 3\theta)$ as a single trigonometric function. **(1 mark)**

b Hence solve $4\cos 3\theta - \sin(90° - 3\theta) = 2$ in the interval $0 \leqslant \theta \leqslant 360°$. Give your answers to 3 significant figures. **(3 marks)**

(E/P) **11** Given that $2\sin 2\theta = \cos 2\theta$:

a Show that $\tan 2\theta = 0.5$. **(1 mark)**

b Hence find the values of θ, to 3 significant figures, in the interval $0 \leqslant \theta \leqslant 2\pi$ for which $2\sin 2\theta = \cos 2\theta$. **(4 marks)**

12 Find all the values of θ in the interval $0 \leqslant \theta \leqslant 360°$ for which:

a $\cos(\theta + 75°) = 0.5$,

b $\sin 2\theta = 0.7$, giving your answers to one decimal place.

(E) **13** Find the values of x in the interval $0 < x < 270°$ which satisfy the equation

$$\frac{\cos 2x + 0.5}{1 - \cos 2x} = 2$$ **(6 marks)**

(E) **14** Find, in radians, the values of θ in the interval $0 < \theta < 2\pi$ for which $2\cos^2\theta - \cos\theta - 1 = \sin^2\theta$

Give your answers to 3 significant figures, where appropriate. **(6 marks)**

(E/P) **15** A teacher asks one of his students to solve the equation $2\sin 3x = 1$ for $-360° \leqslant x \leqslant 360°$. The attempt is shown below:

> $\sin 3x = \frac{1}{2}$
> $3x = 30°$
> $x = 10°$
> Additional solution at $180° - 10° = 170°$

a Identify two mistakes made by the student. **(2 marks)**

b Solve the equation. **(2 marks)**

16 **a** Sketch the graphs of $y = 3\sin x$ and $y = 2\cos x$ on the same set of axes $(0 \leqslant x \leqslant 360°)$.

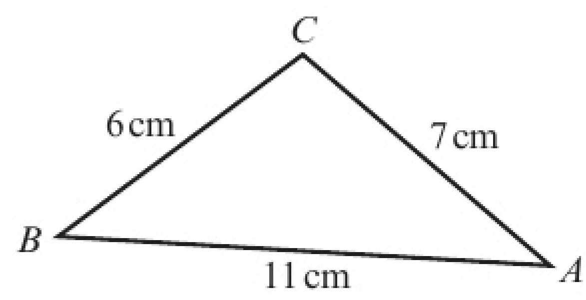
b Write down how many solutions there are in the given range for the equation $3\sin x = 2\cos x$.

c Solve the equation $3\sin x = 2\cos x$ algebraically, giving your answers to one decimal place.

E **17** The diagram shows the triangle ABC with $AB = 11$ cm, $BC = 6$ cm and $AC = 7$ cm.

a Find the exact value of $\cos B$, giving your answer in simplest form. **(3 marks)**

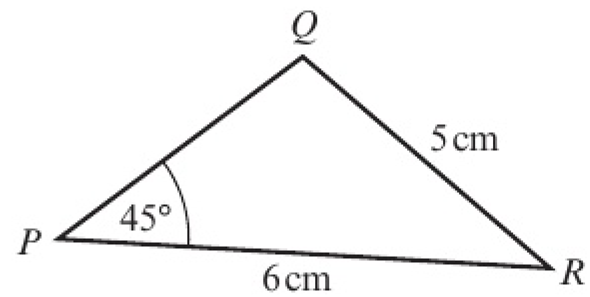
b Hence find the exact value of $\sin B$. **(2 marks)**



E/P **18** The diagram shows triangle PQR with $PR = 6$ cm, $QR = 5$ cm and angle $QPR = 45°$.

a Show that $\sin Q = \frac{3\sqrt{2}}{5}$ **(3 marks)**

b Given that Q is obtuse, find the exact value of $\cos Q$. **(2 marks)**



E/P **19** **a** Show that the equation $3\sin^2 x - \cos^2 x = 2$ can be written as $4\sin^2 x = 3$. **(2 marks)**

b Hence solve the equation $3\sin^2 x - \cos^2 x = 2$ in the interval $-\pi \leqslant x \leqslant \pi$, giving your answers to one decimal place. **(7 marks)**

E **20** Find all the solutions to the equation $3\cos^2 x + 1 = 4\sin x$ in the interval $-360° \leqslant x \leqslant 360°$, giving your answers to 1 decimal place. **(6 marks)**

Challenge

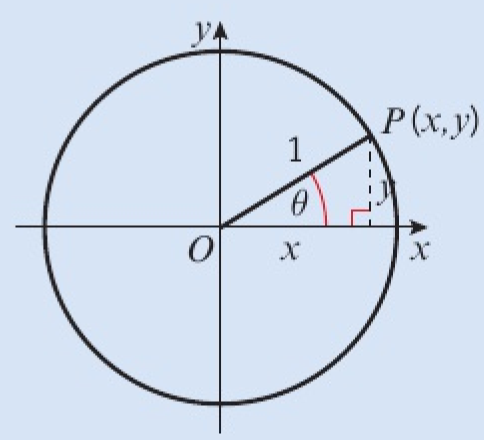
SKILLS INNOVATION

Solve the equation $\tan^4 x - 3\tan^2 x + 2 = 0$ in the interval $0 \leqslant x \leqslant 360°$.

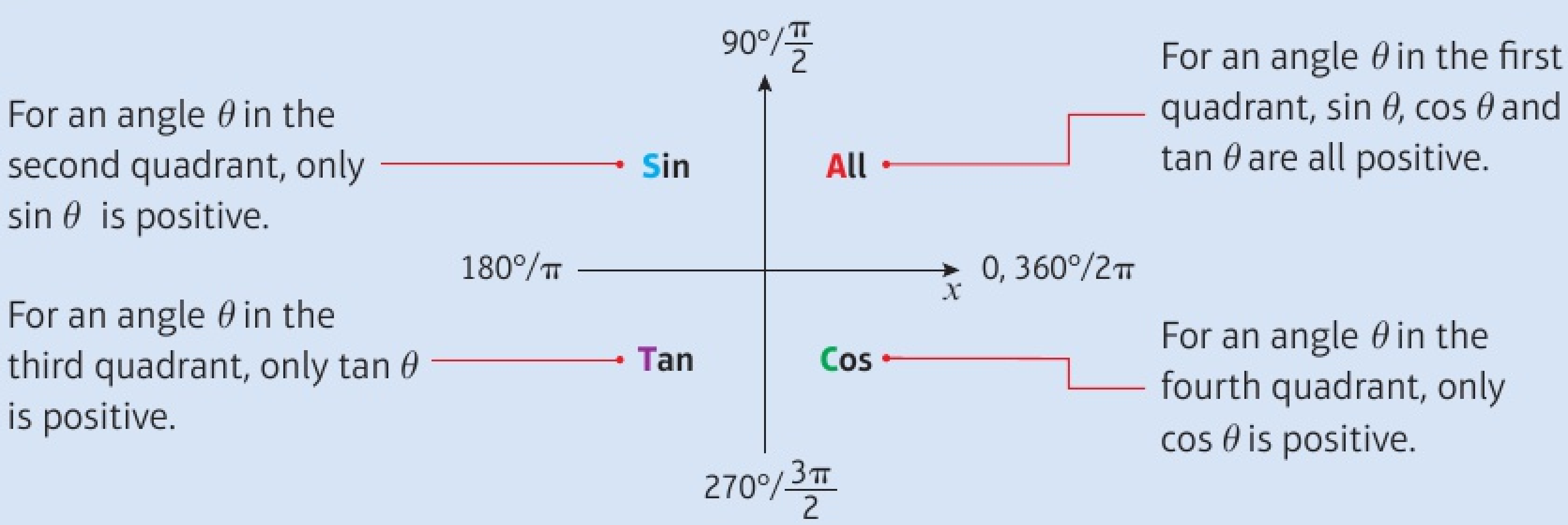
Summary of key points

1 For a point $P(x, y)$ on a unit circle such that OP makes an angle θ with the positive x-axis:

- $\cos\theta = x = x$-coordinate of P
- $\sin\theta = y = y$-coordinate of P
- $\tan\theta = \frac{y}{x} =$ gradient of OP



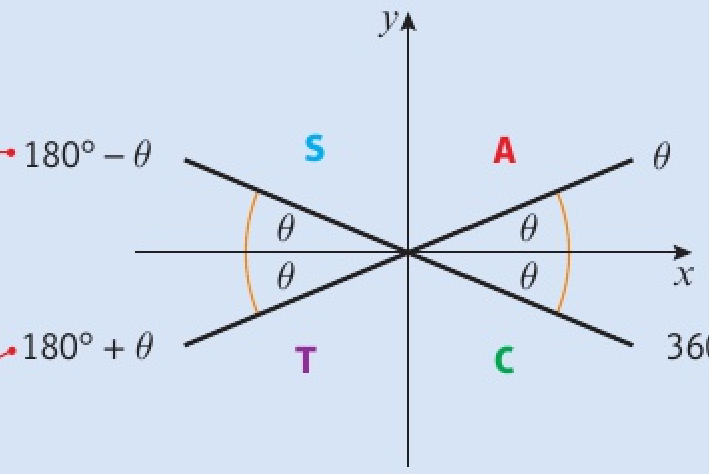
2 You can use the quadrants to determine whether each of the trigonometric ratios is positive or negative.



3 You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding **acute** angle made with the x-axis, θ.

$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

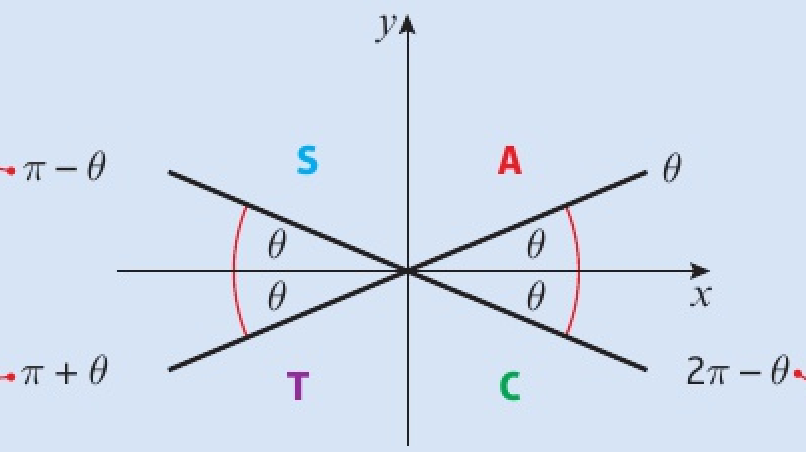
$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$



$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$

$\sin(\pi - \theta) = \sin\theta$
$\cos(\pi - \theta) = -\cos\theta$
$\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$
$\tan(\pi + \theta) = \tan\theta$



$\sin(2\pi - \theta) = -\sin\theta$
$\cos(2\pi - \theta) = \cos\theta$
$\tan(2\pi - \theta) = -\tan\theta$

4 The trigonometric ratios of 30°, 45° 60° and $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ have exact forms, given below:

$\sin 30° = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ $\qquad \cos 30° = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\qquad \tan 30° = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

$\sin 45° = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\qquad \cos 45° = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\qquad \tan 45° = \tan\left(\frac{\pi}{4}\right) = 1$

$\sin 60° = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\qquad \cos 60° = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $\qquad \tan 60° = \tan\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$

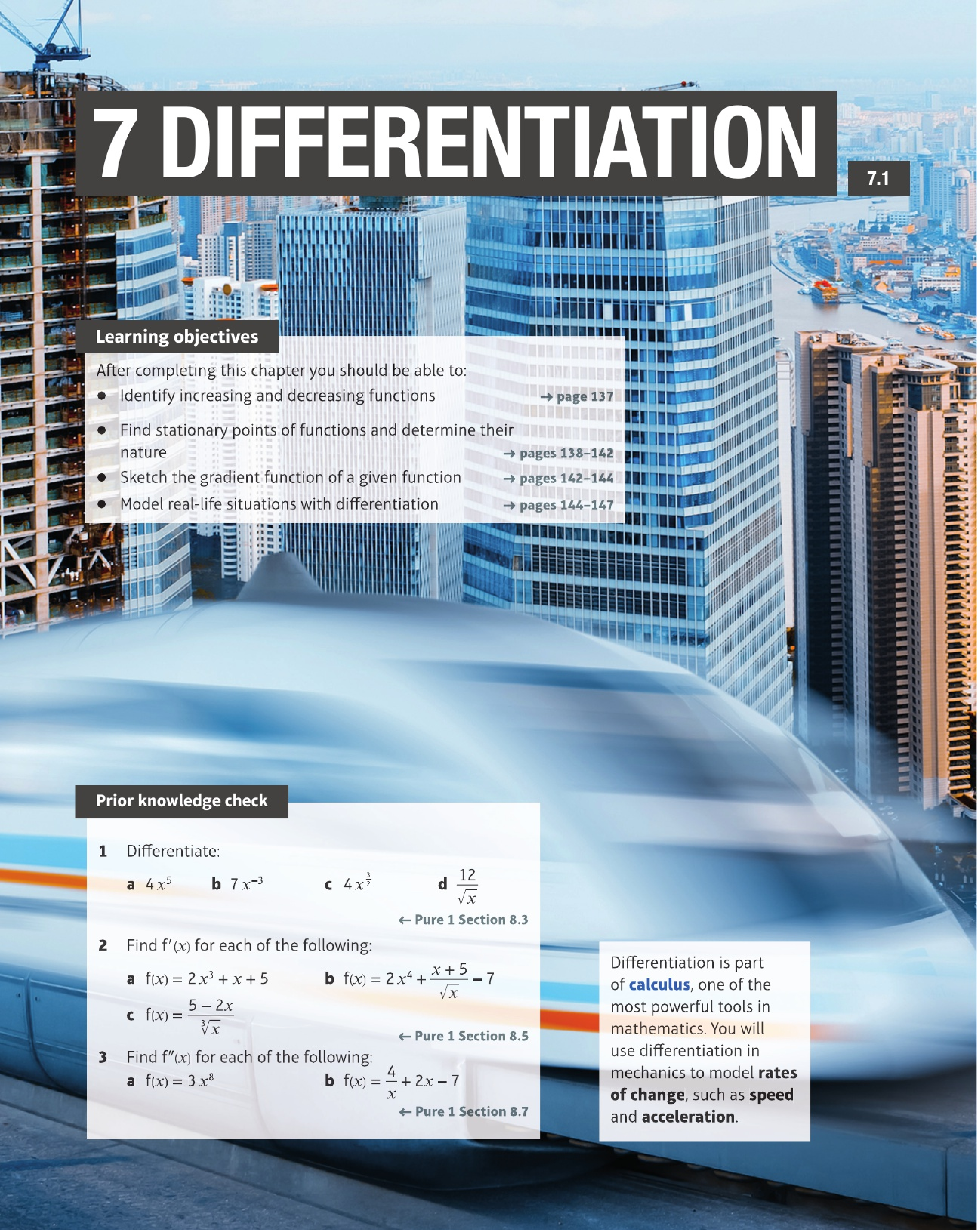
5 For all values of θ, $\sin^2\theta + \cos^2\theta \equiv 1$

6 For all values of θ such that $\cos\theta \neq 0$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

7
- Solutions to $\sin\theta = k$ and $\cos\theta = k$ only exist when $-1 \leqslant k \leqslant 1$
- Solutions to $\tan\theta = p$ exist for all values of p.

8 When you use the inverse trigonometric functions on your calculator, the angle you get is called the **principal value**.

9 Your calculator will give principal values in the following ranges:
- $\cos^{-1}$ in the range $0 \leqslant \theta \leqslant 180°$ in degrees, or $0 \leqslant \theta \leqslant \pi$ in radians
- $\sin^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$ in degress, or $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ in radians
- $\tan^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$ in degress, or $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ in radians



7 DIFFERENTIATION

Learning objectives

After completing this chapter you should be able to:

- Identify increasing and decreasing functions → page 137
- Find stationary points of functions and determine their nature → pages 138–142
- Sketch the gradient function of a given function → pages 142–144
- Model real-life situations with differentiation → pages 144–147

Prior knowledge check

1 Differentiate:

a $4x^5$ **b** $7x^{-3}$ **c** $4x^{\frac{3}{2}}$ **d** $\frac{12}{\sqrt{x}}$

← Pure 1 Section 8.3

2 Find $f'(x)$ for each of the following:

a $f(x) = 2x^3 + x + 5$ **b** $f(x) = 2x^4 + \frac{x+5}{\sqrt{x}} - 7$

c $f(x) = \frac{5-2x}{\sqrt[3]{x}}$

← Pure 1 Section 8.5

3 Find $f''(x)$ for each of the following:

a $f(x) = 3x^8$ **b** $f(x) = \frac{4}{x} + 2x - 7$

← Pure 1 Section 8.7

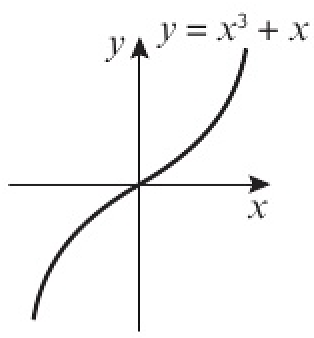
Differentiation is part of **calculus**, one of the most powerful tools in mathematics. You will use differentiation in mechanics to model **rates of change**, such as **speed** and **acceleration**.

7.1 Increasing and decreasing functions

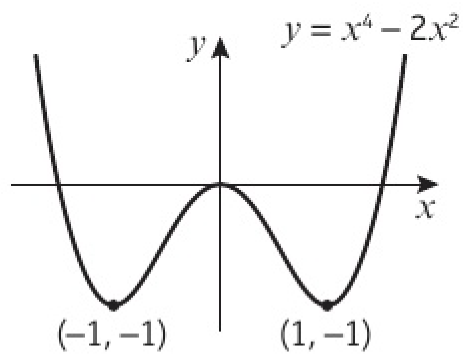
You can use the **derivative** to determine whether a function is **increasing** or **decreasing** on a given interval.

- The function f(x) is **increasing** on the interval [a, b] if f′(x) ⩾ 0 for all values of x such that $a < x < b$.
- The function f(x) is **decreasing** on the interval [a, b] if f′(x) ⩽ 0 for all values of x such that $a < x < b$.

Notation If f′(x) > 0 for all values in the interval the function is said to be **strictly** increasing. If you need to show that a function is increasing or decreasing in your exam you can use either strict or non-strict inequalities.



The function f(x) = $x^3 + x$ is increasing for all real values of x.



The function f(x) = $x^4 - 2x^2$ is increasing on the interval [–1, 0] and decreasing on the interval [0, 1].

Notation The **interval** [a, b] is the set of all real numbers, x, that satisfy $a \leqslant x \leqslant b$.

Example 1 SKILLS INTERPRETATION

Show that the function f(x) = $x^3 + 24x + 3$ is increasing for all real values of x.

f(x) = $x^3 + 24x + 3$
f′(x) = $3x^2 + 24$
$x^2 \geqslant 0$ for all real values of x
So $3x^2 + 24 \geqslant 0$ for all real values of x.
So f(x) is increasing for all real values of x.

First differentiate to obtain the gradient function.

State that the condition for an increasing function is met. In fact f′(x) ⩾ 24 for all real values of x.

Exercise 7A SKILLS INTERPRETATION

1 Find the values of x for which f(x) is an increasing function, given that f(x) equals:

a $3x^2 + 8x + 2$ **b** $4x - 3x^2$ **c** $5 - 8x - 2x^2$ **d** $2x^3 - 15x^2 + 36x$
e $3 + 3x - 3x^2 + x^3$ **f** $5x^3 + 12x$ **g** $x^4 + 2x^2$ **h** $x^4 - 8x^3$

2 Find the values of x for which f(x) is a decreasing function, given that f(x) equals:

a $x^2 - 9x$ **b** $5x - x^2$ **c** $4 - 2x - x^2$ **d** $2x^3 - 3x^2 - 12x$
e $1 - 27x + x^3$ **f** $x + \frac{25}{x}$ **g** $x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$ **h** $x^2(x + 3)$

E/P 3 Show that the function f(x) = $4 - x(2x^2 + 3)$ is decreasing for all $x \in \mathbb{R}$. **(3 marks)**

E/P 4 **a** Given that the function f(x) = $x^2 + px$ is increasing on the interval [–1, 1], find one possible value for p. **(2 marks)**

b State with justification whether this is the only possible value for p. **(1 mark)**

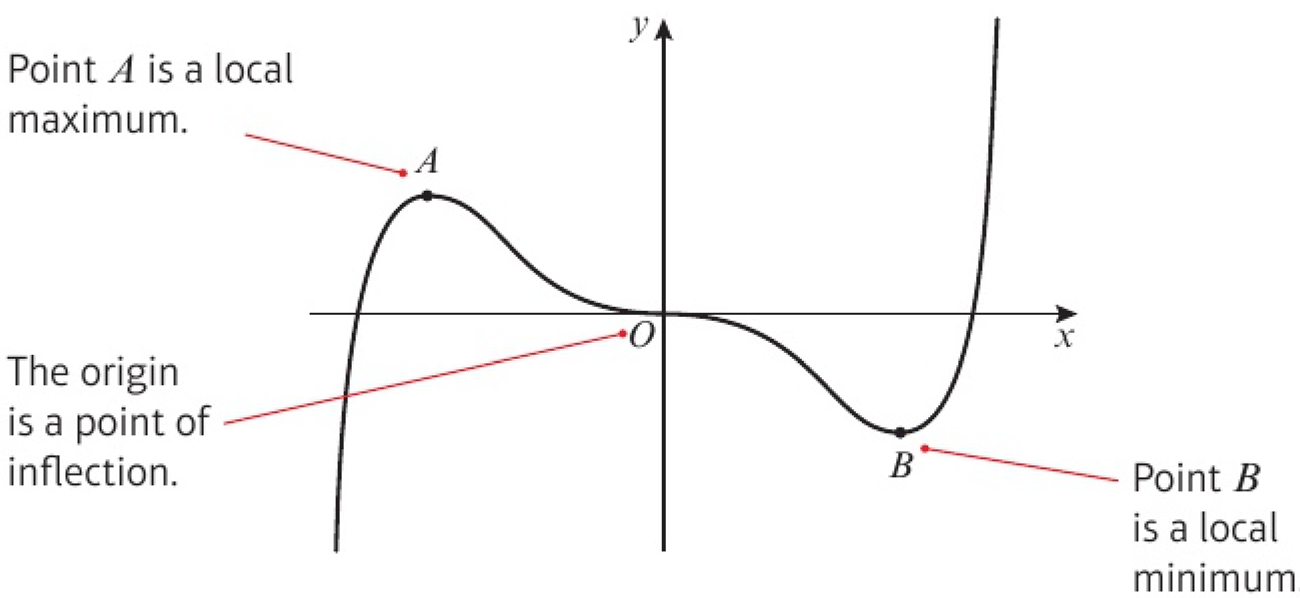
7.2 Stationary points

A **stationary point** on a curve is any point where the curve has **gradient zero**. You can determine whether a stationary point is a **local maximum**, a **local minimum** or a **point of inflection** by looking at the gradient of the curve on either side.

- Any point on the curve $y = f(x)$ where $f'(x) = 0$ is called a **stationary point**. For a small positive value h:

Type of stationary point	$f'(x-h)$	$f'(x)$	$f'(x+h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
Point of inflection	Negative	0	Negative
	Positive	0	Positive

Notation The plural of maximum is **maxima** and the plural of minimum is **minima**.



Notation Point A is called a **local** maximum because it is not the largest value the function can take. It is just the largest value in that immediate vicinity (the area nearest to).

Example 2 SKILLS EXECUTIVE FUNCTION

a Find the coordinates of the stationary point on the curve with equation $y = x^4 - 32x$.

b By considering points on either side of the stationary point, determine whether it is a local maximum, a local minimum or a point of inflection.

a $y = x^4 - 32x$

$\frac{dy}{dx} = 4x^3 - 32$

Let $\frac{dy}{dx} = 0$

Differentiate and let $\frac{dy}{dx} = 0$.

Then $4x^3 - 32 = 0$

$4x^3 = 32$

$x^3 = 8$

$x = 2$

Solve the equation to find the value of x.

So $y = 2^4 - 32 \times 2$

$= 16 - 64$

$= -48$

Substitute the value of x into the original equation to find the value of y.

So $(2, -48)$ is a stationary point.

b Now consider the gradient on either side of $(2, -48)$.

Make a table where you consider a value of x slightly less than 2 and a value of x slightly greater than 2.

Calculate the gradient for each of these values of x close to the stationary point.

Deduce the shape of the curve.

Value of x	$x = 1.9$	$x = 2$	$x = 2.1$
Gradient	−4.56 which is −ve	0	5.04 which is +ve
Shape of curve	\	—	/

From the shape of the curve, the point $(2, -48)$ is a local minimum point.

Online Explore the solution using technology.

In some cases you can use the **second derivative**, $f''(x)$, to determine the nature of a stationary point.

- If a function $f(x)$ has a stationary point when $x = a$, then:
  - if $f''(a) > 0$, the point is a local minimum
  - if $f''(a) < 0$, the point is a local maximum

  If $f''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to look at points on either side to determine its nature.

Hint $f''(x)$ tells you the **rate of change** of the gradient function. When $f'(x) = 0$ and $f''(x) > 0$ the gradient is **increasing** from a negative value to a positive value, so the stationary point is a **minimum**.

Example 3 SKILLS EXECUTIVE FUNCTION

a Find the coordinates of the stationary points on the curve with equation

$y = 2x^3 - 15x^2 + 24x + 6$

b Find $\frac{d^2y}{dx^2}$ and use it to determine the nature of the stationary points.

a $y = 2x^3 - 15x^2 + 24x + 6$

$\frac{dy}{dx} = 6x^2 - 30x + 24$

Differentiate and put the derivative equal to zero.

Putting $6x^2 - 30x + 24 = 0$

$6(x - 4)(x - 1) = 0$

So $x = 4$ or $x = 1$

Solve the equation to obtain the values of x for the stationary points.

When $x = 1$,

$y = 2 - 15 + 24 + 6 = 17$

When $x = 4$,

$y = 2 \times 64 - 15 \times 16 + 24 \times 4 + 6$

$= -10$

Substitute $x = 1$ and $x = 4$ into the original equation of the curve to obtain the values of y which correspond to these values.

So the stationary points are at (1, 17) and (4, −10).

b $\frac{d^2y}{dx^2} = 12x - 30$

Differentiate again to obtain the second derivative.

When $x = 1$, $\frac{d^2y}{dx^2} = -18$ which is < 0

So (1, 17) is a local maximum point.

When $x = 4$, $\frac{d^2y}{dx^2} = 18$ which is > 0

So (4, −10) is a local minimum point.

Substitute $x = 1$ and $x = 4$ into the second derivative expression. If the second derivative is negative then the point is a local maximum point. If it is positive then the point is a local minimum point.

Example 4

a The curve with equation $y = \frac{1}{x} + 27x^3$ has stationary points at $x = \pm a$. Find the value of a.

b Sketch the graph of $y = \frac{1}{x} + 27x^3$.

a $y = x^{-1} + 27x^3$

Write $\frac{1}{x}$ as x^{-1} to differentiate.

$\frac{dy}{dx} = -x^{-2} + 81x^2 = -\frac{1}{x^2} + 81x^2$

When $\frac{dy}{dx} = 0$:

Set $\frac{dy}{dx} = 0$ to determine the x-coordinates of the stationary points.

$-\frac{1}{x^2} + 81x^2 = 0$

$81x^2 = \frac{1}{x^2}$

$81x^4 = 1$

$x^4 = \frac{1}{81}$

$x = \pm\frac{1}{3}$

You need to consider the positive and negative roots: $\left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) = \frac{1}{81}$

So $a = \frac{1}{3}$

b $\frac{d^2y}{dx^2} = 2x^{-3} + 162x = \frac{2}{x^3} + 162x$

When $x = -\frac{1}{3}$, $y = \frac{1}{\left(-\frac{1}{3}\right)} + 27\left(-\frac{1}{3}\right)^3 = -4$

and $\frac{d^2y}{dx^2} = \frac{2}{\left(-\frac{1}{3}\right)^3} + 162\left(-\frac{1}{3}\right) = -108$

which is negative.

So the curve has a local maximum at $\left(-\frac{1}{3}, -4\right)$.

When $x = \frac{1}{3}$,

$y = \frac{1}{\left(\frac{1}{3}\right)} + 27\left(\frac{1}{3}\right)^3 = 4$

and

$\frac{d^2y}{dx^2} = \frac{2}{\left(\frac{1}{3}\right)^3} + 162\left(\frac{1}{3}\right) = 108$

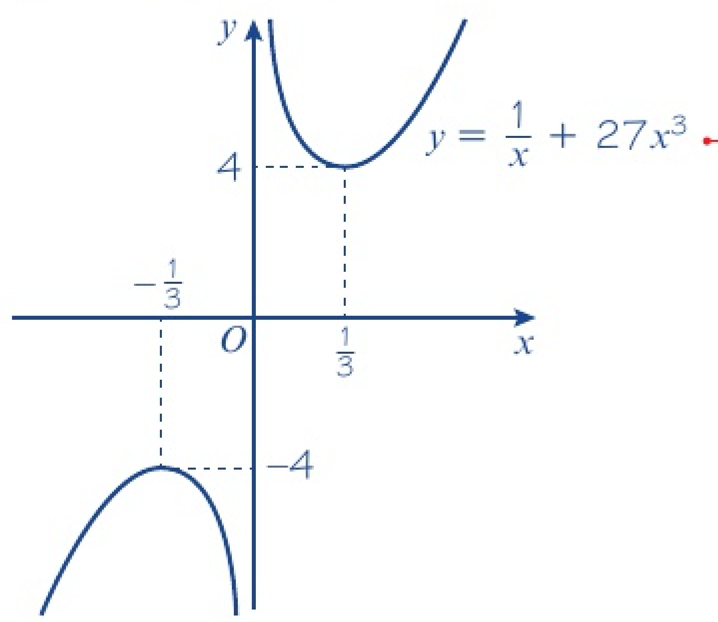
which is positive.

So the curve has a local minimum at $\left(\frac{1}{3}, 4\right)$.

The curve has an asymptote at $x = 0$.

As $x \to \infty$, $y \to \infty$.

As $x \to -\infty$, $y \to -\infty$.



To sketch the curve, you need to find the coordinates of the stationary points and determine their natures. Differentiate your expression for $\frac{dy}{dx}$ to find $\frac{d^2y}{dx^2}$

Substitute $x = -\frac{1}{3}$ and $x = \frac{1}{3}$ into the equation of the curve to find the y-coordinates of the stationary points.

Online Check your solution using your calculator.

$\frac{1}{x} \to \pm\infty$ as $x \to 0$ so $x = 0$ is an asymptote of the curve.

Mark the coordinates of the stationary points on your sketch, and label the curve with its equation.

You could check $\frac{dy}{dx}$ at specific points to help with your sketch:

- When $x = \frac{1}{4}$, $\frac{dy}{dx} = -10.9375$ which is negative.
- When $x = 1$, $\frac{dy}{dx} = 80$ which is positive.

Exercise 7B

SKILLS EXECUTIVE FUNCTION

1 Find the least value of the following functions:

a $f(x) = x^2 - 12x + 8$ **b** $f(x) = x^2 - 8x - 1$

c $f(x) = 5x^2 + 2x$

2 Find the greatest value of the following functions:

a $f(x) = 10 - 5x^2$ **b** $f(x) = 3 + 2x - x^2$

c $f(x) = (6 + x)(1 - x)$

Hint For each part of questions **1** and **2**:

- Find $f'(x)$.
- Set $f'(x) = 0$ and solve to find the value of x at the stationary point.
- Find the corresponding value of $f(x)$.

3 Find the coordinates of the points where the gradient is zero on the curves with the given equations. Establish whether these points are local maximum points, local minimum points or points of inflection in each case.

a $y = 4x^2 + 6x$ **b** $y = 9 + x - x^2$ **c** $y = x^3 - x^2 - x + 1$

d $y = x(x^2 - 4x - 3)$ **e** $y = x + \frac{1}{x}$ **f** $y = x^2 + \frac{54}{x}$

g $y = x - 3\sqrt{x}$ **h** $y = x^{\frac{1}{2}}(x - 6)$ **i** $y = x^4 - 12x^2$

4 Sketch the curves with equations given in question **3** parts **a**, **b**, **c** and **d**, labelling any stationary points with their coordinates.

(P) 5 By considering the gradient on either side of the stationary point on the curve $y = x^3 - 3x^2 + 3x$, show that this point is a point of inflection.
Sketch the curve $y = x^3 - 3x^2 + 3x$.

(P) 6 Find the maximum value and hence the range of values for the function $f(x) = 27 - 2x^4$.

(P) 7 $f(x) = x^4 + 3x^3 - 5x^2 - 3x + 1$

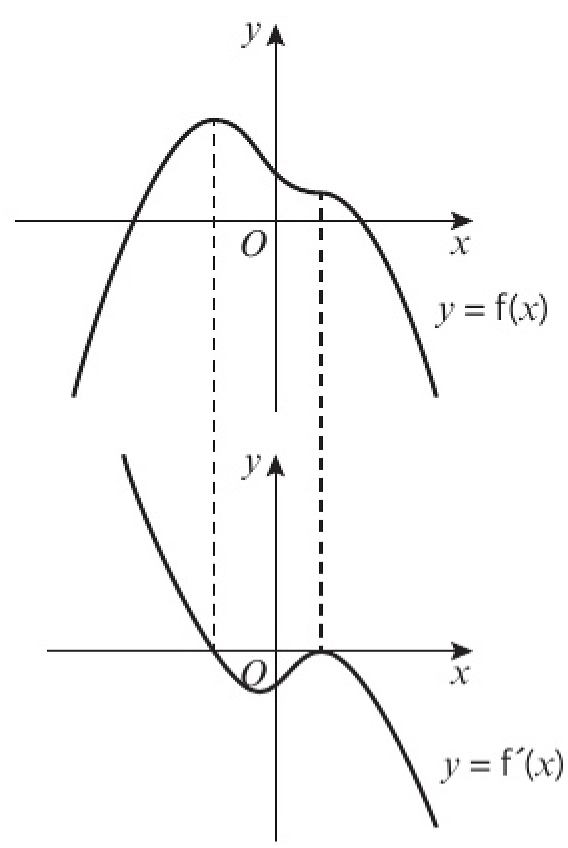
a Find the coordinates of the stationary points of $f(x)$, and determine the nature of each.

b Sketch the graph of $y = f(x)$.

Hint Use the **factor theorem** with small positive integer values of x to find one factor of $f'(x)$. ← Pure 2 Section 7.2

7.3 Sketching gradient functions

You can use the features of a given function to sketch the corresponding gradient function. This table shows you features of the graph of a function, $y = f(x)$, and the graph of its gradient function, $y = f'(x)$, at corresponding values of x.

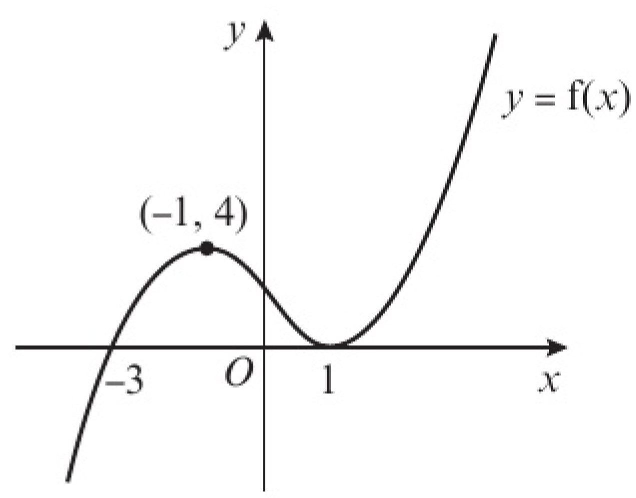


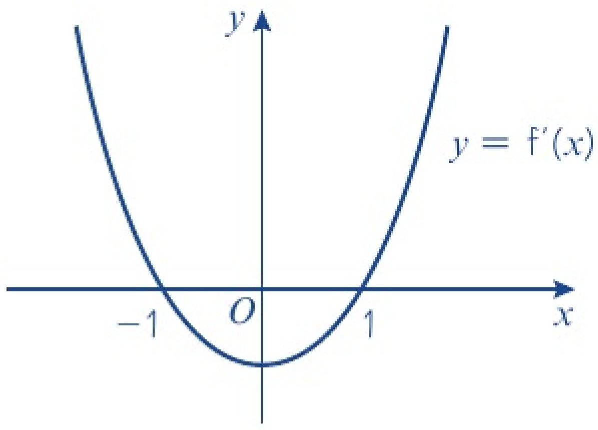
$y = f(x)$	$y = f'(x)$
Maximum or minimum	Cuts the x-axis
Point of inflection	Touches the x-axis
Positive gradient	Above the x-axis
Negative gradient	Below the x-axis
Vertical asymptote	Vertical asymptote
Horizontal asymptote	Horizontal asymptote at the x-axis

Example 5

The diagram shows the curve with equation $y = f(x)$. The curve has stationary points at $(-1, 4)$ and $(1, 0)$, and cuts the x-axis at $(-3, 0)$.

Sketch the gradient function, $y = f'(x)$, showing the coordinates of any points where the curve cuts or meets the x-axis.

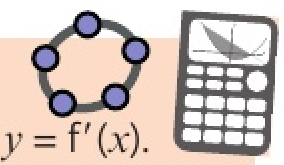




x	$y = \mathrm{f}(x)$	$y = \mathrm{f}'(x)$
$x < -1$	Positive gradient	Above x-axis
$x = -1$	Maximum	Cuts x-axis
$-1 < x < 1$	Negative gradient	Below x-axis
$x = 1$	Minimum	Cuts x-axis
$x > 1$	Positive gradient	Above x-axis

Watch out Ignore any points where the curve $y = \mathrm{f}(x)$ cuts the x-axis. These will not tell you anything about the features of the graph of $y = \mathrm{f}'(x)$.

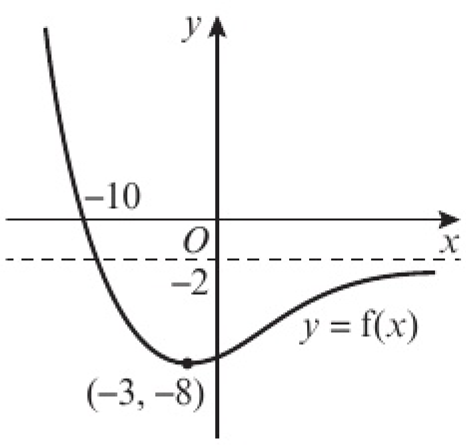
Online Use technology to explore the key features linking $y = \mathrm{f}(x)$ and $y = \mathrm{f}'(x)$.



Example 6

SKILLS ANALYSIS

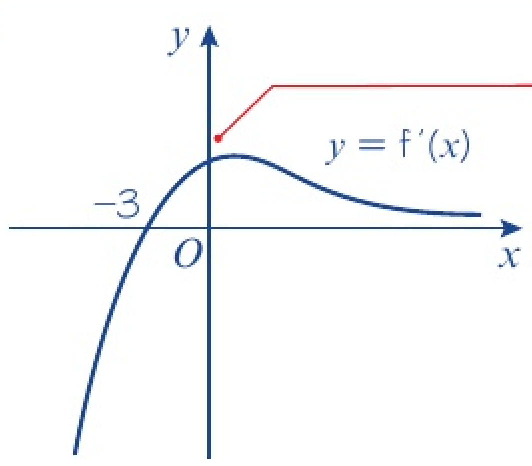
The diagram shows the curve with equation $y = \mathrm{f}(x)$. The curve has an asymptote at $y = -2$ and a **turning point** at $(-3, -8)$.
It cuts the x-axis at $(-10, 0)$.



a Sketch the graph of $y = \mathrm{f}'(x)$.

b State the equation of the asymptote of $y = \mathrm{f}'(x)$.

a



b $y = 0$

Draw your sketch on a separate set of axes. The graph of $y = \mathrm{f}'(x)$ will have the same horizontal scale but will have a different vertical scale.

You don't have enough information to work out the coordinates of the y-intercept, or the local maximum, of the graph of the gradient function.
The graph of $y = \mathrm{f}(x)$ is a smooth curve so the graph of $y = \mathrm{f}'(x)$ will also be a smooth curve.

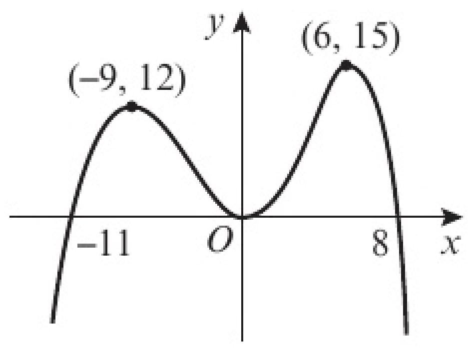
If $y = \mathrm{f}(x)$ has any **horizontal asymptotes** then the graph of $y = \mathrm{f}'(x)$ will have an asymptote at the x-axis.

Exercise 7C

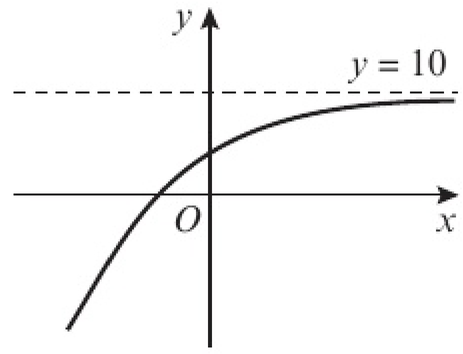
SKILLS ANALYSIS

1 For each graph given, sketch the graph of the corresponding gradient function on a separate set of axes. Show the coordinates of any points where the curve cuts or meets the x-axis, and give the equations of any asymptotes.

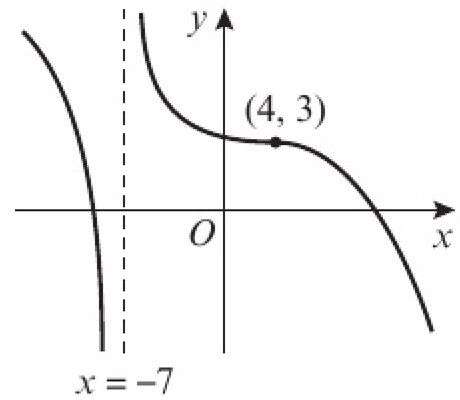
a



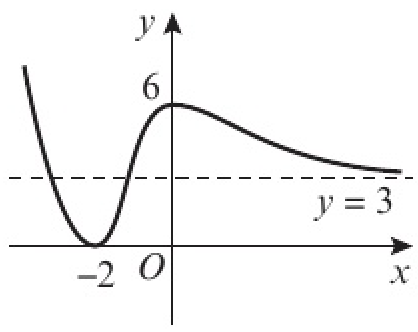
b



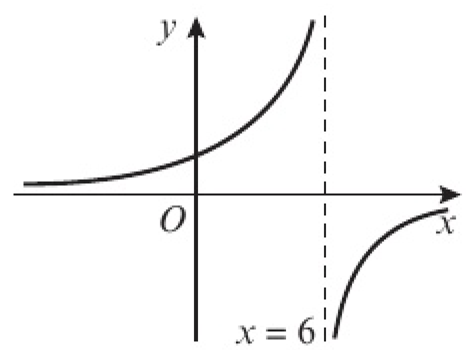
c



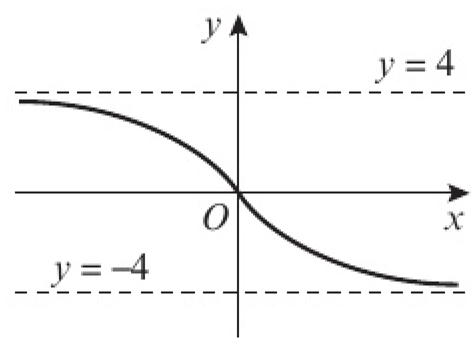
d



e



f



(P) 2 $f(x) = (x + 1)(x - 4)^2$

a Sketch the graph of $y = f(x)$.

b On a separate set of axes, sketch the graph of $y = f'(x)$.

c Show that $f'(x) = (x - 4)(3x - 2)$.

d Use the derivative to determine the exact coordinates of the points where the gradient function cuts the coordinate axes.

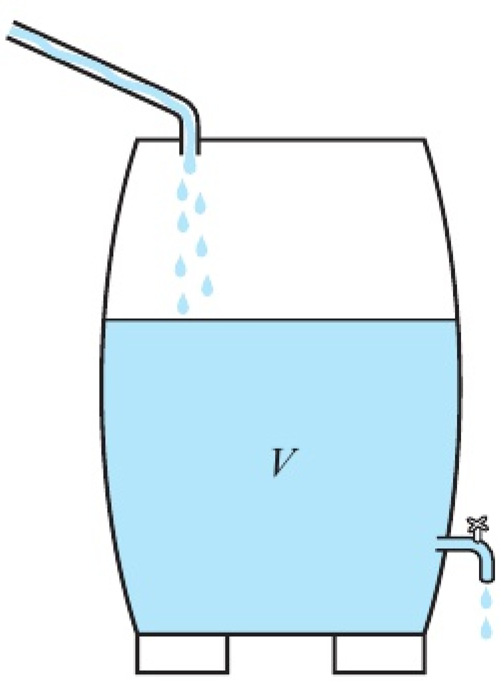
Hint This is an x^3 graph with a positive coefficient of x^3.

← Pure 1 Section 4.1

7.4 Modelling with differentiation

You can think of $\frac{dy}{dx}$ as $\frac{\text{small change in } y}{\text{small change in } x}$. It represents the **rate of change** of y with respect to x.

If you replace y and x with variables that represent real-life quantities, you can use the derivative to model lots of real-life situations involving rates of change.



The volume of water in this water container is constantly changing over time. If V represents the volume of water in the water container in litres, and t represents the time in seconds, then you could model V as a function of t. If $V = f(t)$ then $\frac{dV}{dt} = f'(t)$ would represent the **rate of change** of volume with respect to time. The units of $\frac{dV}{dt}$ would be litres per second.

Example 7

Given that the volume, V cm^3, of an expanding sphere is related to its radius, r cm, by the formula $V = \frac{4}{3}\pi r^3$, find the rate of change of volume with respect to radius at the instant when the radius is 5 cm.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

When $r = 5$, $\frac{dV}{dr} = 4\pi \times 5^2$

$= 314$ (3 s.f.)

So the rate of change is 314 cm^3 per cm.

Differentiate V with respect to r. Remember that π is a constant.

Substitute $r = 5$.

Interpret the answer with units.

Example 8 SKILLS EXECUTIVE FUNCTION

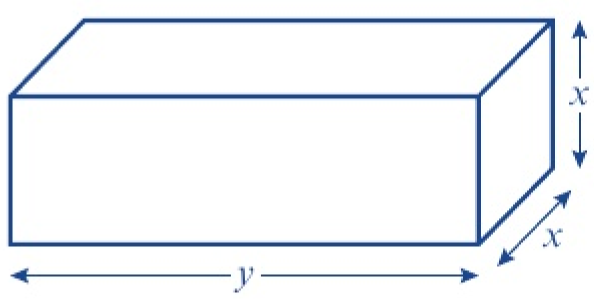
A large tank in the shape of a cuboid is to be made from 54 m^2 of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two opposite vertical faces are squares.

a Show that the volume, V m^3, of the tank is given by $V = 18x - \frac{2}{3}x^3$

b Given that x can vary, use differentiation to find the maximum or minimum value of V.

c Justify that the value of V you have found is a maximum.

a Let the length of the tank be y metres.



Total area, $A = 2x^2 + 3xy$

So $54 = 2x^2 + 3xy$

$y = \frac{54 - 2x^2}{3x}$

But $V = x^2y$

So $V = x^2\left(\frac{54 - 2x^2}{3x}\right)$

$= \frac{x}{3}(54 - 2x^2)$

So $V = 18x - \frac{2}{3}x^3$

Problem-solving

You don't know the length of the tank. Write it as y metres to simplify your working.
You could also draw a sketch to help you find the correct expressions for the surface area and volume of the tank.

Draw a sketch.

Rearrange to find y in terms of x.

Substitute the expression for y into the equation.

Simplify.

b $\frac{dV}{dx} = 18 - 2x^2$

Put $\frac{dV}{dx} = 0$

$0 = 18 - 2x^2$

So $x^2 = 9$

$x = -3$ or 3

But x is a length so $x = 3$

When $x = 3$, $V = 18 \times 3 - \frac{2}{3} \times 3^3$

$= 54 - 18$

$= 36$

$V = 36$ is a maximum or minimum value of V.

Differentiate V with respect to x and put $\frac{dV}{dx} = 0$.

Rearrange to find x.
x is a length so use the positive solution.

Substitute the value of x into the expression for V.

c $\frac{d^2V}{dx^2} = -4x$

When $x = 3$, $\frac{d^2V}{dx^2} = -4 \times 3 = -12$

This is negative, so $V = 36$ is the maximum value of V.

Find the second derivative of V.

$\frac{d^2V}{dx^2} < 0$ so $V = 36$ is a maximum.

Exercise 7D

1 Find $\frac{d\theta}{dt}$ where $\theta = t^2 - 3t$.

2 Find $\frac{dA}{dr}$ where $A = 2\pi r$.

3 Given that $r = \frac{12}{t}$, find the value of $\frac{dr}{dt}$ when $t = 3$.

4 The surface area, A cm², of an expanding sphere of radius r cm is given by $A = 4\pi r^2$. Find the rate of change of the area with respect to the radius at the instant when the radius is 6 cm.

5 The displacement, s metres, of a car from a fixed point at time t seconds is given by $s = t^2 + 8t$. Find the rate of change of the displacement with respect to time at the instant when $t = 5$.

(P) **6** A rectangular garden is fenced on three sides, and the house forms the fourth side of the rectangle.

a Given that the total length of the fence is 80 m, show that the area, A, of the garden is given by the formula $A = y(80 - 2y)$, where y is the distance from the house to the end of the garden.

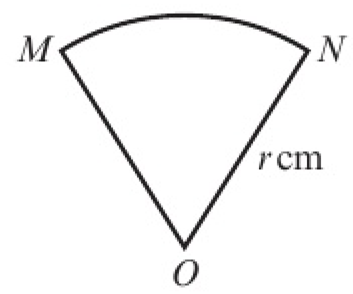
b Given that the area is a maximum for this length of fence, find the dimensions of the enclosed garden, and the area which is enclosed.

P **7** A closed cylinder has total surface area equal to 600π.

a Show that the volume, $V\text{ cm}^3$, of this cylinder is given by the formula $V = 300\pi r - \pi r^3$, where r cm is the radius of the cylinder.

b Find the maximum volume of such a cylinder.

P **8** A sector of a circle has area 100 cm^2.

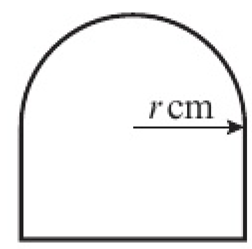


a Show that the perimeter of this **sector** is given by the formula

$$P = 2r + \frac{200}{r},\ r > \sqrt{\frac{100}{\pi}}$$

b Find the minimum value for the perimeter.

E/P **9** A shape consists of a rectangular base with a semicircular top, as shown.



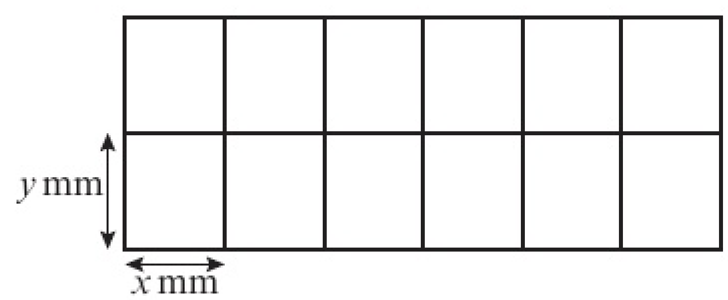
a Given that the perimeter of the shape is 40 cm, show that its area, $A\text{ cm}^2$, is given by the formula

$$A = 40r - 2r^2 - \frac{\pi r^2}{2}$$

where r cm is the radius of the semicircle. **(2 marks)**

b Hence find the maximum value for the area of the shape. **(4 marks)**

E/P **10** The shape shown is a wire frame in the form of a large rectangle split by parallel lengths of wire into 12 smaller equal-sized rectangles.



a Given that the total length of wire used to complete the whole frame is 1512 mm, show that the area of the whole shape, $A\text{ mm}^2$, is given by the formula

$$A = 1296x - \frac{108x^2}{7}$$

where x mm is the width of one of the smaller rectangles. **(4 marks)**

b Hence find the maximum area which can be enclosed in this way. **(4 marks)**

Chapter review 7

SKILLS EXECUTIVE FUNCTION

E **1** Given that $y = x^{\frac{3}{2}} + \frac{48}{x},\ x > 0$

a find the value of x and the value of y when $\frac{dy}{dx} = 0$. **(5 marks)**

b show that the value of y which you found in part **a** is a minimum. **(2 marks)**

2 A curve has equation $y = x^3 - 5x^2 + 7x - 14$. Determine, by calculation, the coordinates of the stationary points of the curve.

E/P **3** The function f, defined for $x \in \mathbb{R},\ x > 0$, is such that:

$$f'(x) = x^2 - 2 + \frac{1}{x^2}$$

a Find the value of $f''(x)$ at $x = 4$. **(4 marks)**

b Prove that f is an increasing function. **(3 marks)**

E **4** A curve has equation $y = x^3 - 6x^2 + 9x$. Find the coordinates of its local maximum. **(4 marks)**

5 $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 20$

a Find the coordinates of the stationary points of $f(x)$, and determine the nature of each of them.

b Sketch the graph of $y = f(x)$.

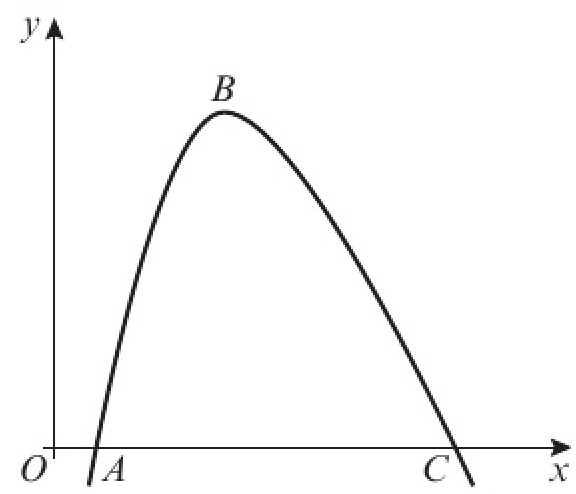
E **6** The diagram shows part of the curve with equation $y = f(x)$, where:

$$f(x) = 200 - \frac{250}{x} - x^2,\ x > 0$$

The curve cuts the x-axis at the points A and C.
The point B is the maximum point of the curve.

a Find $f'(x)$. **(3 marks)**

b Use your answer to part **a** to calculate the coordinates of B. **(4 marks)**



E/P **7** The diagram shows the part of the curve with equation $y = 5 - \frac{1}{2}x^2$ for which $y > 0$.

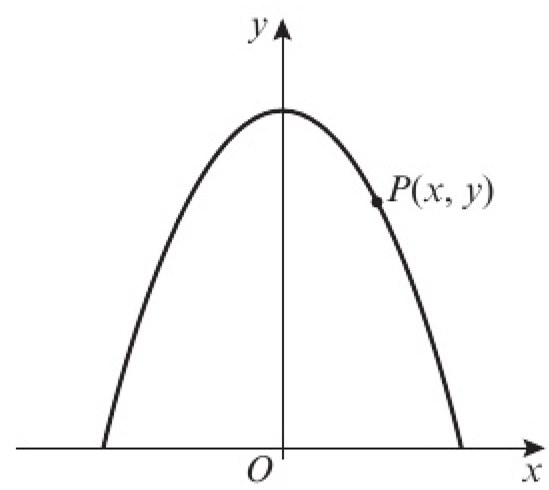
The point $P(x, y)$ lies on the curve and O is the origin.

a Show that $OP^2 = \frac{1}{4}x^4 - 4x^2 + 25$. **(3 marks)**

Taking $f(x) = \frac{1}{4}x^4 - 4x^2 + 25$:

b Find the values of x for which $f'(x) = 0$. **(4 marks)**

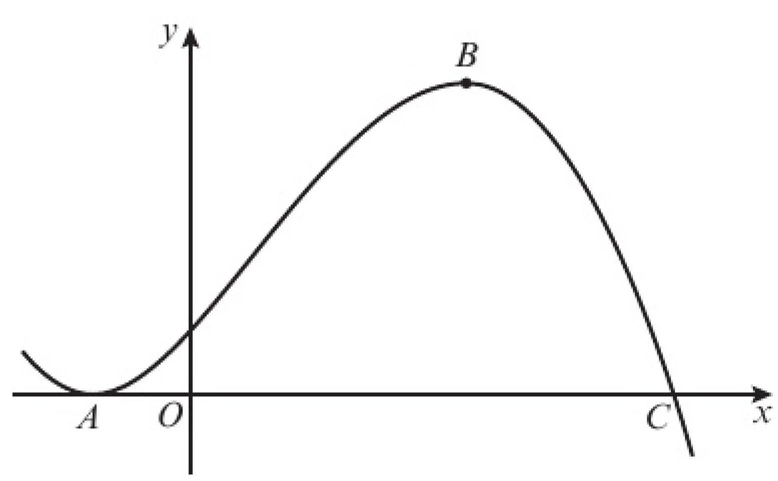
c Hence, or otherwise, find the minimum distance from O to the curve, showing that your answer is a minimum. **(4 marks)**



E **8** The diagram shows part of the curve with equation $y = 3 + 5x + x^2 - x^3$. The curve touches the x-axis at A and crosses the x-axis at C. The points A and B are stationary points on the curve.

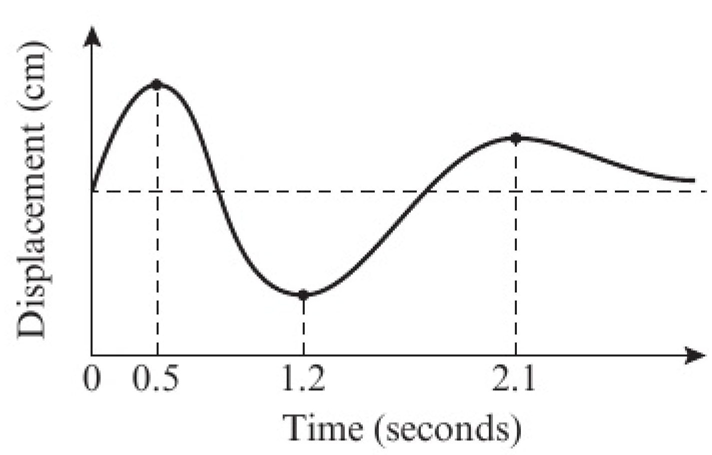
a Show that C has coordinates $(3, 0)$. **(1 mark)**

b Using calculus and showing all your working, find the coordinates of A and B. **(5 marks)**



P **9** The motion of a damped spring is modelled using this graph.

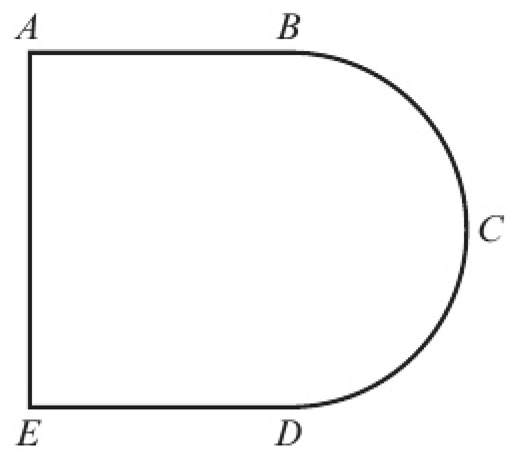
On a separate graph, sketch the gradient function for this model. Choose suitable labels and units for each axis, and indicate the coordinates of any points where the gradient function crosses the horizontal axis.



10 The volume, $V\text{ cm}^3$, of a tin of radius r cm is given by the formula $V = \pi(40r - r^2 - r^3)$. Find the positive value of r for which $\frac{dV}{dr} = 0$, and find the value of V which corresponds to this value of r.

(P) **11** The total surface area, $A\text{ cm}^2$, of a cylinder with a fixed volume of 1000 cm^3 is given by the formula $A = 2\pi x^2 + \frac{2000}{x}$, where x cm is the radius. Show that when the rate of change of the area with respect to the radius is zero, $x^3 = \frac{500}{\pi}$

(E/P) **12** A wire is bent into the plane shape $ABCDE$ as shown. Shape $ABDE$ is a rectangle and BCD is a semicircle with diameter BD. The area of the region enclosed by the wire is $R\text{ m}^2$, $AE = x$ metres, and $AB = ED = y$ metres. The total length of the wire is 2 m.



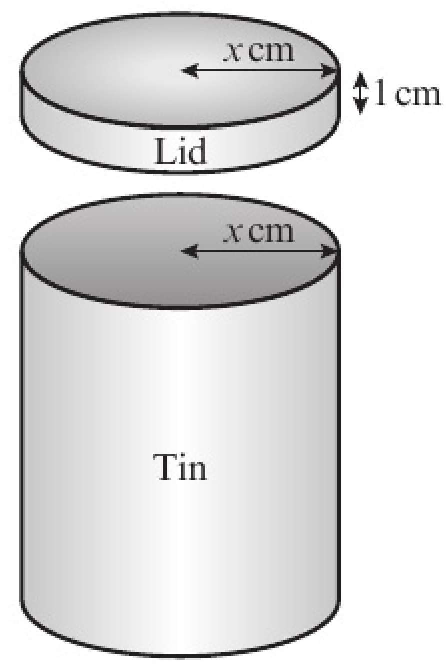
a Find an expression for y in terms of x. **(3 marks)**

b Prove that $R = \frac{x}{8}(8 - 4x - \pi x)$. **(4 marks)**

Given that x can vary, using calculus and showing your working:

c find the maximum value of R. (You do not have to prove that the value you obtain is a maximum.) **(5 marks)**

(E/P) **13** A cylindrical biscuit tin has a close-fitting lid which overlaps the tin by 1 cm, as shown. The radii of the tin and the lid are both x cm. The tin and the lid are made from a thin sheet of metal of area $80\pi\text{ cm}^2$ and there is no wastage. The volume of the tin is $V\text{ cm}^3$.



a Show that $V = \pi(40x - x^2 - x^3)$. **(5 marks)**

Given that x can vary:

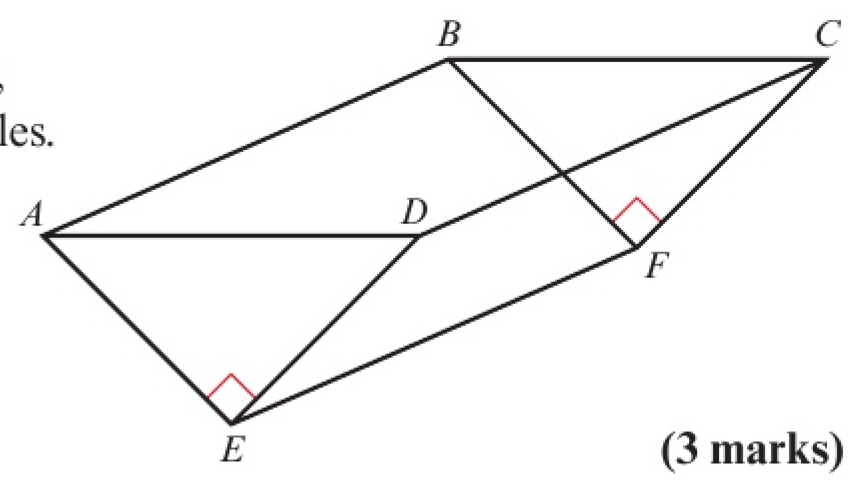
b use differentiation to find the positive value of x for which V is stationary. **(3 marks)**

c Prove that this value of x gives a maximum value of V. **(2 marks)**

d Find this maximum value of V. **(1 mark)**

e Determine the percentage of the sheet metal used in the lid when V is a maximum. **(2 marks)**

(E) **14** The diagram shows an open tank for storing water, $ABCDEF$. The sides $ABFE$ and $CDEF$ are rectangles. The triangular ends ADE and BCF are isosceles, and $\angle AED = \angle BFC = 90°$. The ends ADE and BCF are vertical and EF is horizontal.



Given that $AD = x$ metres:

a show that the area of triangle ADE is $\frac{1}{4}x^2\text{ m}^2$ **(3 marks)**

Given also that the capacity of the container is 4000 m^3 and that the total area of the two triangular and two rectangular sides of the container is $S\text{ m}^2$:

b show that $S = \frac{x^2}{2} + \frac{16\,000\sqrt{2}}{x}$ **(4 marks)**

Given that x can vary:

c use calculus to find the minimum value of S **(6 marks)**

d justify that the value of S you have found is a minimum. **(2 marks)**

Challenge

SKILLS CREATIVITY

Define functions that are:

a both increasing and decreasing for all $x \in \mathbb{R}$

b strictly increasing on [0, 1] and strictly decreasing on [1, 2]

c increasing for $0 < x < 1$, but neither increasing nor decreasing for $0 \leqslant x \leqslant 1$

d increasing on the rational numbers and decreasing on the irrational numbers.

Summary of key points

1
- The function f(x) is **increasing** on the interval $[a, b]$ if $f'(x) \geqslant 0$ for all values of x such that $a < x < b$.
- The function f(x) is **decreasing** on the interval $[a, b]$ if $f'(x) \leqslant 0$ for all values of x such that $a < x < b$.

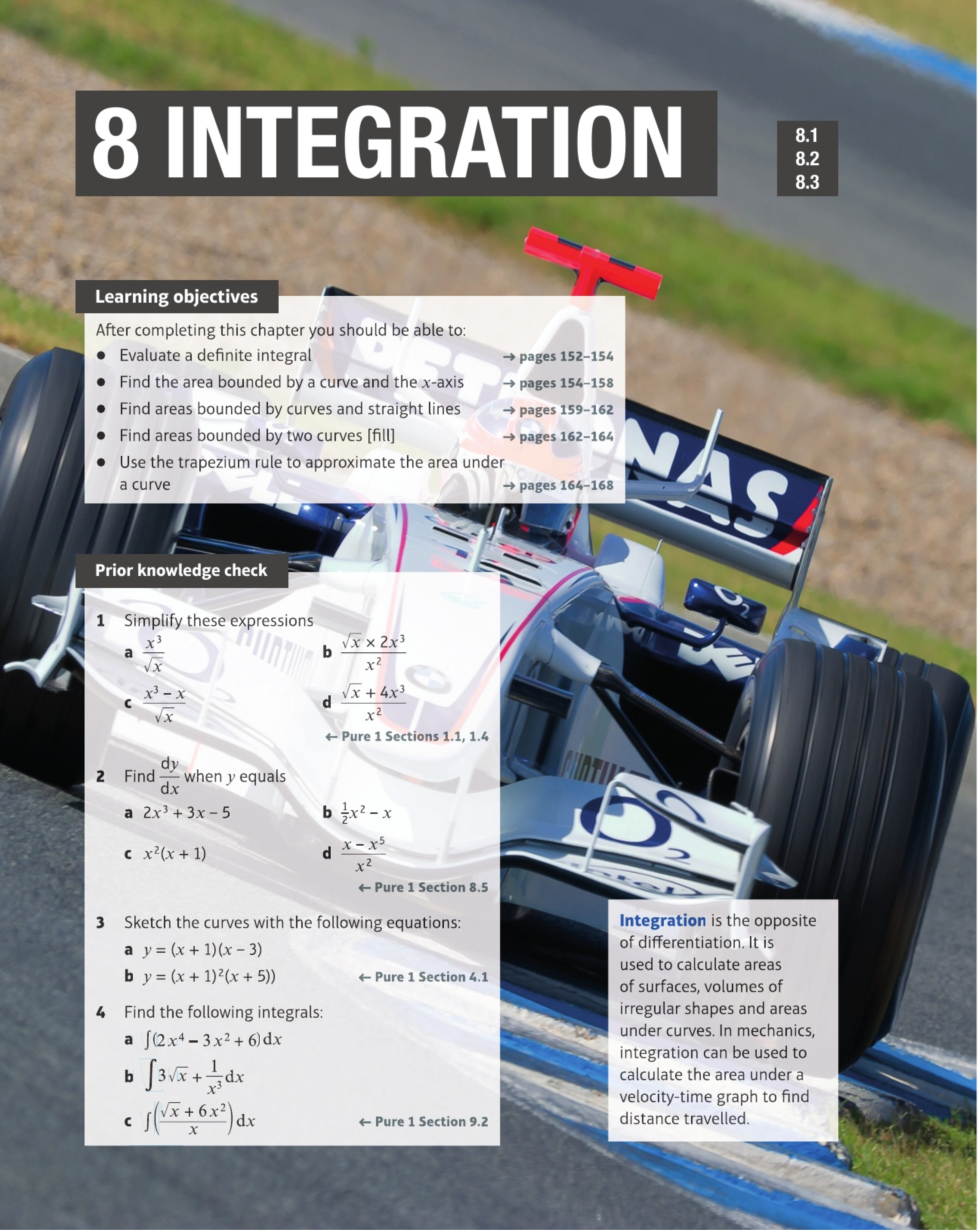
2 Differentiating a function $y = f(x)$ twice gives you the second order derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$

3 Any point on the curve $y = f(x)$ where $f'(x) = 0$ is called a **stationary point**. For a small positive value h:

Type of stationary point	$f'(x - h)$	$f'(x)$	$f'(x + h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
Point of inflection	Negative	0	Negative
	Positive	0	Positive

4 If a function f(x) has a stationary point when $x = a$, then:
- if $f''(a) > 0$, the point is a local minimum
- if $f''(a) < 0$, the point is a local maximum.

If $f''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to look at points on either side to determine its nature.



8 INTEGRATION

8.1
8.2
8.3

Learning objectives

After completing this chapter you should be able to:

- Evaluate a definite integral → **pages 152–154**
- Find the area bounded by a curve and the x-axis → **pages 154–158**
- Find areas bounded by curves and straight lines → **pages 159–162**
- Find areas bounded by two curves [fill] → **pages 162–164**
- Use the trapezium rule to approximate the area under a curve → **pages 164–168**

Prior knowledge check

1 Simplify these expressions

a $\dfrac{x^3}{\sqrt{x}}$ **b** $\dfrac{\sqrt{x}\times 2x^3}{x^2}$

c $\dfrac{x^3 - x}{\sqrt{x}}$ **d** $\dfrac{\sqrt{x} + 4x^3}{x^2}$

← **Pure 1 Sections 1.1, 1.4**

2 Find $\dfrac{dy}{dx}$ when y equals

a $2x^3 + 3x - 5$ **b** $\frac{1}{2}x^2 - x$

c $x^2(x + 1)$ **d** $\dfrac{x - x^5}{x^2}$

← **Pure 1 Section 8.5**

3 Sketch the curves with the following equations:

a $y = (x + 1)(x - 3)$

b $y = (x + 1)^2(x + 5))$

← **Pure 1 Section 4.1**

4 Find the following integrals:

a $\int(2x^4 - 3x^2 + 6)\,dx$

b $\int 3\sqrt{x} + \dfrac{1}{x^3}\,dx$

c $\int\left(\dfrac{\sqrt{x} + 6x^2}{x}\right)dx$

← **Pure 1 Section 9.2**

Integration is the opposite of differentiation. It is used to calculate areas of surfaces, volumes of irregular shapes and areas under curves. In mechanics, integration can be used to calculate the area under a velocity-time graph to find distance travelled.

8.1 Definite integrals

You can calculate an integral between two **limits**. This is called a **definite integral**. A definite integral usually produces a **value** whereas an indefinite integral always produces a **function**.

Here are the steps for integrating the function $3x^2$ between the limits $x = 1$ and $x = 2$.

$$\int_1^2 3x^2\,dx = [x^3]_1^2$$
$$= (2^3) - (1^3)$$
$$= 8 - 1$$
$$= 7$$

The limits of the integral are from $x = 1$ to $x = 2$.

Write the integral in [ ] brackets.

Write this step in ( ) brackets.

Evaluate the integral at the upper limit.

Evaluate the integral at the lower limit.

There are three stages when you work out a definite integral:

Write the definite integral **statement** with its limits, a and b.	**Integrate**, and write the integral in square brackets	**Evaluate** the definite integral by working out f(b) – f(a).
$\int_a^b \ldots dx$	$[\ldots]_a^b$	$(\ldots) - (\ldots)$

■ If $f'(x)$ is the derivative of $f(x)$ for all values of x in the interval $[a, b]$, then the definite integral is defined as $\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$.

Problem-solving

The relationship between the derivative and the integral is called the **fundamental theorem of calculus**.

Example 1

Evaluate

$$\int_0^1 (x^{\frac{1}{3}} - 1)^2\,dx$$

$$\int_0^1 (x^{\frac{1}{3}} - 1)^2\,dx$$
$$= \int_0^1 (x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1)\,dx$$
$$= \left[\frac{x^{\frac{5}{3}}}{\frac{5}{3}} - 2\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + x\right]_0^1$$
$$= \left[\frac{3}{5}x^{\frac{5}{3}} - \frac{3}{2}x^{\frac{4}{3}} + x\right]_0^1$$
$$= \left(\frac{3}{5} - \frac{3}{2} + 1\right) - (0 + 0 + 0)$$
$$= \frac{1}{10}$$

First multiply out the bracket to put the expression in a form ready to be integrated.

For definite integrals you don't need to include $+c$ in your square brackets.

Simplify each term.

Example 2 SKILLS PROBLEM-SOLVING

Given that P is a constant and $\int_1^5 (2Px + 7)\,dx = 4P^2$, show that there are two possible values for P and find these values.

$\int_1^5 (2Px + 7)\,dx = [Px^2 + 7x]_1^5$
$= (25P + 35) - (P + 7)$
$= 24P + 28$
$24P + 28 = 4P^2$
$4P^2 - 24P - 28 = 0$
$P^2 - 6P - 7 = 0$
$(P + 1)(P - 7) = 0$
$P = -1$ or 7

Divide every term by 4 to simplify.

Problem-solving
You are integrating with respect to x so treat P as a constant. Find the definite integral in terms of P then set it equal to $4P^2$. The fact that the question asks for 'two possible values' gives you a clue that the resulting equation will be quadratic.

Exercise 8A SKILLS INTERPRETATION

Watch out You must not use a calculator to work out definite integrals in your exam. You need to use calculus and show clear algebraic working.

1 Evaluate the following definite integrals:

a $\int_2^5 x^3\,dx$ **b** $\int_1^3 x^4\,dx$

c $\int_0^4 \sqrt{x}\,dx$ **d** $\int_1^3 \frac{3}{x^2}\,dx$

2 Evaluate the following definite integrals:

a $\int_1^2 \left(\frac{2}{x^3} + 3x\right)dx$ **b** $\int_0^2 (2x^3 - 4x + 5)\,dx$ **c** $\int_4^9 \left(\sqrt{x} - \frac{6}{x^2}\right)dx$ **d** $\int_1^8 (x^{-\frac{1}{3}} + 2x - 1)\,dx$

3 Evaluate the following definite integrals:

a $\int_1^3 \frac{x^3 + 2x^2}{x}\,dx$ **b** $\int_3^6 \left(x - \frac{3}{x}\right)^2 dx$ **c** $\int_0^1 x^2\left(\sqrt{x} + \frac{1}{x}\right)dx$ **d** $\int_1^4 \frac{2 + \sqrt{x}}{x^2}\,dx$

E/P 4 Given that A is a constant and $\int_1^4 (6\sqrt{x} - A)\,dx = A^2$, show that there are two possible values for A and find these values. **(5 marks)**

E 5 Use calculus to find the value of $\int_1^9 (2x - 3\sqrt{x})\,dx$. **(5 marks)**

E 6 Evaluate $\int_4^{12} \frac{2}{\sqrt{x}}\,dx$, giving your answer in the form $a + b\sqrt{3}$, where a and b are integers. **(4 marks)**

E/P 7 Given that $\int_1^k \frac{1}{\sqrt{x}}\,dx = 3$,

calculate the value of k. **(4 marks)**

Problem-solving
You might encounter a definite integral with an unknown in the limits. Here, you can find an expression for the definite integral in terms of k then set that expression equal to 3.

8 The speed, v m s^{-1}, of a train at time t seconds is given by $v = 20 + 5t$, $0 \leqslant t \leqslant 10$.

The distance, s metres, travelled by the train in 10 seconds is given by $s = \int_0^{10}(20 + 5t)\,dt$. Find the value of s.

Challenge

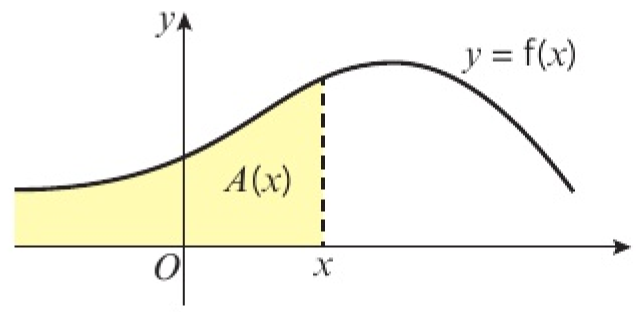
SKILLS CREATIVITY

Given that $\int_k^{3k} \frac{3x+2}{8}\,dx = 7$ and $k > 0$, calculate the value of k.

8.2 Areas under curves

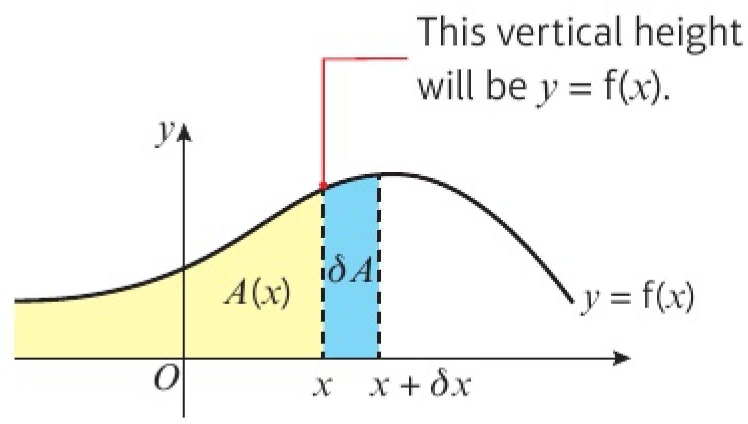
Definite integration can be used to find the area under a curve.

For any curve with equation $y = f(x)$, you can define the area under the curve to the left of x as a function of x called $A(x)$. As x increases, this area $A(x)$ also increases (since x moves further to the right).



If you look at a small increase in x, say δx, then the area increases by an amount $\delta A = A(x + \delta x) - A(x)$.

This increase in the δA is approximately rectangular and of magnitude $y\delta x$. (As you make δx smaller any error between the actual area and this will be negligible.)



So you have $\qquad \delta A \approx y\delta x$

or $\qquad \frac{\delta A}{\delta x} \approx y$

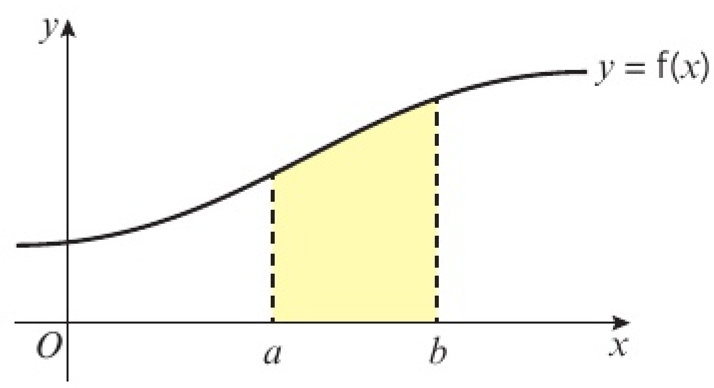
and if you take the limit $\lim_{\delta x \to 0}\left(\frac{\delta A}{\delta x}\right)$ then you will see that $\frac{dA}{dx} = y$.

Now if you know that $\frac{dA}{dx} = y$, then to find A you have to integrate, giving $A = \int y\,dx$.

- **The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by**

  Area = $\int_a^b y\,dx$

  where $y = f(x)$ is the equation of the curve.

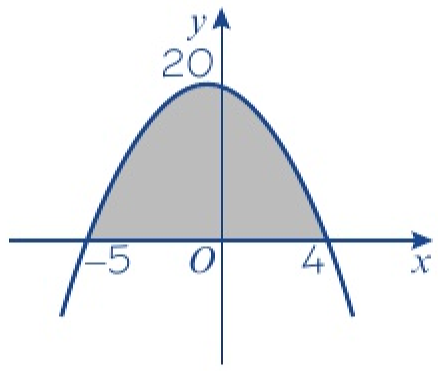


Example 3 SKILLS INTERPRETATION

Find the area of the finite region between the curve with equation $y = 20 - x - x^2$ and the x-axis.

$y = 20 - x - x^2 = (4 - x)(5 + x)$

Factorise the expression.



Draw a sketch of the graph. $x = 4$ and $x = -5$ are the points of intersection of the curve and the x-axis.

$$\text{Area} = \int_{-5}^{4} (20 - x - x^2)\,dx$$
$$= \left[20x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-5}^{4}$$
$$= \left(80 - 8 - \frac{64}{3}\right) - \left(-100 - \frac{25}{2} + \frac{125}{3}\right)$$
$$= \frac{243}{2}$$

You don't normally need to give units when you are finding areas on graphs.

Exercise 8B SKILLS PROBLEM-SOLVING

1 Find the area between the curve with equation $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ in each of the following cases:

a $f(x) = -3x^2 + 17x - 10;\quad a = 1,\quad b = 3$

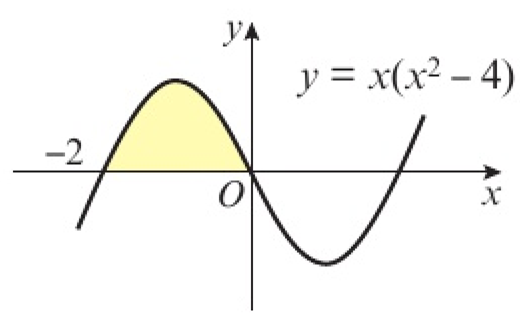
b $f(x) = 2x^3 + 7x^2 - 4x;\quad a = -3,\quad b = -1$

c $f(x) = -x^4 + 7x^3 - 11x^2 + 5x;\quad a = 0,\quad b = 4$

d $f(x) = \frac{8}{x^2};\quad a = -4,\quad b = -1$

Hint For part **c**, $f(x) = -x(x - 1)^2(x - 5)$.

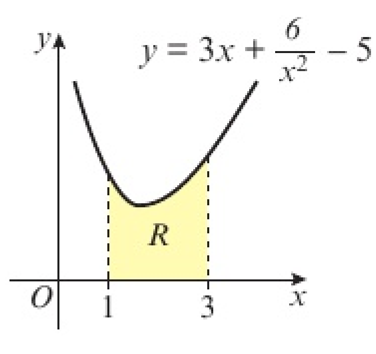
2 The sketch shows part of the curve with equation $y = x(x^2 - 4)$. Find the area of the shaded region.



3 The diagram shows a sketch of the curve with equation $y = 3x + \frac{6}{x^2} - 5,\ x > 0$.

The region R is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 3$.

Find the area of R.

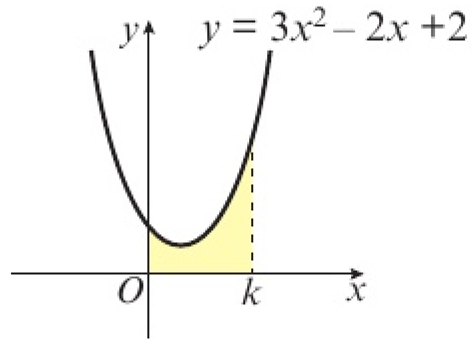


4 Find the area of the finite region between the curve with equation $y = (3 - x)(1 + x)$ and the x-axis.

5 Find the area of the finite region between the curve with equation $y = x(x - 4)^2$ and the x-axis.

(P) 6 Find the area of the finite region between the curve with equation $y = 2x^2 - 3x^3$ and the x-axis.

(P) 7 The shaded area under the graph of the function $f(x) = 3x^2 - 2x + 2$, bounded by the curve, the x-axis and the lines $x = 0$ and $x = k$, is 8. Work out the value of k.



Problem-solving

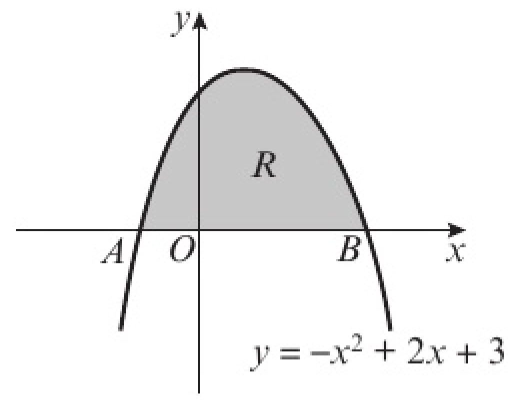
$\int_0^k (3x^2 - 2x + 2)\,dx = 8$

(E) 8 The finite region R is bounded by the x-axis and the curve with equation $y = -x^2 + 2x + 3$, $x \geqslant 0$.

The curve meets the x-axis at points A and B.

a Find the coordinates of point A and point B. **(2 marks)**

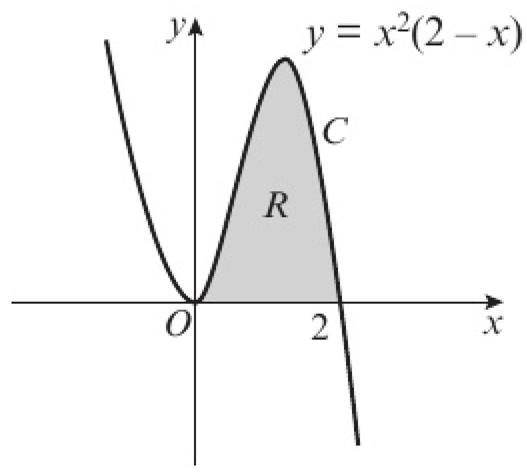
b Find the area of the region R. **(4 marks)**



(E) 9 The graph shows part of the curve C with equation $y = x^2(2 - x)$.

The region R, shown shaded, is bounded by C and the x-axis.

Use calculus to find the exact area of R. **(5 marks)**



Watch out If a question says "use calculus" then you need to use integration or differentiation, and show clear algebraic working.

8.3 Areas under the x-axis

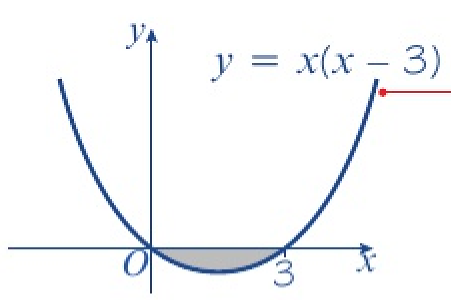
You need to be careful when you are finding areas below the x-axis.

- **When the area bounded by a curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative answer.**

Example 4

Find the area of the finite region bounded by the curve $y = x(x - 3)$ and the x-axis.

When $x = 0$, $y = 0$
When $y = 0$, $x = 0$ or 3



Online Check your solution using your calculator.

First sketch the curve.
It is ∪-shaped and crosses the x-axis at 0 and 3.

Area $= \int_0^3 x(x-3)\,dx$

$= \int_0^3 (x^2 - 3x)\,dx$

$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$

$= \left(\frac{27}{3} - \frac{27}{2}\right) - (0 - 0)$

$= -\frac{27}{6}$ or $-\frac{9}{2}$ or -4.5

So the area is 4.5

The limits on the integral will therefore be 0 and 3.

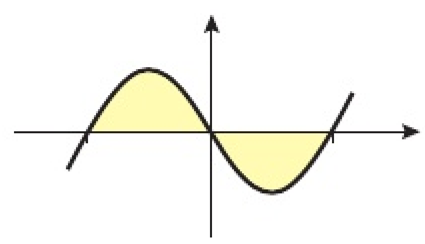
Multiply out the brackets.

Integrate as usual.

The area is below the x-axis so the definite integral is negative.

State the area as a positive quantity.

The following example shows that great care must be taken if you are trying to find an area which straddles the x-axis such as the shaded region.



For examples of this type you need to draw a sketch, unless one is given in the question.

Example 5 SKILLS PROBLEM-SOLVING

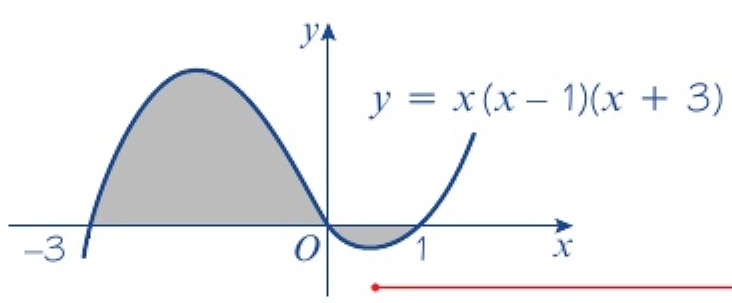
Sketch the curve with equation $y = x(x-1)(x+3)$ and find the area of the finite region bounded by the curve and the x-axis.

When $x = 0$, $y = 0$
When $y = 0$, $x = 0$, 1 or -3

Find out where the curve cuts the axes.

$x \to \infty$, $y \to \infty$
$x \to -\infty$, $y \to -\infty$

Find out what happens to y when x is large and positive or large and negative.



Problem-solving

Always draw a sketch, and use the points of intersection with the x-axis as the limits for your integrals.

The area is given by $\int_{-3}^{0} y\,dx - \int_0^1 y\,dx$

Since the area between $x = 0$ and 1 is below the axis the integral between these points will give a negative answer.

Now $\int y\,dx = \int (x^3 + 2x^2 - 3x)\,dx$

Multiply out the brackets.

$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]$

So $\int_{-3}^{0} y\,dx = (0) - \left(\frac{81}{4} - \frac{2}{3} \times 27 - \frac{3}{2} \times 9\right)$

$= \frac{45}{4}$

and $\int_0^1 y\,dx = \left(\frac{1}{4} + \frac{2}{3} - \frac{3}{2}\right) - (0)$

$= -\frac{7}{12}$

So the area required is $\frac{45}{4} + \frac{7}{12} = \frac{71}{6}$

Watch out If you try to calculate the area as a single definite integral, the positive and negative areas will partly cancel each other out.

Exercise 8C

SKILLS PROBLEM-SOLVING

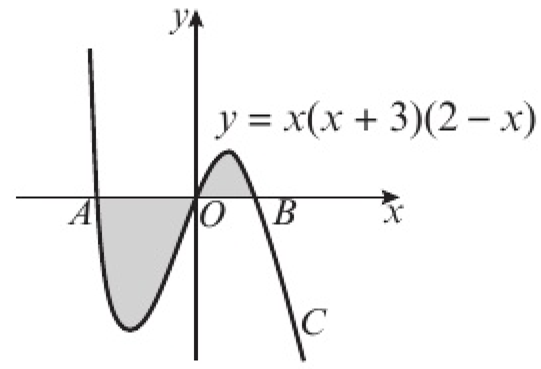
1 Sketch the following and find the total area of the finite region or regions bounded by the curves and the x-axis:

a $y = x(x + 2)$ **b** $y = (x + 1)(x - 4)$ **c** $y = (x + 3)x(x - 3)$

d $y = x^2(x - 2)$ **e** $y = x(x - 2)(x - 5)$

E **2** The graph shows a sketch of part of the curve C with equation $y = x(x + 3)(2 - x)$.

The curve C crosses the x-axis at the origin O and at points A and B.



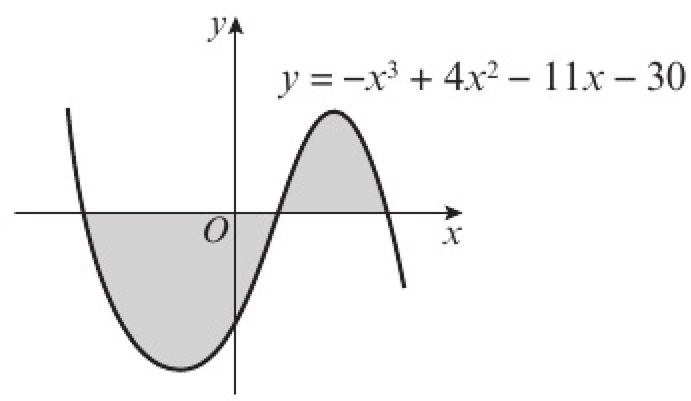
a Write down the x-coordinates of A and B. **(1 mark)**

The finite region, shown shaded, is bounded by the curve C and the x-axis.

b Use integration to find the total area of the finite shaded region. **(7 marks)**

3 $f(x) = -x^3 + 4x^2 + 11x - 30$

The graph shows a sketch of part of the curve with equation $y = -x^3 + 4x^2 + 11x - 30$.



a Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

b Write $f(x)$ in the form $(x + 3)(Ax^2 + Bx + C)$.

c Hence, factorise $f(x)$ completely.

d Hence, determine the x-coordinates where the curve intersects the x-axis.

e Hence, determine the total shaded area shown on the sketch.

Challenge

SKILLS CREATIVITY

1 Given that $f(x) = x(3 - x)$, find the area of the finite region bounded by the x-axis and the curve with equation

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = af(x)$

d $y = f(x + a)$ **e** $y = f(ax)$.

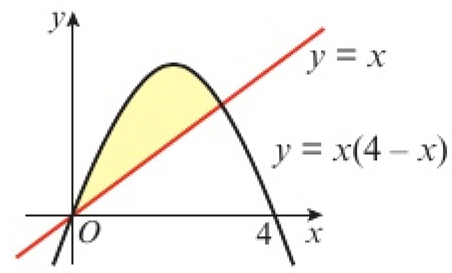
8.4 Areas between curves and lines

- You can use definite integration together with areas of trapeziums and triangles to find more complicated areas on graphs.

Example 6

The diagram shows a sketch of part of the curve with equation $y = x(4 - x)$ and the line with equation $y = x$.

Find the area of the region bounded by the curve and the line.



$x(4 - x) = x$

$3x - x^2 = 0$

$x(3 - x) = 0$

$x = 0$ or 3

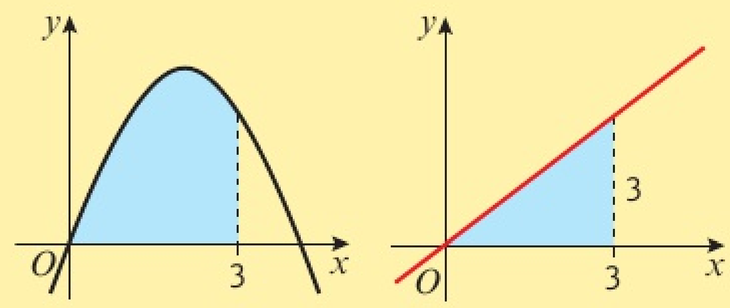
First, find the x-coordinate of the points of intersection of the curve $y = x(4 - x)$ and the line $y = x$.

Area beneath curve $= \int_0^3 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^3$

$= 9$

Shaded area = area beneath curve – area beneath triangle



$\left[2x^2 - \frac{x^3}{3}\right]_0^3 = \left(18 - \frac{27}{3}\right) - (0 - 0) = 18 - 9$

Area beneath triangle $= \frac{1}{2} \times 3 \times 3$

$= \frac{9}{2}$

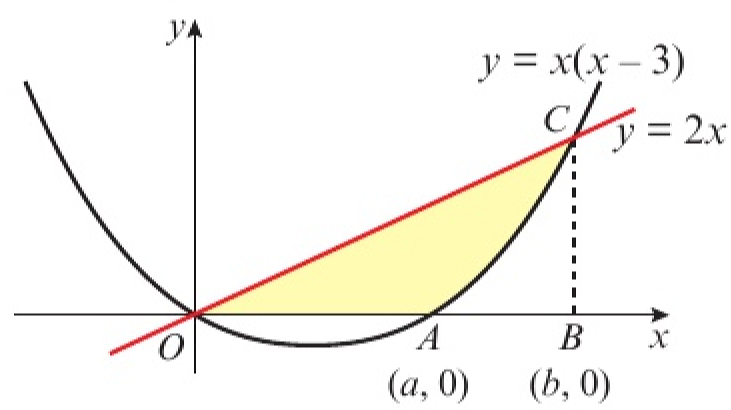
Shaded area $= 9 - \frac{9}{2} = \frac{9}{2}$

Example 7

SKILLS EXECUTIVE FUNCTION

The diagram shows a sketch of the curve with equation $y = x(x - 3)$ and the line with equation $y = 2x$.

Find the area of the shaded region OAC.



The required area is given by:

Area of triangle $OBC - \int_a^b x(x - 3)\,dx$

The curve cuts the x-axis at $x = 3$ (and $x = 0$) so $a = 3$.

The curve meets the line $y = 2x$ when $2x = x(x - 3)$.

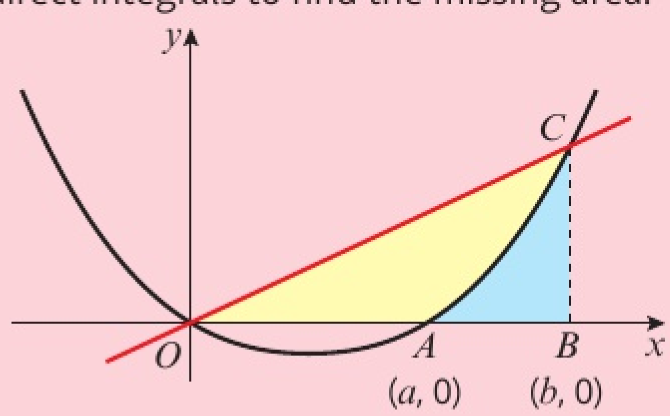
So $0 = x^2 - 5x$

$0 = x(x - 5)$

$x = 0$ or 5, so $b = 5$

Problem-solving

Look for ways of combining triangles, trapeziums and direct integrals to find the missing area.



The point C is (5, 10).

Area of triangle $OBC = \frac{1}{2} \times 5 \times 10 = 25$.

Area between curve, x-axis and the line $x = 5$ is

$$\int_3^5 x(x-3)\,dx = \int_3^5 (x^2 - 3x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_3^5$$

$$= \left(\frac{125}{3} - \frac{75}{2}\right) - \left(\frac{27}{3} - \frac{27}{2}\right)$$

$$= \left(\frac{25}{6}\right) - \left(-\frac{27}{6}\right)$$

$$= \frac{26}{3}$$

Shaded region is therefore $= 25 - \frac{26}{3} = \frac{49}{3}$

Substituting $x = 5$ into the equation of the line gives $y = 2 \times 5 = 10$.

Work out the definite integral separately. This will help you avoid making errors in your working.

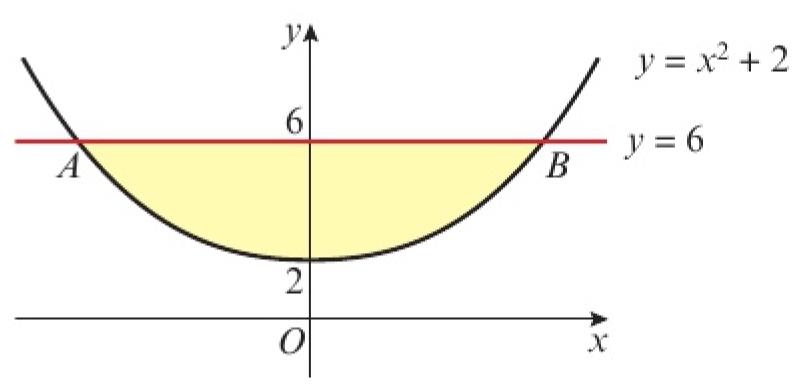
Exercise 8D

SKILLS EXECUTIVE FUNCTION

1 The diagram shows part of the curve with equation $y = x^2 + 2$ and the line with equation $y = 6$. The line cuts the curve at the points A and B.

a Find the coordinates of the points A and B.

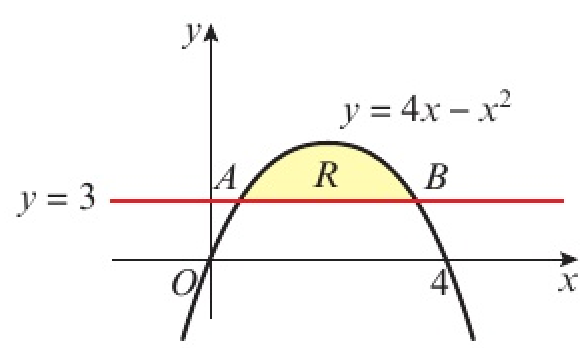
b Find the area of the finite region bounded by line AB and the curve.



2 The diagram shows the finite region, R, bounded by the curve with equation $y = 4x - x^2$ and the line $y = 3$. The line cuts the curve at the points A and B.

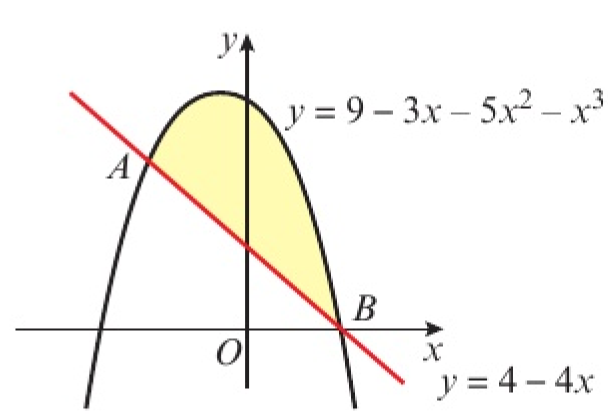
a Find the coordinates of the points A and B.

b Find the area of R.



(P) 3 The diagram shows a sketch of part of the curve with equation $y = 9 - 3x - 5x^2 - x^3$ and the line with equation $y = 4 - 4x$. The line cuts the curve at the points $A\,(-1, 8)$ and $B(1, 0)$.

Find the area of the shaded region between AB and the curve.

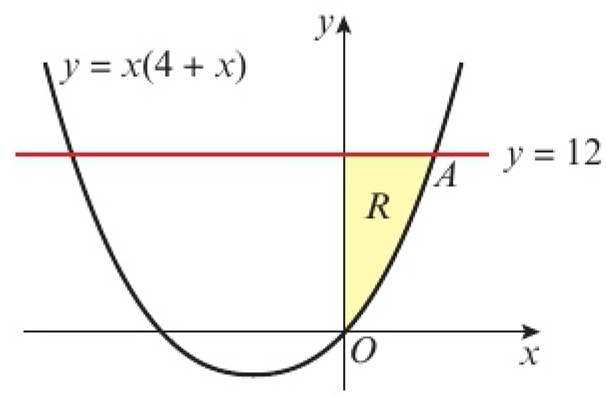


(P) **4** Find the area of the finite region bounded by the curve with equation $y = (1 - x)(x + 3)$ and the line $y = x + 3$.

5 The diagram shows the finite region, R, bounded by the curve with equation $y = x(4 + x)$, the line with equation $y = 12$ and the y-axis.

a Find the coordinates of the point A where the line meets the curve.

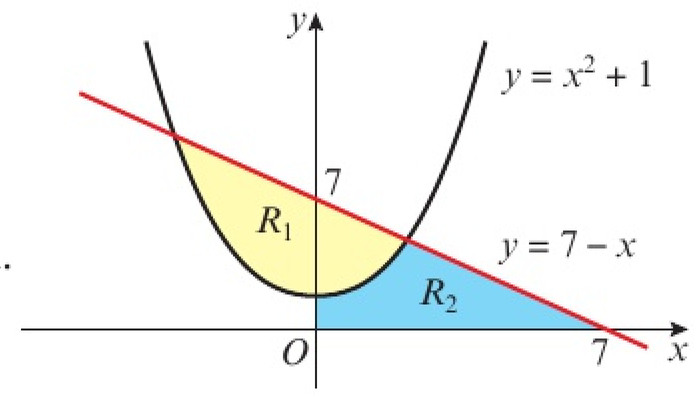
b Find the area of R.



(P) **6** The diagram shows a sketch of part of the curve with equation $y = x^2 + 1$ and the line with equation $y = 7 - x$. The finite region, R_1 is bounded by the line and the curve. The finite region, R_2 is below the curve and the line and is bounded by the positive x- and y-axes as shown in the diagram.

a Find the area of R_1.

b Find the area of R_2.



(P) **7** The curve C has equation $y = x^{\frac{2}{3}} - \dfrac{2}{x^{\frac{1}{3}}} + 1$.

a Verify that C crosses the x-axis at the point $(1, 0)$.

b Show that the point $A(8, 4)$ also lies on C.

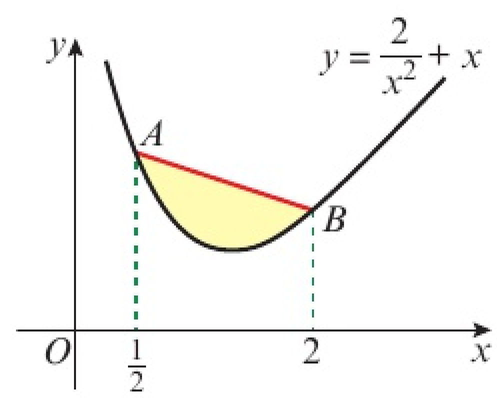
c The point B is $(4, 0)$. Find the equation of the line through AB.

The finite region R is bounded by C, AB and the positive x-axis.

d Find the area of R.

(P) **8** The diagram shows part of a sketch of the curve with equation $y = \dfrac{2}{x^2} + x$. The points A and B have x-coordinates $\frac{1}{2}$ and 2 respectively.

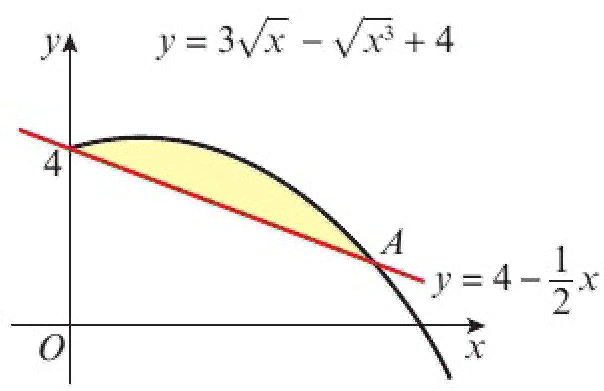
Find the area of the finite region between AB and the curve.



(P) **9** The diagram shows part of the curve with equation $y = 3\sqrt{x} - \sqrt{x^3} + 4$ and the line with equation $y = 4 - \frac{1}{2}x$.

a Verify that the line and the curve cross at the point $A(4, 2)$.

b Find the area of the finite region bounded by the curve and the line.

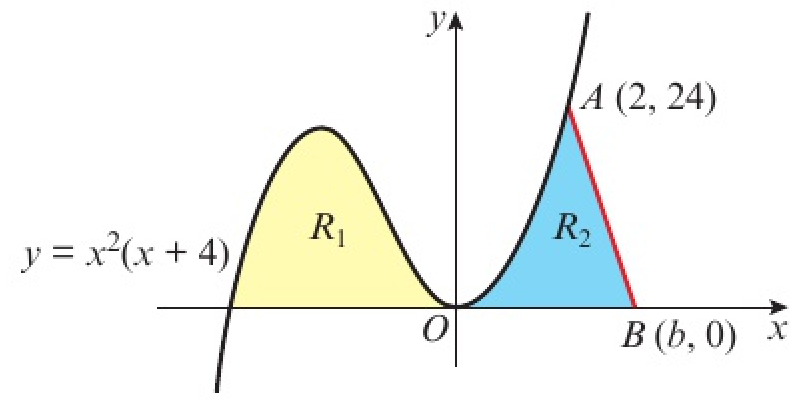


(P) **10** The sketch shows part of the curve with equation $y = x^2(x + 4)$. The finite region R_1 is bounded by the curve and the negative x-axis. The finite region R_2 is bounded by the curve, the positive x-axis and AB, where $A(2, 24)$ and $B(b, 0)$.

The area of R_1 = the area of R_2.

a Find the area of R_1.

b Find the value of b.



Problem-solving

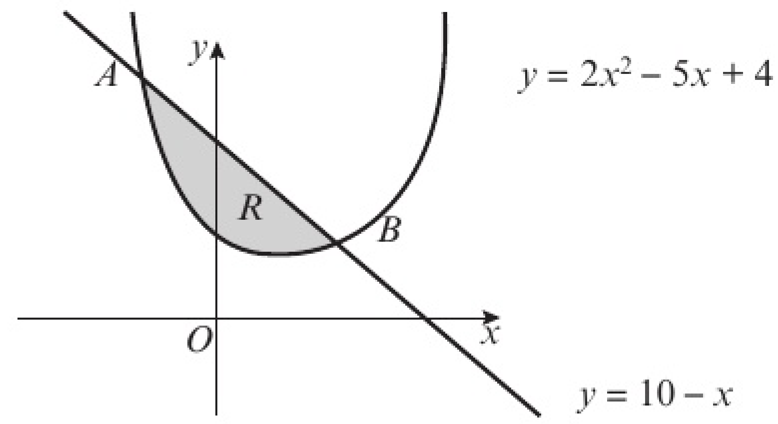
Split R_2 into two areas by drawing a vertical line at $x = 2$.

(E/P) **11** The line with equation $y = 10 - x$ cuts the curve with equation $y = 2x^2 - 5x + 4$ at the points A and B, as shown.

a Find the coordinates of A and the coordinates of B. **(5 marks)**

The shaded region R is bounded by the line and the curve as shown.

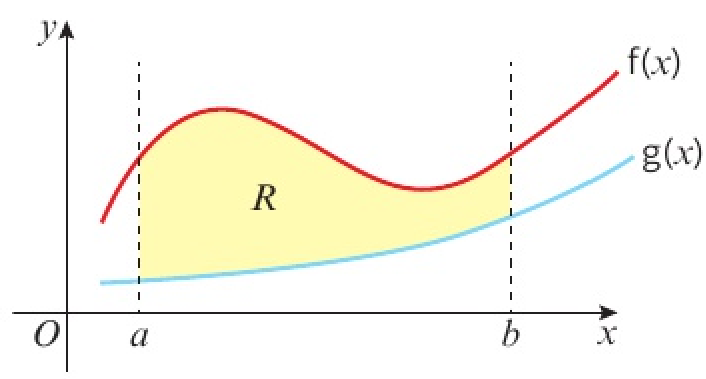
b Find the exact area of R. **(6 marks)**



8.5 Areas between two curves

- The area bounded by two curves can be found using integration:

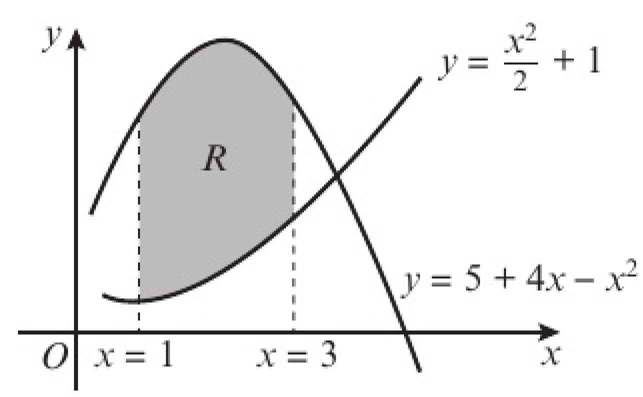
$$\text{Area of } \boldsymbol{R} = \int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \mathrm{g}(x)\,\mathrm{d}x$$



Watch out You can only use this formula if the two curves do not intersect between a and b.

Example 8

The diagram shows a sketch of part of the curve with equation $y = 5 + 4x - x^2$ and part of the curve with equation $y = \frac{x^2}{2} + 1$.



Area of $R = \int_1^3 (5 + 4x - x^2)\,dx - \int_1^3 \left(\frac{x^2}{2} + 1\right)dx$

Use the formula with the limits.

Area of $R = \left[5x + 2x^2 - \frac{x^3}{3}\right]_1^3 - \left[\frac{x^3}{6} + x\right]_1^3$

Integrate.

$R = \left[\left(5 \times 3 + 2 \times 3^2 - \frac{3^3}{3}\right) - \left(1 \times 3 + 2 \times 1^2 - \frac{1^3}{3}\right)\right] - \left[\left(\frac{3^3}{6} + 3\right) - \left(\frac{1^3}{6} + 1\right)\right]$

Substitute in the limits and evaluate.

Area of $R = 11$ (units^2)

- **You will need to find the limits yourself if you are required to find the area bounded by two curves.**

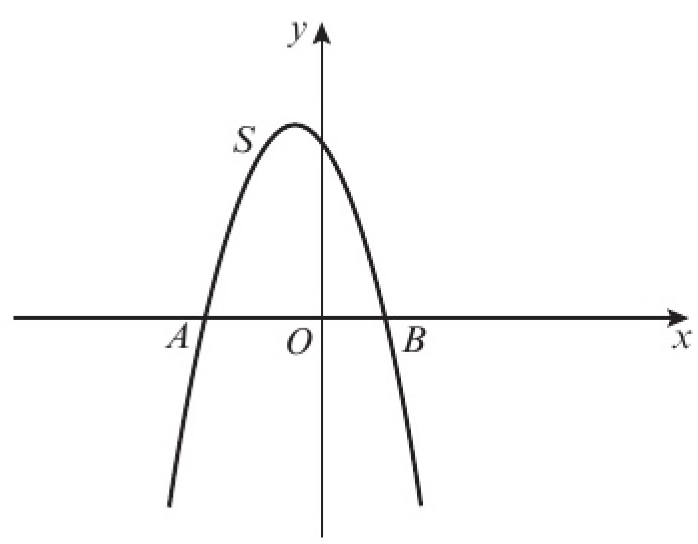
Example 9

The diagram below shows a sketch of part of the curve S with equation $y = 8 - 2x - x^2$

The curve T with equation $y = x^2 + x + 6$ intersects S at two points.

a Find the x-coordinates of the points of intersection of S and T.

b Use calculus to find the area of the finite region bounded by S and T.



a $8 - 2x - x^2 = x^2 + x + 6$

$\Rightarrow 2x^2 + 3x - 2 = 0 \Rightarrow (2x - 1)(x + 2) = 0$

$\Rightarrow x = -2$ or $x = \frac{1}{2}$

Set $S = T$ and solve to find the two points of intersection.

b Area $= \int_{-2}^{0.5} (8 - 2x - x^2) - (x^2 + x + 6)\,dx$

The area of the required region is the area below curve S – the area below curve T.

Area $= \int_{-2}^{0.5} (2 - 3x - 2x^2)\,dx = \left[2x - \frac{3x^2}{2} - \frac{2x^3}{3}\right]_{-2}^{0.5}$

Because the limits are the same for **both** equations you can combine them into one.

Area $= \left[\left(2 \times 0.5 - \frac{3 \times 0.5^2}{2} - \frac{2 \times 0.5^3}{3}\right) - \left(2 \times (-2) - \frac{3 \times -2^2}{2} - \frac{2 \times (-2)^3}{3}\right)\right]$

Area $= \frac{125}{24}$

Integrate and evaluate.

Exercise 8E SKILLS ANALYSIS

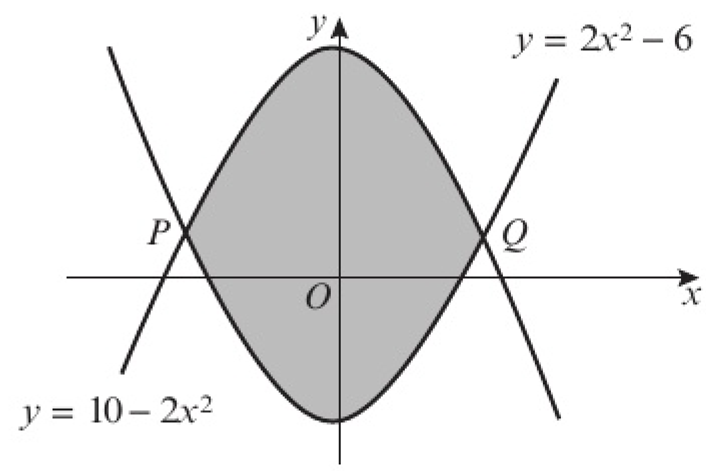
1 Find, using calculus, the finite area bounded by the curve $y = 2x^2 - 8x - 10$ the curve $y = \frac{x^2}{2} - 2x - 1$ and the lines $x = 1$ and $x = 3$.

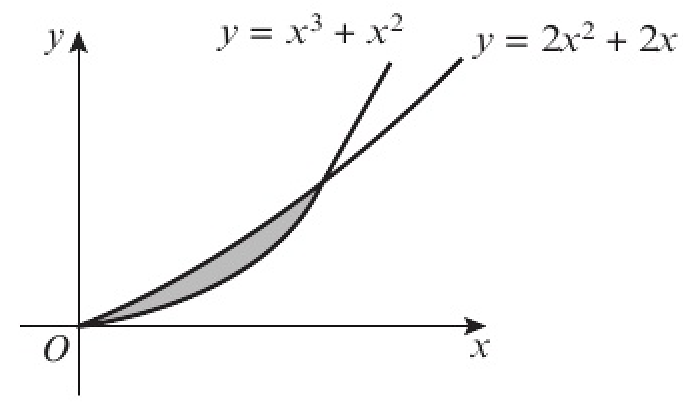
2 The curve C with equation $y = x^2$ and the curve S with equation $y = 2x^2 - 25$ intersect at two points.

Using algebraic integration calculate the finite region enclosed by C and S.

3 The curves with equations $y = 2x^2 - 6$ and $y = 10 - 2x^2$ intersect at points P and Q as shown in the diagram opposite.

Use calculus to calculate the area of the finite region enclosed by these curves.



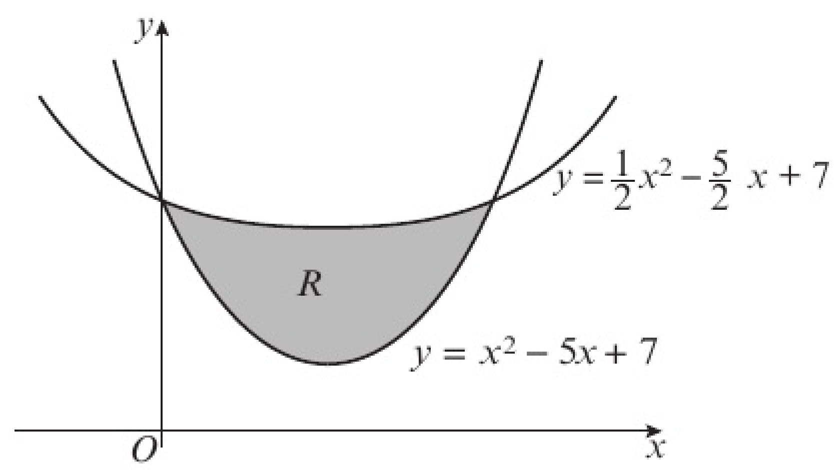


4 The diagram above shows part of the curves $y = x^3 + x^2$ and $y = 2x^2 + 2x$.

Use calculus to calculate the area of the finite region enclosed.

5 The curve with equation $y = x^2 - 5x + 7$ cuts the curve with equation $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$. The shaded region R is bounded by the curves as shown.

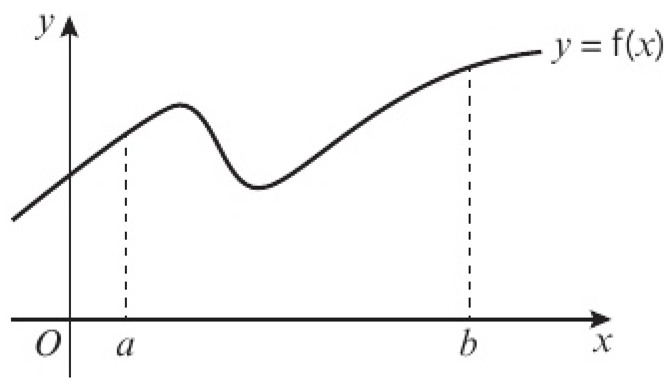
Find the exact area of R.



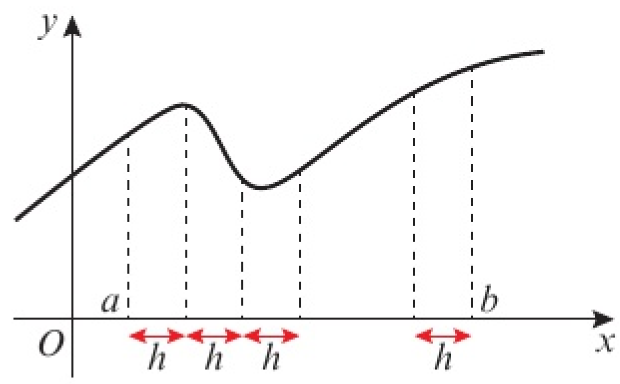
8.6 The trapezium rule

If you cannot integrate a function algebraically, you can use a numerical method to approximate the area beneath a curve.

Consider the curve $y = \mathrm{f}(x)$:

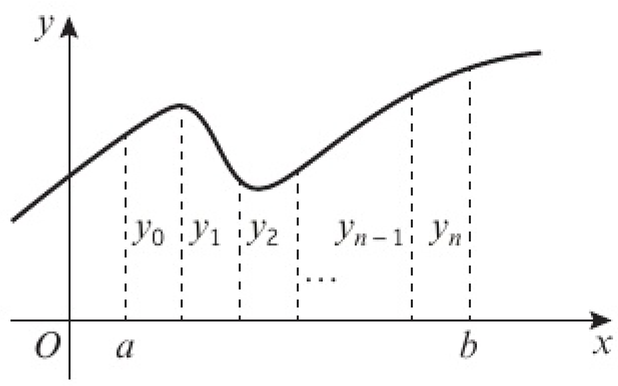


To approximate the area given by $\int_a^b y \, dx$, you can divide the area up into n equal strips. Each strip will be of width h, where $h = \frac{b-a}{n}$



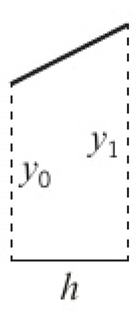
Next you calculate the value of y for each value of x that forms a boundary of one of the strips. So you find y for $x = a$, $x = a + h$, $x = a + 2h$, $x = a + 3h$ and so on up to $x = b$.
You can label these values $y_0, y_1, y_2, y_3, \ldots, y_n$.

Hint Notice that for n strips there will be $n + 1$ values of x and $n + 1$ values of y.



Finally you join adjacent points to form n trapeziums and approximate the original area by the sum of the areas of these n trapeziums.

That the area of a trapezium like this:



is given by $\frac{1}{2}(y_0 + y_1)h$. The required area under the curve is therefore given by:

$$\int_a^b y \, dx \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \ldots + \tfrac{1}{2}h(y_{n-1} + y_n)$$

Factorising gives:

$$\int_a^b y \, dx \approx \tfrac{1}{2}h(y_0 + y_1 + y_1 + y_2 + y_2 + \ldots + y_{n-1} + y_{n-1} + y_n)$$

or $$\int_a^b y \, dx \approx \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n)$$

This formula is given in the formula booklet but you will need to know how to use it.

- The trapezium rule:

$$\int_a^b y\,dx \approx \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n)$$

where $h = \dfrac{b-a}{n}$ and $y_i = f(a + ih)$.

Example 10 SKILLS INTERPRETATION

Use the trapezium rule with

a 4 strips **b** 8 strips

to estimate the area under the curve with equation $y = \sqrt{2x+3}$ between the lines $x = 0$ and $x = 2$.

a Each strip will have width $\dfrac{2-0}{4} = 0.5$

x	0	0.5	1	1.5	2
$y = \sqrt{2x+3}$	1.732	2	2.236	2.449	2.646

First work out the value of y at the boundaries of each of your strips.

It is very helpful to put your working in a table.

So area $= \frac{1}{2} \times 0.5 \times (1.732 + 2(2 + 2.236 + 2.449) + 2.646)$
$= \frac{1}{2} \times 0.5 \times (17.748)$
$= 4.437$ or 4.44

b Each strip will have width $\dfrac{2-0}{8} = 0.25$

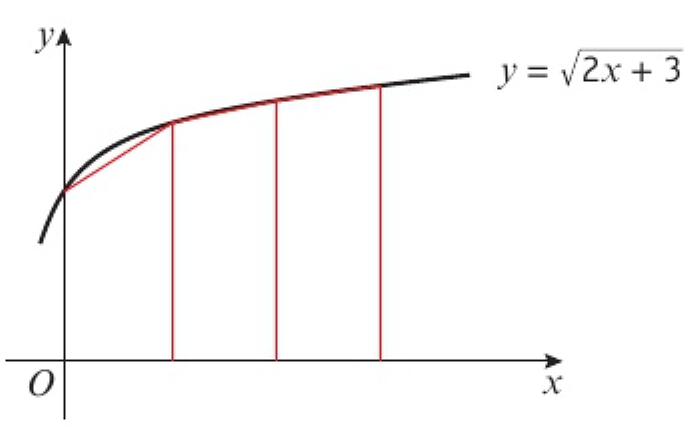
x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$y = \sqrt{2x+3}$	1.732	1.871	2	2.121	2.236	2.345	2.449	2.550	2.646

Values of y.

So area $= \frac{1}{2} \times 0.25 \times (1.732 + 2(1.871 + 2 + 2 + 2.121 + 2.236 + 2.345 + 2.449 + 2.550) + 2.646)$
$= \frac{1}{2} \times 0.25 \times (35.522)$
$= 4.440\,25$ or 4.44

The actual area in this case is 4.441 368… and you can see (if you look at the calculations to 3 decimal places) in the above example that increasing the number of strips (or reducing their width) should improve the accuracy of the approximation.

A sketch of $y = \sqrt{2x+3}$ looks like this:



The curve is concave (bends 'outwards') so each of the trapeziums is entirely below the curve. The trapezium rule will give an **underestimate** in this case.

Graphical calculators can be used to evaluate definite integrals. Calculators usually use a slightly different method from the trapezium rule to carry out these calculations and they will generally be more accurate. So, although the calculator can provide a useful check, you should remember that the trapezium rule is being used to estimate the value and you should not expect this estimate to be the same as the answer from a graphical calculator.

Watch out If you are using the trapezium rule in your exam, you show your values of x_i and y_i and how you have substituted them into the formula.

Exercise 8F

SKILLS INTERPRETATION

1 Copy and complete the table below and use the trapezium rule to estimate $\int_1^3 \left(\frac{1}{x^2+1}\right) dx$:

x	1	1.5	2	2.5	3
$y = \frac{1}{x^2+1}$	0.5	0.308		0.138	

2 Use the table below to estimate $\int_1^{2.5} \sqrt{2x-1}\, dx$ with the trapezium rule:

x	1	1.25	1.5	1.75	2	2.25	2.5
$y = \sqrt{2x-1}$	1	1.225			1.732		2

3 Copy and complete the table below and use it, together with the trapezium rule, to estimate $\int_1^3 \sqrt{x^3+1}\, dx$:

x	0	0.5	1	1.5	2
$y = \sqrt{x^3+1}$	1	1.061	1.414		

4 Use the trapezium rule with 6 strips to estimate $\int_1^3 \frac{1}{\sqrt{x^2+1}} dx$.

5 **a** Copy and complete the table below and use the trapezium rule to estimate the area bounded by the curve, the x-axis and the lines $x = -1$ and $x = 1$.

x	−1	−0.6	−0.2	0.2	0.6	1
$y = \frac{1}{x+2}$	1	0.714			0.385	

b State, with a reason, whether your answer in part **a** is an overestimate or an underestimate.

(P) 6 **a** Sketch the curve with equation $y = x^3 + 1$ for $-2 < x < 2$.

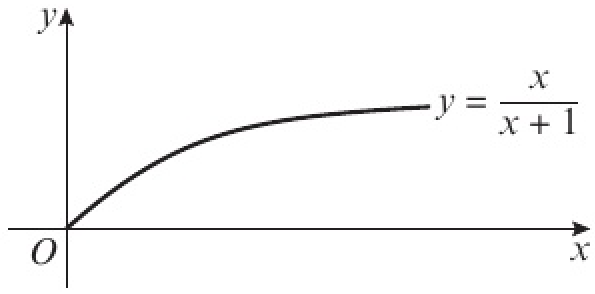
b Use the trapezium rule with 4 strips to estimate the value of $\int_{-1}^1 x^3 + 1\, dx$.

c Use integration to find the exact value of $\int_{-1}^1 (x^3 + 1)\, dx$.

d Comment on your answers to parts **b** and **c**.

(P) 7 Use the trapezium rule with 4 strips to estimate $\int_0^2 \sqrt{3^x - 1}\, dx$.

P **8** The sketch shows part of the curve with equation $y = \frac{x}{x+1}$, $x \geqslant 0$.



a Use the trapezium rule with 6 strips to estimate $\int_1^3 \left(\frac{x}{x+1}\right) dx$.

b With reference to the above sketch, state, with a reason, whether the answer in part **a** is an overestimate or an underestimate.

P **9** **a** Use the trapezium rule with n strips to estimate $\int_0^2 \sqrt{x}\ dx$ in the cases **i** $n = 4$ **ii** $n = 6$.

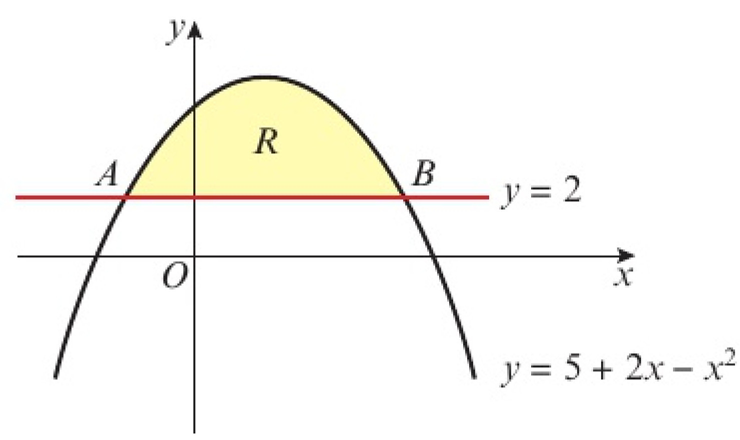
b Compare your answers from part **a** with the exact value of the integral and calculate the percentage error in each case.

E **10** **a** Use the trapezium rule with 8 strips to estimate $\int_0^2 2^x dx$. **(5 marks)**

b With reference to a sketch of $y = 2^x$ explain whether your answer in part **a** is an underestimate or overestimate of $\int_0^2 2^x dx$. **(2 marks)**

Chapter review 8

1 The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.



a Find the x-coordinates of A and B.

b The shaded region R is bounded by the curve and the line. Find the area of R.

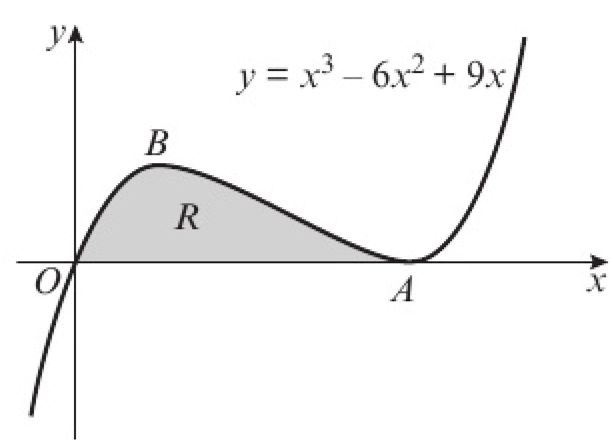
E/P **2** **a** Find $\int (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)dx$. **(4 marks)**

b Use your answer to part **a** to evaluate

$$\int_1^4 (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)dx$$

giving your answer as an exact fraction. **(2 marks)**

E **3** The diagram shows part of the curve with equation $y = x^3 - 6x^2 + 9x$. The curve touches the x-axis at A and has a local maximum at B.



a Show that the equation of the curve may be written as $y = x(x-3)^2$, and hence write down the coordinates of A. **(2 marks)**

b Find the coordinates of B. **(2 marks)**

c The shaded region R is bounded by the curve and the x-axis. Find the area of R. **(6 marks)**

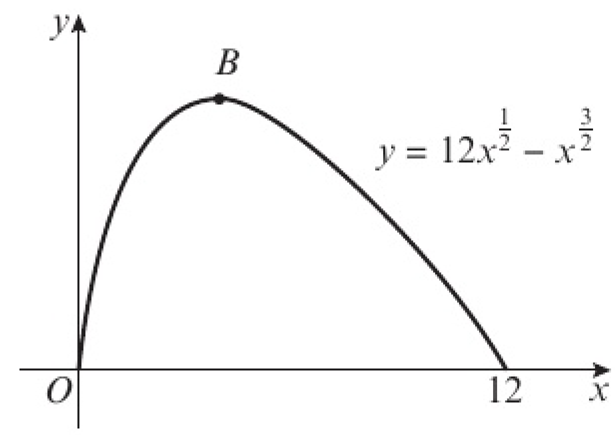
E **4** Consider the function $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}, x > 0$.

a Find $\frac{dy}{dx}$. **(2 marks)**

b Find $\int y\,dx$. **(3 marks)**

c Hence show that $\int_1^3 y\,dx = A + B\sqrt{3}$, where A and B are integers to be found. **(2 marks)**

E/P **5** The diagram shows a sketch of the curve with equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$ for $0 \leqslant x \leqslant 12$.

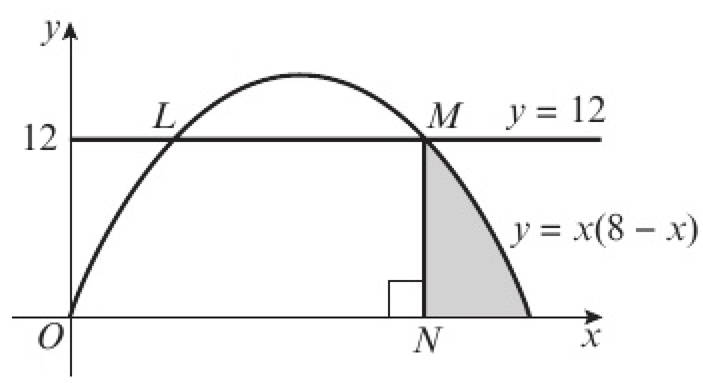


a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$. **(2 marks)**

b At the point B on the curve the tangent to the curve is parallel to the x-axis. Find the coordinates of the point B. **(2 marks)**

c Find, to 3 significant figures, the area of the finite region bounded by the curve and the x-axis. **(6 marks)**

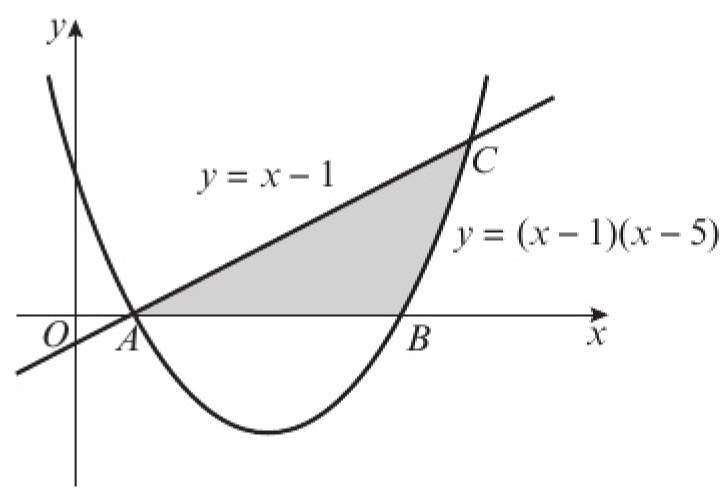
E/P **6** The diagram shows the curve C with equation $y = x(8 - x)$ and the line with equation $y = 12$ which meet at the points L and M.



a Determine the coordinates of the point M. **(2 marks)**

b Given that N is the foot of the perpendicular from M on to the x-axis, calculate the area of the shaded region which is bounded by NM, the curve C and the x-axis. **(6 marks)**

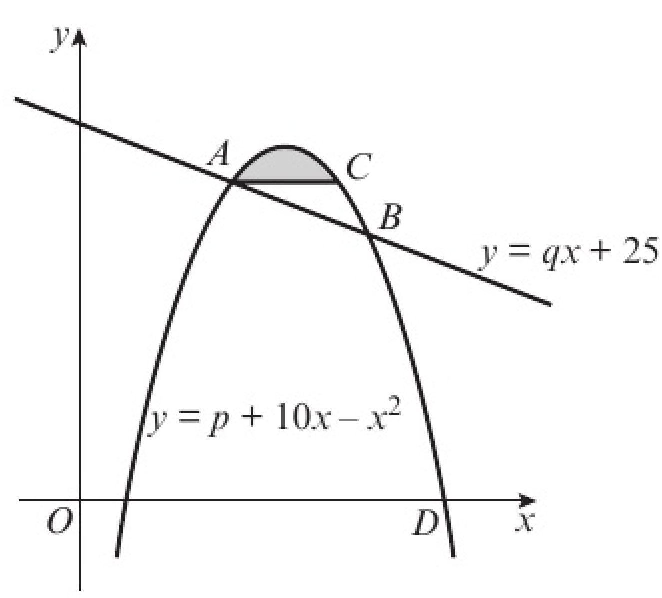
E/P **7** The diagram shows the line $y = x - 1$ meeting the curve with equation $y = (x - 1)(x - 5)$ at A and C. The curve meets the x-axis at A and B.



a Write down the coordinates of A and B and find the coordinates of C. **(4 marks)**

b Find the area of the shaded region bounded by the line, the curve and the x-axis. **(6 marks)**

E/P **8** The diagram shows part of the curve with equation $y = p + 10x - x^2$, where p is a constant, and part of the line l with equation $y = qx + 25$, where q is a constant. The line l cuts the curve at the points A and B. The x-coordinates of A and B are 4 and 8 respectively. The line through A parallel to the x-axis intersects the curve again at the point C.



a Show that $p = -7$ and calculate the value of q. **(3 marks)**

b Calculate the coordinates of C. **(2 marks)**

c The shaded region in the diagram is bounded by the curve and the line segment AC. Using integration and showing all your working, calculate the area of the shaded region. **(6 marks)**

E/P **9** Given that A is constant and $\int_4^9 \left(\frac{3}{\sqrt{x}} - A\right) dx = A^2$ show that there are two possible values for A and find these values. **(5 marks)**

E/P **10** $f'(x) = \frac{(2 - x^2)^3}{x^2}$, $x \neq 0$.

a Show that $f'(x) = 8x^{-2} - 12 + Ax^2 + Bx^4$, where A and B are constants to be found. **(3 marks)**

b Find $f''(x)$.

Given that the point $(-2, 9)$ lies on the curve with equation $y = f(x)$,

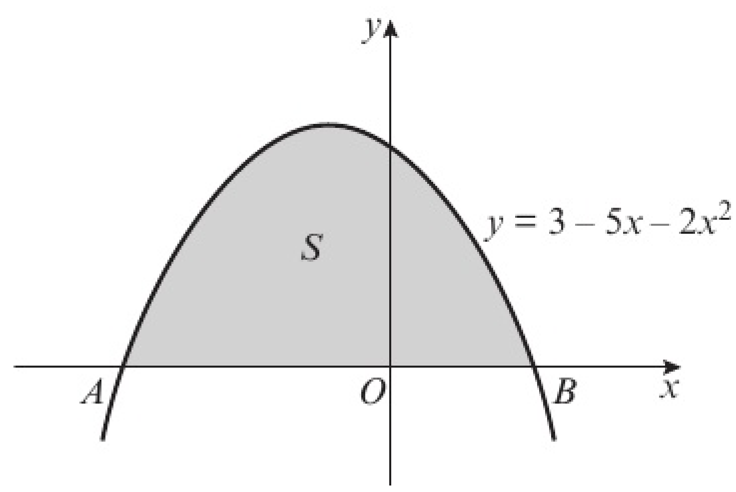
c find $f(x)$. **(5 marks)**

E **11** The finite region S, which is shown shaded, is bounded by the x-axis and the curve with equation $y = 3 - 5x - 2x^2$.

The curve meets the x-axis at points A and B.

a Find the coordinates of point A and point B. **(2 marks)**

b Find the area of the region S. **(4 marks)**



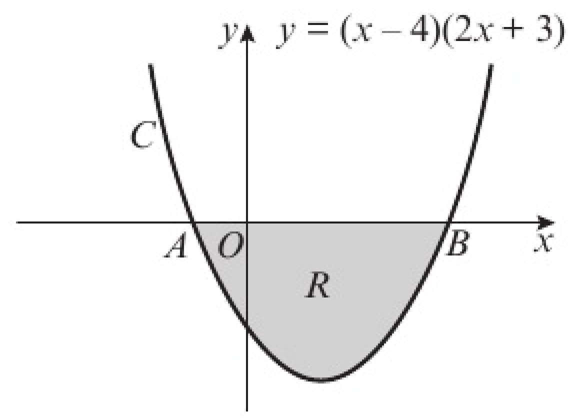
E **12** The graph shows a sketch of part of the curve C with equation $y = (x - 4)(2x + 3)$.

The curve C crosses the x-axis at the points A and B.

a Write down the x-coordinates of A and B. **(1 mark)**

The finite region R, shown shaded, is bounded by C and the x-axis.

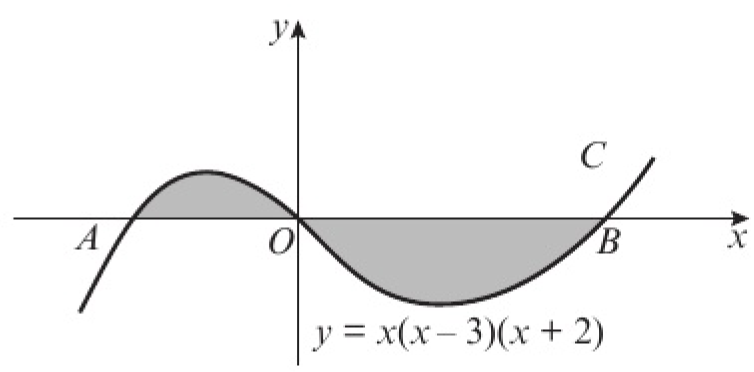
b Use integration to find the area of R. **(6 marks)**



E **13** The graph shows a sketch of part of the curve C with equation $y = x(x - 3)(x + 2)$.

The curve crosses the x-axis at the origin O and the points A and B.

a Write down the x-coordinates of the points A and B. **(1 mark)**



The finite region shown shaded is bounded by the curve C and the x-axis.

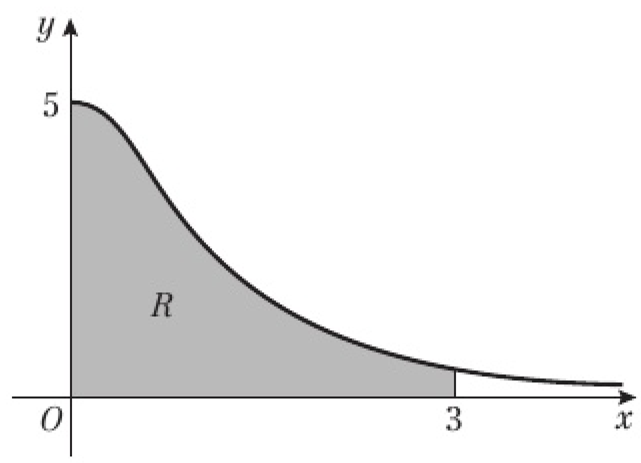
b Use integration to find the total area of this region. **(7 marks)**

E **14** $y = \frac{5}{x^2 + 1}$

a Complete the table below, giving the missing values of y to 3 decimal places.

x	0	0.5	1	1.5	2	2.5	3
$y = \frac{5}{x^2 + 1}$	5	4	2.5		1		0.5

(2 marks)



The graph shows the region R which is bounded by the curve with equation $y = \frac{5}{x^2 + 1}$, the x-axis and and the lines $x = 0$ and $x = 3$.

b Use the trapezium rule, with all the values of y from your table, to find an approximate value for the area of R. **(4 marks)**

c Use your answer from part **b** to find an approximate value for

$$\int_0^3 \left(4 + \frac{5}{x^2 + 1}\right) dx$$

giving your answer to 2 decimal places. **(2 marks)**

E **15** $y = \sqrt{3^x + x}$

a Complete the table below, giving the values of y to 3 decimal places.

x	0	0.25	0.5	0.75	1
$y = \sqrt{3^x + x}$	1	1.251			2

(2 marks)

b Use the trapezium rule with all of the values of y from your table to find an approximation for the value of $\int_0^1 \sqrt{3^x + x}\, dx$.

You must show clearly how you obtained your answer. **(4 marks)**

16 The curve A with equation $y = 8 + 4x - x^2$ and the curve B with equation $y = x^2 - 4x + 8$ intersect at two points M and N.

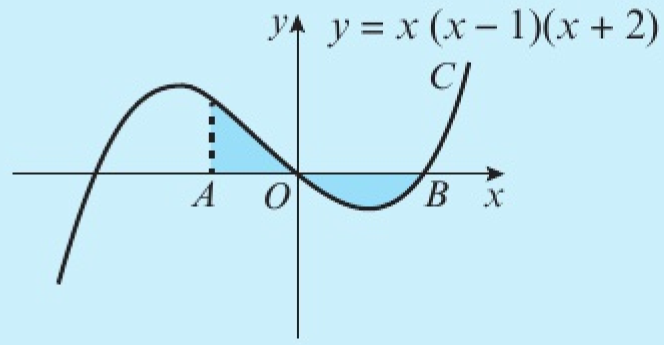
a Find the coordinates of M and the coordinates of N.

b Use calculus to find the area of the finite region enclosed by A and B.

Challenge

SKILLS
CREATIVITY

The graph shows a sketch of part of the curve C with equation $y = x(x - 1)(x + 2)$.
The curve C crosses the x-axis at the origin O and at point B.
The shaded areas above and below the x-axis are equal.



a Show that the x-coordinate of A satisfies the equation

$$(x - 1)^2(3x^2 + 10x + 5) = 0$$

b Hence find the exact coordinates of A, and interpret geometrically the other roots of this equation.

Summary of key points

1 If $f'(x)$ is the derivative of $f(x)$ for all values of x in the interval $[a, b]$, then the definite integral is defined as $\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$

2 The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b y\,dx$$

where $y = f(x)$ is the equation of the curve.

3 When the area bounded by a curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative answer.

4 You can use definite integration together with areas of trapeziums and triangles to find more complicated areas on graphs.

5 The trapezium rule is:

$$\int_a^b y\,dx = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n)$$

where $h = \dfrac{b - a}{n}$ and $y_i = f(a + ih)$

Review exercise 2

E/P **1** The 4th, 5th and 6th terms in an arithmetic sequence are:

$12 - 7k, 3k^2, k^2 - 10k$

a Find two possible values of k. **(3)**

Given that the sequence contains only integer terms,

b find the first term and the common difference. **(2)**

← Pure 2 Section 5.1

E/P **2** **a** Find, in terms of p, the 30th term of the arithmetic sequence

$(19p - 18), (17p - 8), (15p + 2), \ldots$

giving your answer in its simplest form. **(2)**

b Given $S_{31} = 0$, find the value of p. **(3)**

← Pure 2 Sections 5.1, 5.2

E/P **3** The second term of a geometric sequence is 256. The eighth term of the same sequence is 900. The common ratio is r, $r > 0$.

a Show that r satisfies the equation

$6 \ln r + \ln\left(\frac{64}{225}\right) = 0$ **(3)**

b Find the value of r correct to 3 significant figures. **(3)**

← Pure 2 Section 5.3

E/P **4** A geometric series has first term 4 and common ratio r. The sum of the first three terms of the series is 7.

a Show that $4r^2 + 4r - 3 = 0$. **(3)**

b Find the two possible values of r. **(2)**

Given that r is positive,

c find the sum to infinity of the series. **(2)**

← Pure 2 Sections 5.4, 5.5

E/P **5** The fourth, fifth and sixth terms of a geometric series are x, 3 and $x + 8$.

a Find the two possible values of x and the corresponding values of the common ratio. **(4)**

Given that the sum to infinity of the series exists,

b find the first term **(1)**

c find the sum to infinity of the series. **(2)**

← Pure 2 Sections 5.3, 5.5

E/P **6** A sequence $a_1, a_2, a_3, \ldots$ is defined by

$a_1 = k,$

$a_{n+1} = 3a_n + 5, n \geqslant 1$

where k is a positive integer.

a Write down an expression for a_2 in terms of k. **(1)**

b Show that $a_3 = 9k + 20$. **(2)**

c **i** Find $\sum_{r=1}^{4} a_r$ in terms of k. **(2)**

ii Show that $\sum_{r=1}^{4} a_r$ is divisible by 10. **(2)**

← Pure 2 Sections 5.6, 5.7

E/P **7** A geometric series is given by

$6 - 24x + 96x^2 - \ldots.$

The series is convergent.

a Write down a condition on x. **(1)**

Given that $\sum_{r=1}^{\infty} 6 \times (-4x)^{r-1} = \frac{24}{5}$

b Calculate the value of x. **(5)**

← Pure 2 Sections 5.5, 5.6

E/P **8** The first three terms of a geometric sequence are $10, \frac{50}{6}$ and $\frac{250}{36}$.

a Find the sum to infinity of the series. **(3)**

Given that the sum to k terms of the series is greater than 55,

b show that $k > \dfrac{\log\left(\frac{1}{12}\right)}{\log\left(\frac{5}{6}\right)}$ **(4)**

c find the smallest possible value of k. **(1)**

← Pure 2 Sections 3.4, 5.4, 5.5

E/P **9** At the end of year 1, a company employs 2400 people. A model predicts that the number of employees will increase by 6% each year, forming a geometric sequence.

a Find the predicted number of employees after 4 years, giving your answer to the nearest 10. **(3)**

The company expects to expand in this way until the total number of employees first exceeds 6000 at the end of a year, N.

b Show that $(N - 1)\log 1.06 > \log 2.5$. **(3)**

c Find the value of N. **(2)**

The company has a charity scheme by which they match any employee charity contribution exactly.

d Given that the average employee charity contribution is £5 each year, find the total charity donation over the 10-year period from the end of year 1 to the end of year 10. Give your answer to the nearest £1000. **(3)**

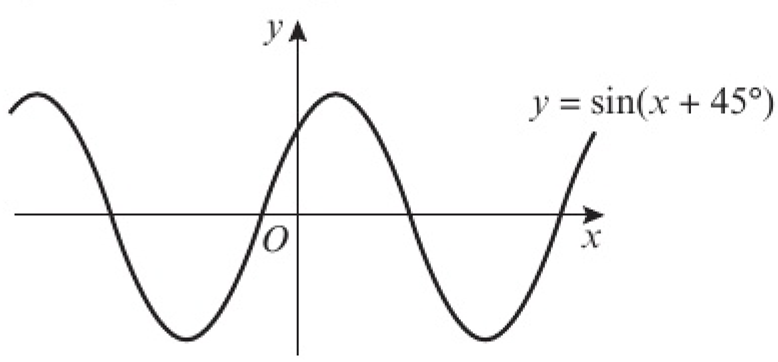
← Pure 2 Section 3.4, 5.8

E **10** **a** On the same set of axes, in the interval $0 \leqslant x \leqslant 360°$, sketch the graphs of $y = \tan(x - 90°)$ and $y = \sin x$. Label clearly any points at which the graphs cross the coordinate axes. **(5)**

b Hence write down the number of solutions in the interval $0 \leqslant x < 360°$ of the equation $\tan(x - 90°) = \sin x$. **(1)**

← Pure 2 Section 6.1, 6.6

E **11** The graph shows the curve $y = \sin(x + 45°)$, $-360° \leqslant x \leqslant 360°$.



a Given that $\sin\theta = \cos\theta$, find the value of $\tan\theta$. **(2)**

b Find the values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$ for which $\sin\theta = \cos\theta$. **(1)**

← Pure 2 Section 6.1

E **12** **a** Given that $\sin\theta = \cos\theta$, find the value of $\tan\theta$. **(1)**

b Find the values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$ for which $\sin\theta = \cos\theta$. **(2)**

← Pure 2 Sections 6.4

E **13** Find all the values of θ in the interval $0 \leqslant x < 360°$ for which $3\tan^2 x = 1$. **(4)**

← Pure 2 Section 6.4

E **14** Find all the values of θ in the interval $0 \leqslant \theta \leqslant 360°$ for which $2\sin\left(\theta - \frac{\pi}{6}\right) = \sqrt{3}$. **(4)**

← Pure 2 Section 6.5

E **15** **a** Show that the equation $2\cos^2 x = 4 - 5\sin x$ may be written as $2\sin^2 x - 5\sin x + 2 = 0$. **(2)**

b Hence solve, for $0 \leqslant x < 360°$, the equation $2\cos^2 x = 4 - 5\sin x$. **(4)**

E **16** Find all the solutions in the interval $0 \leqslant \theta \leqslant 2\pi$ of $2\tan^2 x - 4 = 5\tan x$.

Give each solution in radians to 3 significant figures.

E **17** Find all of the solutions in the interval $0 \leqslant x < 360°$ of $5\sin^2 x = 6(1 - \cos x)$ giving each solution, in degrees, to one decimal place. **(7)**

← Pure 2 Section 6.6

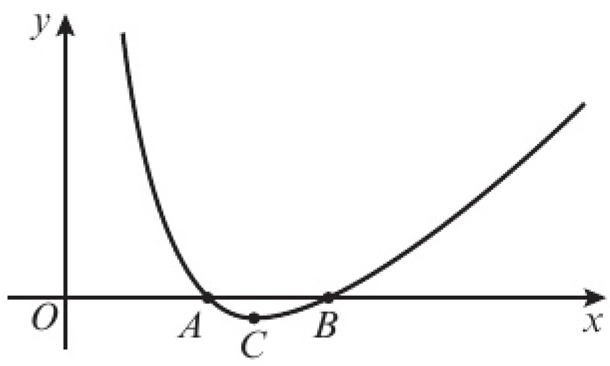
E/P **18** Prove that $\cos^2 x(\tan^2 x + 1) = 1$ for all values of x where $\cos x$ and $\tan x$ are defined. **(4)**

← Pure 2 Sections 6.6

E/P **19** Prove that the function $f(x) = x^3 - 12x^2 + 48x$ is increasing for all $x \in \mathbb{R}$. **(3)**

← Pure 2 Section 7.1

E/P **20** The diagram shows part of the curve with equation $y = x + \frac{2}{x} - 3$. The curve crosses the x-axis at A and B and the point C is the minimum point of the curve.



a Find the coordinates of A and B. **(2)**

b Find the exact coordinates of C, giving your answers in surd form. **(4)**

← Pure 2 Section 7.2

E/P **21** A company makes solid cylinders of variable radius r cm and constant volume 128π cm^3.

a Show that the surface area of the cylinder is given by $S = \frac{256\pi}{r} + 2\pi r^2$. **(2)**

b Find the minium value for the surface area of the cylinder. **(4)**

← Pure 2 Section 7.4

E **22** Given that $y = 3x^2 + 4\sqrt{x}$, $x > 0$, find

a $\frac{dy}{dx}$ **(2)**

b $\frac{d^2y}{dx^2}$ **(2)**

c $\int y\,dx$ **(3)**

← Pure 2 Sections 7.1, 7.2, 8.2

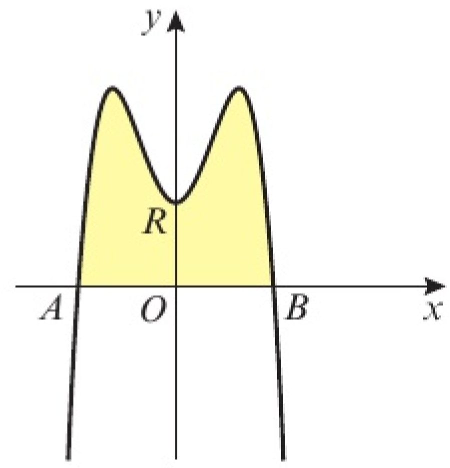
23 Use calculus to evaluate $\int_1^8 (x^{\frac{1}{3}} - x^{-\frac{1}{3}})\,dx$.

← Pure 2 Section 8.1

E/P **24** Given that $\int_0^6 (x^2 - kx)\,dx = 0$, find the value of the constant k. **(3)**

← Pure 2 Section 8.1

E/P **25** The diagram shows a section of the curve with equation $y = -x^4 + 3x^2 + 4$. The curve intersects the x-axis at points A and B. The finite region R, which is shown shaded, is bounded by the curve and the x-axis.

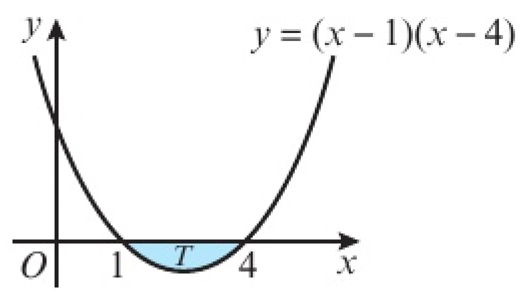


a Show that the equation $-x^4 + 3x^2 + 4 = 0$ only has two solutions, and hence or otherwise find the coordinates of A and B. **(3)**

b Find the area of the region R. **(4)**

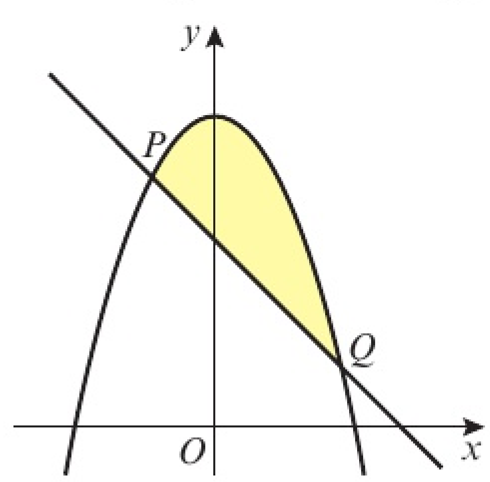
← Pure 2 Sections 8.2, 8.5

E **26** The diagram shows the shaded region T which is bounded by the curve $y = (x - 1)(x - 4)$ and the x-axis.
Find the area of the shaded region T. **(4)**



← Pure 2 Section 8.3

E/P **27** The diagram shows the curve with equation $y = 5 - x^2$ and the line with equation $y = 3 - x$. The curve and the line intersect at the points P and Q.



a Find the coordinates of P and Q. **(3)**

b Find the area of the finite region between PQ and the curve. **(6)**

← Pure 2 Section 8.4

E **28** **a** Sketch the graph of $y = \frac{1}{x}$, $x > 0$. **(2)**

b Copy and complete the table, giving your values of $\frac{1}{x}$ to 3 decimal places. **(2)**

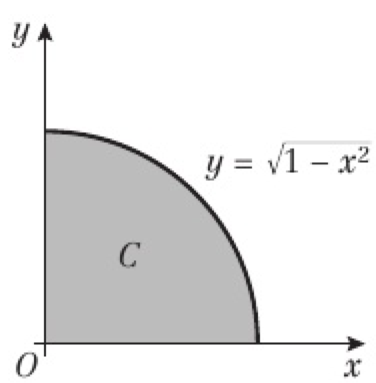
x	1	1.2	1.4	1.6	1.8	2
$\frac{1}{x}$	1					0.5

c Use the trapezium rule, with all the values from your table, to find an estimate for the value of $\int_1^2 \frac{1}{x}\,dx$. **(3)**

d Is this an overestimate or an underestimate for the value of $\int_1^2 \frac{1}{x}\,dx$? Give a reason for your answer. **(2)**

← Pure 2 Section 8.5

E **29** The diagram shows the shaded region C which is bounded by the circle $y = \sqrt{1 - x^2}$ and the coordinate axes.



a Use the trapezium rule with 10 strips to find an estimate, to 3 decimal places, for the area of the shaded region C.

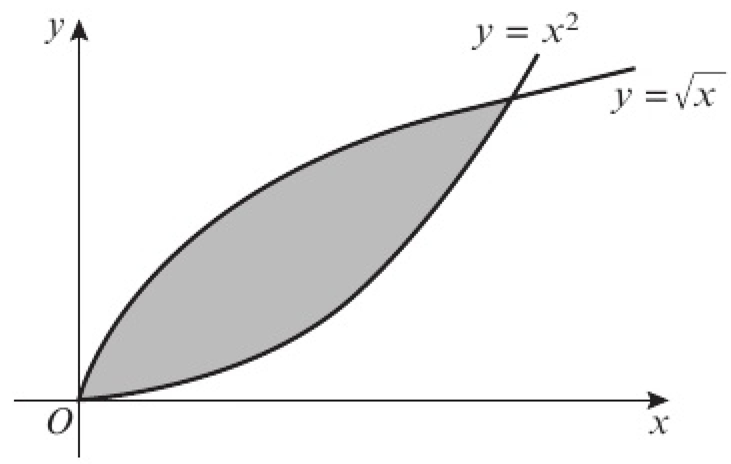
The actual area of C is $\frac{\pi}{4}$ **(5)**

b Calculate the percentage error in your calculation for the area of C. **(2)**

← Pure 2 Section 8.5

E **30** The diagram below shows part of the curves of $y = x^2$ and $y = \sqrt{x}$.

Use calculus to find the area of the finite region enclosed by the curves.



Challenge

SKILLS CREATIVITY

1 Given that $a_{n+1} = a_n + k$, $a_1 = m$ and $\sum_{i=6}^{11} a_i = \sum_{i=12}^{15} a_i$, show that $m = \frac{5}{2}k$. ← Pure 2 Section 5.6

2 Solve for $0° \leqslant x \leqslant 360°$ the equation $2\sin^3 x - \sin x + 1 = \cos^2 x$. ← Pure 2 Section 6.6

Exam practice

Mathematics
International Advanced Subsidiary/ Advanced Level Pure Mathematics 2

Time: 1 hour 30 minutes
You must have: Mathematical Formulae and Statistical Tables, Calculator
Answer ALL questions

---

1 Prove, by exhaustion, that if n is an integer and $2 \leqslant n \leqslant 7$, then $A = n^2 + 2$ is not divisible by 4. **(4)**

2 Given that a and b are positive constants, solve the simultaneous equations

$$\log_6 a + \log_6 b = 2$$

$$\frac{a}{b} = 144$$

Show each step of your working giving exact values for a and b. **(6)**

3 $f(x) = 2x^3 - 3px^2 + x + 4p$

Given that $(x - 4)$ is a factor of $f(x)$,

a show that the value of p is 3. **(2)**

Using this value of p,

b find the remainder when $f(x)$ is divided by $(x + 2)$ **(2)**

c factorise $f(x)$ completely. **(3)**

4 Figure 1 shows a sketch of part of the graph of $y = (1 + x^2)^5$, $x \geqslant 0$.

Complete the table below giving your values of y rounded to 4 decimal places. **(2)**

x	0	0.1	0.2	0.3	0.4
y	1.0000		1.2167		2.1003

Use the trapezium rule with 4 strips to estimate the approximate value, to 3 decimal places, for

$$\int_0^{0.4} (1 + x^2)^5 \, dx$$ **(4)**

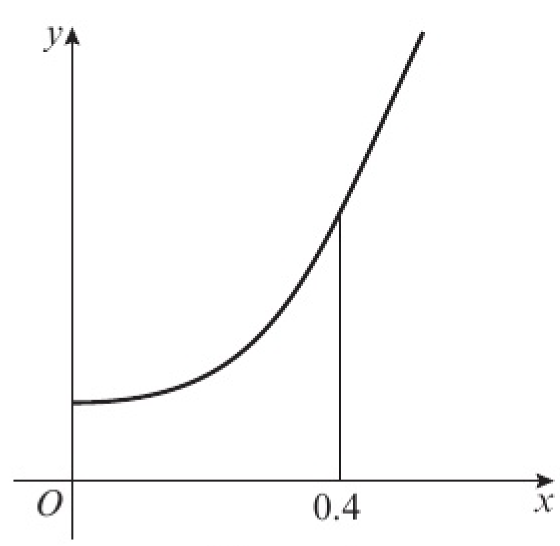


Figure 1

5 The nth term of a geometric series is t_n and the common ratio is r.

Given that $t_3 + t_6 = \frac{28}{81}$ and $t_3 - t_6 = \frac{76}{405}$

a **i** show that $r = \frac{2}{3}$

ii find the first term of the series. **(5)**

b Find the sum to infinity of this geometric series. **(2)**

6 A circle with centre O has equation $x^2 - 2x + y^2 + 10y - 19 = 0$.

a **i** Find the coordinates of O.

ii Find the radius of the circle. **(4)**

Point P has coordinates $(7, -2)$.

b Verify that P lies on the circle. **(1)**

c Find the equation of the tangent to the circle at P.
Give your answer in the form $ax + by + c = 0$ where a, b and c are integers. **(4)**

7 Two numbers x and y are such that $3x + y = 15$.

The sum of the squares of $2x$ and y is S.

a Show that $S = 18x^2 - 90x + 225$. **(3)**

Using calculus,

b find the value of x for which S is a minimum, justifying that this value of x gives a minimum value of S. **(4)**

c Find the minimum value of S. **(2)**

8 **a** Find the first 4 terms of the binomial expansion, in ascending powers of x, of

$$\left(1 - \frac{x}{4}\right)^9$$

giving each term in its simplest form. **(3)**

b Use your expansion to estimate the value of $(0.975)^9$, giving your answer to 4 decimal places. **(3)**

9 The line with equation $y = 3x + 10$ intersects the curve with equation $y = -x^2 + x + 13$ at the points P and Q as shown in Figure 2.

a Use algebra to find the coordinates of P and the coordinates of Q. **(4)**

The shaded region S is bounded by the line and the curve as shown in Figure 2.

b Use calculus to find the exact area of S. **(7)**

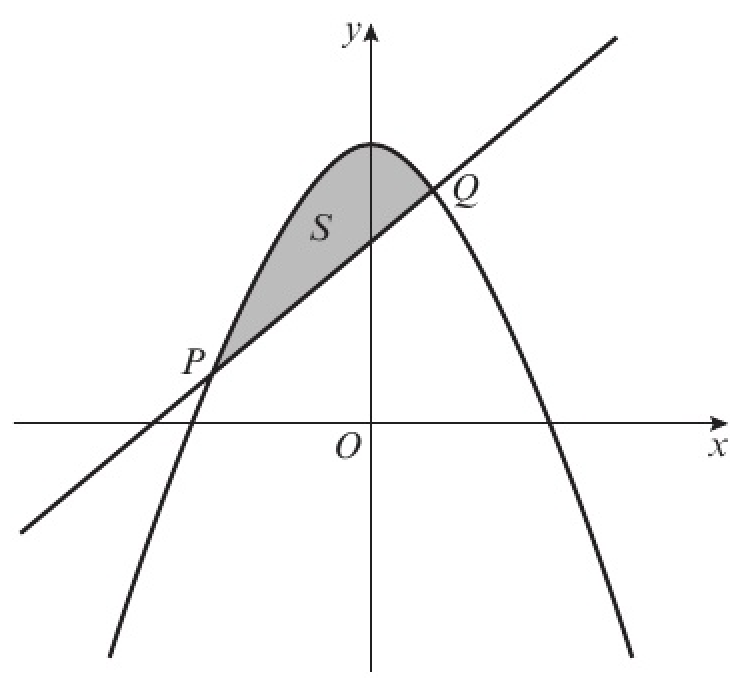


Figure 2

10 **a** Solve for $0 \leqslant x \leqslant 180°$, giving your answers in degrees to 1 decimal place,

$$2\tan(2x + 30°) = 3$$ **(4)**

b Find, for $0 \leqslant x \leqslant \pi$, all the solutions of

$$6\cos^2 x + \sin x - 4 = 0$$

giving your answers in radians to 3 significant figures.

You must show clearly how you obtained your answers. **(6)**

TOTAL FOR PAPER: 75 MARKS

GLOSSARY

acute (angle) – is an angle that is less than 90°.

adjacent – is the side of a right-angled triangle that is next to a given angle and is not the **hypotenuse**.

algebraic – a mathematical expression consisting of numbers, operations (add, subtract, etc.) and letters representing unknown values.

arc – of a circle is part of the **circumference** of a circle.

arithmetic sequence – an ordered set of numbers where the difference between consecutive terms is constant

arithmetic series – is the sum of the terms in an **arithmetic sequence**.

asymptote – is a line that a curve approaches but never quite reaches.

binomial – is an algebraic expression of the sum or difference of two terms.

For example; $(a + b)^n$ is the general form of a binomial expression

bisector – is the line that divides a line segment or angle into two equal parts.

calculus – is the mathematical study of continuous change

circle – is the set of all points in a plane that are the same distance from a given point; the centre.

chord – is a straight line segment whose **endpoints** both lie on the circle.

circumference – of a circle is the distance around the circle.

coefficient – is a numerical or constant quantity placed before and multiplying the variable in an algebraic expression.

For example: 4 is the coefficient of $4x^3$

common Ratio – is the constant ratio between two consecutive terms in a geometric sequence or series.

conjecture – A statement believed to be true that is yet to be proved

consecutive – is something that follows directly after the thing before it.

constant – is a fixed value in an expression.

coordinates – a set of values that show an exact position. The first number represents a point on the x-axis; the second number represents a point on the y-axis in a 2 dimensional grid.

cosine – is the **trigonometric** function that is equal to the ratio of the side **adjacent** to an acute angle (in a right-angled triangle) to the **hypotenuse**.

decreasing function – is a graph that moves downward as it moves from left to right. That is, the gradient is negative at all points in a decreasing function.

definite integral – is an integral having a definite (fixed) value.

derivative – is a way to represent the rate of change of a mathematical function.

diameter – is any straight line that passes through the centre of the circle and whose endpoints both lie on the circle.

difference – is the result of subtracting one number from another. The **common difference** in an arithmetic sequence or series is the constant difference between two consecutive terms.

differentiation – is the instantaneous rate of change of a function with respect to one of its variables.

discriminant – is an expression that allows one to determine whether a quadratic equation has two, one or no solutions.

disproof by counter-example – is a way of proving that a given statement cannot be true by showing a way that is contrary to the statement.

('Contrary' is C1 vocabulary.)

divisor – a number or algebraic expression by which another number or algebraic expression is to be divided.

endpoint of a line segment – is the point at the start or end of part of a straight line.

equation – is a statement that values of two mathematical expressions are equal.

establish – means work out or prove.

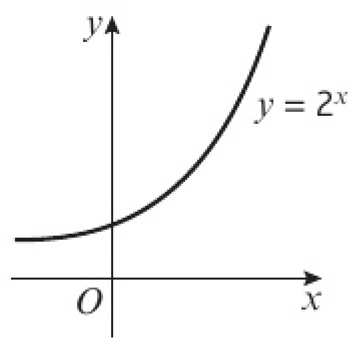
evaluate – a definite integral is to substitute numerical values for each variable and perform the arithmetical operations.

even number – is an **integer** that is divisible by 2 without leaving a remainder.

expansion – is a mathematical expression written in an extended form.
For example, the expansion of $(x + y)^2$
is $x^3 + 3x^2y + 3xy^2 + y^3$

exponent – is the power to which a given number or expression is to be raised.
For example, the exponent of 2^3 is 3.

exponential – something is said to increase or decrease exponentially if its rate of change is expressed using exponents. A graph of such a rate is a curve that continually becomes steeper or shallower. For example:



factorial – denoted by $n!$ Is the product of all positive integers less than or equal to n.
For example, $5! = 5 \times 4 \times 3 \times 2 \times 1$.

finite – is a value that is bounded; not infinite.

fraction – a mathematical expression representing the division of one whole number by another.

geometric sequence – a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the **common ratio**.

geometric series – is the sum of the terms in a **geometric sequence**.

hypotenuse – is the longest side of a right-angled triangle.

identity – is an equality between expressions that is true for all values of the variables in those expressions.

increasing function – is a graph that moves upwards as it moves from left to right. That is, the gradient is positive at all points in an increasing function.

indices – is another name for exponents.

integer – is a whole number (1, 2, etc.).

integration – is one of the two operations of calculus, the other being differentiation, its inverse.

logarithm – is the power to which the base number must be raised in order to get a particular number.
For example, $\log_2 32 = 5 \Rightarrow 2^5 = 32$.

mid point of a line segment – a **point** on a line segment that divides it into two equal parts.

***n*th term** – is the expression that will allow us to calculate the term that is in the nth position in the sequence or series.
For example, the nth term of the arithmetic sequence 2, 6, 10, 14, 18, 22, … is $4n - 2$.

obtuse (angle) – is an angle that is greater than 90° but less than 180°.

odd number – an integer which is not a multiple of 2.

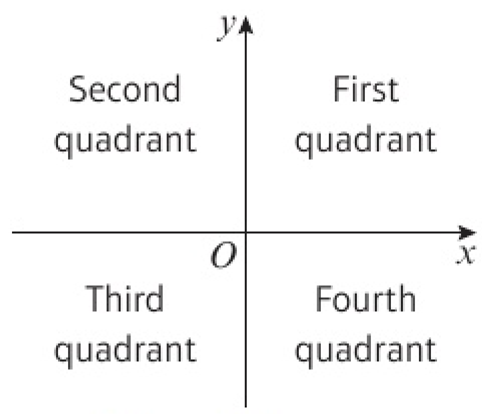
perpendicular – means at right angles. A line meeting another at 90°.

point – marks a location but has no size itself.

polynomial – is an expression of two or more algebraic terms with positive whole number **indices**.
For example, $2x + 6x^2 + 7x^6$ is a polynomial.

proof by exhaustion – proving a mathematical statement is true by showing that it is true for each and every case that could possibly be considered.

quadrant – the area defined by a set of two-dimensional axes is divided into four quadrants:



quadrilateral – a polygon with four edges and four **vertices**.

quotient – a result obtained by dividing one quantity by another.

radius – a line segment from the centre of a circle to its perimeter.

reflex (angle) – is an angle that is greater than 180° but less than 360°.

remainder – the amount left over after dividing one integer or algebraic expression by another.

sector – of a circle is the part of a circle enclosed by two **radii** and their **arc**.

segment – of a circle is the region bounded by a chord and its arc.

sine – is the trigonometric function that is equal to the ratio of the side opposite an acute angle (in a right-angled triangle) to the hypotenuse.

solve an equation – to determine the value(s) of the variable.

stationary point – is the point on a function where the gradient is zero.

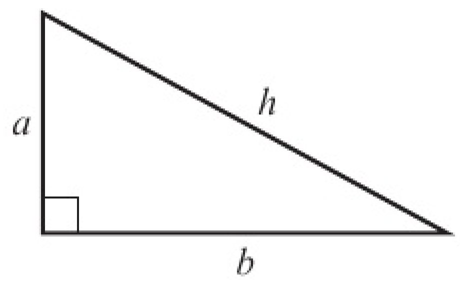
substitute – to replace a variable in an expression with a value or another representation.

symmetry – is when a shape looks the same following a transformation such as reflection or rotation.

tangent – of a curve is the straight line that just touches the curve at a point, matching the gradient of the curve at that point.

tangent – is the trigonometric function that is equal to the ratio of the side opposite an acute angle (in a right-angled triangle) to the adjacent.

theorem – a mathematical statement that has been proved on the basis of previously **established** statements.
For example, Pythagoras' Theorem
$h^2 = a^2 + b^2$



trapezium rule – is a numerical method for approximating the area of the region under the graph of a function.

trigonometric – means related to triangles and their sides and angles.

turning point – is a point at which $\frac{dy}{dx}$ changes sign. It is also known as a maximum, a minimum or a stationary point. However, not all stationary points are turning points. (A point of inflection is a stationary point but not a turning point.)

unit circle – is a circle with a radius of 1 unit.

vertices – the points at which the sides of a geometrical shape meet.

ANSWERS

CHAPTER 1

Prior knowledge check

1 **a** $15x^7$ **b** $\frac{x}{3y}$

2 **a** $(x-6)(x+4)$ **b** $(3x-5)(x-4)$

3 **a** 8567 **b** 1652

4 **a** $y = 1 - 3x$ **b** $y = \frac{1}{2}x - 7$

5 **a** $(x-1)^2 - 21$ **b** $2(x+1)^2 + 13$

Exercise 1A

1 **a** $4x^3 + 5x - 7$ **b** $2x^4 + 9x^2 + x$
c $-x^3 + 4x + \frac{6}{x}$ **d** $7x^4 - x^2 - \frac{4}{x}$
e $4x^3 - 2x^2 + 3$ **f** $3x - 4x^2 - 1$
g $\frac{7x^2}{5} - \frac{x^3}{5} - \frac{2}{5x}$ **h** $2x - 3x^3 + 1$
i $\frac{x^7}{2} - \frac{9x^3}{2} + 2x^2 - \frac{3}{x}$ **j** $3x^8 + 2x^5 - \frac{4x^3}{3} + \frac{2}{3x}$

2 **a** $x+3$ **b** $x+4$ **c** $x+3$
d $x+7$ **e** $x+5$ **f** $x+4$
g $\frac{x-4}{x-3}$ **h** $\frac{x+2}{x+4}$ **i** $\frac{x+4}{x-6}$
j $\frac{2x+3}{x-5}$ **k** $\frac{2x-3}{x+1}$ **l** $\frac{x-2}{x+2}$
m $\frac{2x+1}{x-2}$ **n** $\frac{x+4}{3x+1}$ **o** $\frac{2x+1}{2x-3}$

3 $a = 1, b = 4, c = -2$

Exercise 1B

1 **a** $(x+1)(x^2+5x+3)$ **b** $(x+4)(x^2+6x+1)$
c $(x+2)(x^2-3x+7)$ **d** $(x-3)(x^2+4x+5)$
e $(x-5)(x^2-3x-2)$ **f** $(x-7)(x^2+2x+8)$

2 **a** $(x+4)(6x^2+3x+2)$ **b** $(x+2)(4x^2+x-5)$
c $(x+3)(2x^2-2x-3)$ **d** $(x-6)(2x^2-3x-4)$
e $(x+6)(-5x^2+3x+5)$ **f** $(x-2)(-4x^2+x-1)$

3 **a** $x^3 + 3x^2 - 4x + 1$ **b** $4x^3 + 2x^2 - 3x - 5$
c $-3x^3 + 3x^2 - 4x - 7$ **d** $-5x^4 + 2x^3 + 4x^2 - 3x + 7$

4 **a** $x^3 + 2x^2 - 5x + 4$ **b** $x^3 - x^2 + 3x - 1$
c $2x^3 + 5x + 2$ **d** $3x^4 + 2x^3 - 5x^2 + 3x + 6$
e $2x^4 - 2x^3 + 3x^2 + 4x - 7$ **f** $4x^4 - 3x^3 - 2x^2 + 6x - 5$
g $5x^3 + 12x^2 - 6x - 2$ **h** $3x^4 + 5x^3 + 6$

5 **a** $x^2 - 2x + 5$ **b** $2x^2 - 6x + 1$
c $-3x^2 - 12x + 2$

6 **a** $x^2 + 4x + 12$ **b** $2x^2 - x + 5$
c $-3x^2 + 5x + 10$

7 Divide $x^3 + 2x^2 - 5x - 10$ by $(x+2)$ to give $(x^2 - 5)$. So $x^3 + 2x^2 - 5x - 10 = (x+2)(x^2-5)$.

8 **a** -8 **b** -7 **c** -12

9 $f(1) = 3 - 2 + 4 = 5$

10 $f(-1) = 3 + 8 + 10 + 3 - 25 = -1$

11 $(x+4)(5x^2 - 20x + 7)$

12 $3x^2 + 6x + 4$

13 $x^2 + x + 1$

14 $x^3 - 2x^2 + 4x - 8$

15 14

16 **a** -200 **b** $(x+2)(x-7)(3x+1)$

17 **a** **i** 30 **ii** 0 **b** $x = -3, x = -4, x = 1$

18 **a** $a = 1, b = 2, c = -3$
b $f(x) = (2x-1)(x+3)(x-1)$
c $x = 0.5, x = -3, x = 1$

19 **a** $a = 3, b = 2, c = 1$
b Quadratic has no real solutions so only $\frac{1}{4}$ is a solution

Exercise 1C

1 **a** $f(1) = 0$ **b** $f(-3) = 0$ **c** $f(4) = 0$

2 $(x-1)(x+3)(x+4)$

3 $(x+1)(x+7)(x-5)$

4 $(x-5)(x-4)(x+2)$

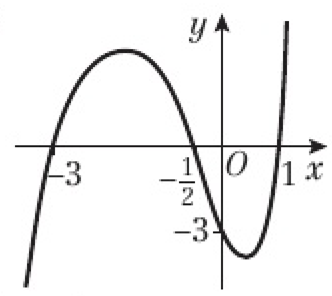
5 $(x-2)(2x-1)(x+4)$

6 $f\left(\frac{1}{2}\right) = 2 \times \left(\frac{1}{2}\right)^3 + 17 \times \left(\frac{1}{2}\right)^2 + 31 \times \left(\frac{1}{2}\right) - 20 = 0$

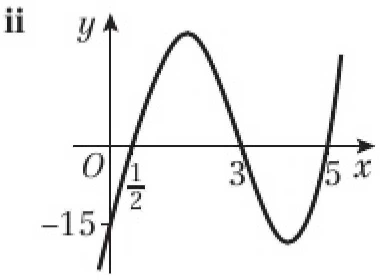
hence $2x - 1$ is a factor

7 **a** $(x+1)(x-5)(x-6)$ **b** $(x-2)(x+1)(x+2)$
c $(x-5)(x+3)(x-2)$

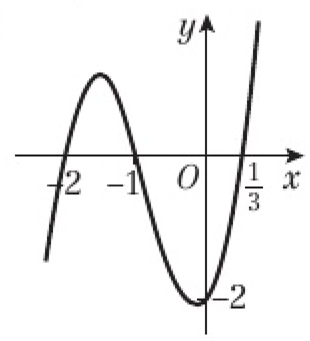
8 **a** **i** $(x-1)(x+3)(2x+1)$ **ii**



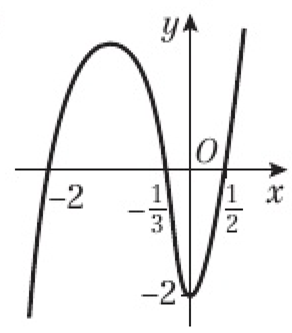
b **i** $(x-3)(x-5)(2x-1)$ **ii**



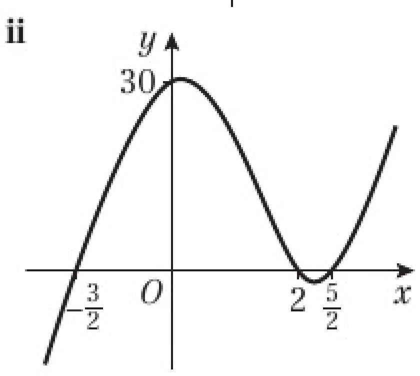
c **i** $(x+1)(x+2)(3x-1)$ **ii**



d **i** $(x+2)(2x-1)(3x+1)$ **ii**



e **i** $(x-2)(2x-5)(2x+3)$ **ii**



9 $(2x+1)(x+3)(x-1)$

10 2

11 -16

12 $p = 3, q = 7$
13 $c = 2, d = 3$
14 $p = 6, q = -13$
15 $g = 3, h = -7$
16 $b = 2 \quad (3x + 2)(x - 3)(x + 1)$
17 **a** $f(4) = 0$
b $f(x) = (x - 4)(3x^2 + 6)$
For $3x^2 + 6 = 0$, $b^2 - 4ac = -72$ so there are no real roots. Therefore, 4 is the only real root of $f(x) = 0$.
18 **a** $f(-2) = 0$ **b** $(x + 2)(2x + 1)(2x - 3)$
c $x = -2, x = -\frac{1}{2}$ and $x = 1\frac{1}{2}$
19 **a** $f(2) = 0$ **b** $x = 0, x = 2, x = -\frac{1}{3}$ and $x = \frac{1}{3}$

Challenge

a $f(1) = 2 - 5 - 42 - 9 + 54 = 0$
$f(-3) = 162 + 135 - 378 + 27 + 54 = 0$
b $2x^4 - 5x^3 - 42x^2 - 9x + 54$
$= (x - 1)(x + 3)(x - 6)(2x + 3)$
$x = 1\ x = -3, x = 6, x = -1.5$

Exercise 1D

1 **a** 27 **b** −6 **c** 0 **d** 1 **e** $2\frac{1}{4}$
f 8 **g** 14 **h** 0 **i** $-15\frac{1}{3}$ **j** 20.52
2 −1
3 18
4 30
7 −9
8 $8\frac{8}{27}$
9 $a = 5, b = -8$
10 $p = 8, q = 3$

Exercise 1E

1 $n^2 - n = n(n - 1)$
If n is even, $n - 1$ is odd and even × odd = even
If n is odd, $n - 1$ is even and odd × even = even
2 $\frac{x}{(1 + \sqrt{2})} \times \frac{(1 - \sqrt{2})}{(1 - \sqrt{2})} = \frac{x(1 - \sqrt{2})}{(1 - 2)} = \frac{x - x\sqrt{2}}{-1} = x\sqrt{2} - x$
3 $(x + \sqrt{y})(x - \sqrt{y}) = x^2 - x\sqrt{y} + x\sqrt{y} - y = x^2 - y$
4 $(2x - 1)(x + 6)(x - 5) = (2x - 1)(x^2 + x - 30)$
$= 2x^3 + x^2 - 61x + 30$
5 LHS $= x^2 + bx$, using completing the square,
$\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$
6 $x^2 + 2bx + c = 0$, using completing the square
$(x + b)^2 + c - b^2 = 0$
$(x + b)^2 = b^2 - c$
$x + b = \pm\sqrt{b^2 - c}$
$x = -b \pm \sqrt{b^2 - c}$
7 $\left(x - \frac{2}{x}\right)^3 = \left(x - \frac{2}{x}\right)\left(x^2 - 4 + \frac{4}{x^2}\right) = x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$
8 $\left(x^3 - \frac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{\frac{-5}{2}}\right) = x^{\frac{9}{2}} + x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{-\frac{7}{2}} = x^{\frac{9}{2}} - x^{-\frac{7}{2}}$
$= x^{\frac{1}{2}}\left(x^4 - \frac{1}{x^4}\right)$
9 $3n^2 - 4n + 10 = 3\left[n^2 - \frac{4}{3}n + \frac{10}{3}\right] = 3\left[\left(n - \frac{2}{3}\right)^2 + \frac{10}{3} - \frac{4}{9}\right]$
$= 3\left(n - \frac{2}{3}\right)^2 + \frac{26}{3}$
The minimum value is $\frac{26}{3}$ so $3n^2 - 4n + 10$ is always positive.
10 $-n^2 - 2n - 3 = -[n^2 + 2n + 3] = -[(n + 1)^2 + 3 - 1]$
$= -(n + 1)^2 - 2$
The maximum value is −2 so $-n^2 - 2n - 3$ is always negative.
11 $x^2 + 8x + 20 = (x + 4)^2 + 4$
The minimum value is 4 so $x^2 + 8x + 20$ is always greater than or equal to 4.
12 $kx^2 + 5kx + 3 = 0$, $b^2 - 4ac < 0$, $25k^2 - 12k < 0$,
$k(25k - 12) < 0$, $0 < k < \frac{12}{25}$.
When $k = 0$ there are no real roots, so $0 \leqslant k < \frac{12}{25}$
13 $px^2 - 5x - 6 = 0$, $b^2 - 4ac > 0$, $25 + 24p > 0$, $p > -\frac{25}{24}$
14 Gradient $AB = -\frac{1}{2}$, gradient $BC = 2$,
Gradient AB × gradient $BC = -\frac{1}{2} \times 2 = -1$,
so AB and BC are perpendicular.
15 Gradient $AB = 3$, gradient $BC = \frac{1}{4}$, gradient $CD = 3$, gradient $AD = \frac{1}{4}$
Gradient AB = gradient CD so AB and CD are parallel.
Gradient BC = gradient AD so BC and AD are parallel.
16 Gradient $AB = \frac{1}{3}$, gradient $BC = 3$, gradient $CD = \frac{1}{3}$, gradient $AD = 3$
Gradient AB = gradient CD so AB and CD are parallel.
Gradient BC = gradient AD so BC and AD are parallel.
Length $AB = \sqrt{10}$, $BC = \sqrt{10}$, $CD = \sqrt{10}$ and $AD = \sqrt{10}$, so all four sides are equal.
17 Gradient $AB = -3$, gradient $BC = \frac{1}{3}$,
Gradient AB × gradient $BC = -3 \times \frac{1}{3} = -1$, so AB and BC are perpendicular.
Length $AB = \sqrt{40}$, $BC = \sqrt{40}$, $AB = BC$
18 $(x - 1)^2 + y^2 = k$, $y = ax$, $(x - 1)^2 + a^2x^2 = k$,
$x^2(1 + a^2) - 2x + 1 - k = 0$
$b^2 - 4ac > 0$, $k > \frac{a^2}{1 + a^2}$.
19 $x = 2$. There is only one solution so the line $4y - 3x + 26 = 0$ only touches the circle in one place so is the tangent to the circle.
20 Area of square $= (a + b)^2 = a^2 + 2ab + b^2$
Shaded area $= 4\left(\frac{1}{2}ab\right)$
Area of smaller square: $a^2 + 2ab + b^2 - 2ab$
$= a^2 + b^2 = c^2$

Challenge

1 The equation of the circle is $(x - 3)^2 + (y - 5)^2 = 25$ and all four points satisfy this equation.
2 $2k + 1 = 1 \times (2k + 1) = ((k + 1) - k)((k + 1) + k) = (k + 1)^2 - k^2$

Exercise 1F

1 3, 4, 5, 6, 7 and 8 are not divisible by 10
2 3, 5, 7, 11, 13, 17, 19, 23 are prime numbers. 9, 15, 21, 25 are the product of two prime numbers.
3 $1^2 + 2^2 = 5$, $2^2 + 3^2 =$ odd, $3^2 + 4^2 =$ odd, $4^2 + 5^2 =$ odd, $5^2 + 6^2 =$ odd, $6^2 + 7^2 =$ odd, $7^2 + 8^2 = 113$
4 $(3n)^3 = 27n^3 = 9n(3n^2)$ which is a multiple of 9
$(3n + 1)^3 = 27n^3 + 27n^2 + 9n + 1 = 9n(3n^2 + 3n + 1) + 1$
which is one more than a multiple of 9
$(3n + 2)^3 = 27n^3 + 54n^2 + 36n + 8 = 9n(3n^2 + 6n + 4) + 8$
which is one less than a multiple of 9
5 **a** For example, when $n = 2$, $2^4 - 2 = 14$, 14 is not divisible by 4.
b Any square number
c For example, when $n = \frac{1}{2}$
d For example, when $n = 1$
6 **a** Assuming that x and y are positive
b e.g. $x = 0, y = 0$
7 $(x + 5)^2 \geqslant 0$ for all real values of x, and
$(x + 5)^2 + 2x + 11 = (x + 6)^2$, so $(x + 6)^2 \geqslant 2x + 11$

8 If $a^2 + 1 \geqslant 2a$ (a is positive, so multiplying both sides by a does not reverse the inequality), then $a^2 - 2a + 1 \geqslant 0$, and $(a - 1)^2 \geqslant 0$, which we know is true.

9 **a** $(p + q)^2 = p^2 + 2pq + q^2 = (p - q)^2 + 4pq$
$(p - q)^2 \geqslant 0$ since it is a square, so $(p + q)^2 \geqslant 4pq$
$p > 0, q > 0 \Rightarrow p + q > 0 \Rightarrow p + q \geqslant \sqrt{4pq}$
b e.g. $p = q = -1$: $p + q = -2$, $\sqrt{4pq} = 2$

10 **a** Starts by assuming the inequality is true: i.e. negative $\geqslant$ positive
b e.g. $x = y = -1$: $x + y = -2$, $\sqrt{x^2 + y^2} = \sqrt{2}$
c $(x + y)^2 = x^2 + 2xy + y^2 > x^2 + y^2$ since $x > 0$, $y > 0 \Rightarrow 2xy > 0$
As $x + y > 0$, can take square roots: $x + y \geqslant \sqrt{x^2 + y^2}$

Chapter review 1

1 **a** $x^3 - 7$ **b** $\frac{x + 4}{x - 1}$ **c** $\frac{2x - 1}{2x + 1}$

2 $3x^2 + 5$

3 $2x^2 - 2x + 5$

4 **a** When $x = 3$, $2x^3 - 2x^2 - 17x + 15 = 0$
b $A = 2, B = 4, C = -5$

5 $7\frac{1}{4}$

6 **a** When $x = 2$, $x^3 + 4x^2 - 3x - 18 = 0$
b $p = 1, q = 3$

7 $(x - 2)(x + 4)(2x - 1)$

8 7

9 **a** $p = 1, q = -15$ **b** $(x + 3)(2x - 5)$

10 **a** $r = 3, s = 0$ **b** $x(x + 1)(x + 3)$

11 **a** $(x - 1)(x + 5)(2x + 1)$ **b** $-5, -\frac{1}{2}, 1$

12 **a** When $x = 2$, $x^3 + x^2 - 5x - 2 = 0$
b $2, -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$

13 $\frac{1}{2}, 3$

14 -2

15 **a** When $x = -4$, $f(x) = 0$
b $x = -4$, $x = 1$ and $x = 5$

16 **a** $f(\frac{2}{3}) = 0$, therefore $(3x - 2)$ is a factor of $f(x)$
$a = 2, b = 7$ and $c = 3$
b $(3x - 2)(2x + 1)(x + 3)$
c $x = \frac{2}{3}, -\frac{1}{2}, -3$

17 $\frac{x - y}{(\sqrt{x} - \sqrt{y})} \times \frac{(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})} = \frac{(x - y)(\sqrt{x} + \sqrt{y})}{x - y} = \sqrt{x} + \sqrt{y}$

18 $n^2 - 8n + 20 = (n - 4)^2 + 4$, 4 is the minimum value so $n^2 - 8n + 20$ is always positive

19 Gradient $AB = \frac{1}{2}$, gradient $BC = -2$, gradient $CD = \frac{1}{2}$, gradient $AD = -2$
AB and BC, BC and CD, CD and AD and AB and AD are all perpendicular
Length $AB = \sqrt{5}$, $BC = \sqrt{5}$, $CD = \sqrt{5}$ and $AD = \sqrt{5}$, all four sides are equal

20 $1 + 3 =$ even, $3 + 5 =$ even, $5 + 7 =$ even, $7 + 9 =$ even

21 For example when $n = 6$

22 $\left(x - \frac{1}{x}\right)(x^{\frac{4}{3}} + x^{-\frac{2}{3}}) = x^{\frac{7}{3}} + x^{\frac{1}{3}} - x^{\frac{1}{3}} - x^{-\frac{5}{3}} = x^{\frac{1}{3}}\left(x^2 - \frac{1}{x^2}\right)$

23 RHS $= (x + 4)(x - 5)(2x + 3) = (x + 4)(2x^2 - 7x - 15)$
$= 2x^3 + x^2 - 43x - 60 =$ LHS

24 $x^2 - kx + k = 0$, $b^2 - 4ac = 0$, $k^2 - 4k = 0$, $k(k - 4) = 0$, $k = 4$.

25 The distance between opposite edges
$= 2\sqrt{\sqrt{3}^2 - \left(\frac{\sqrt{3}}{2}\right)^2} = 2\sqrt{\frac{9}{4}} = 3$ which is rational.

26 **a** $(2n + 2)^2 - (2n)^2 = 8n + 4 = 4(2n + 1)$ is always divisible by 4.
b Yes, $(2n + 1)^2 - (2n - 1)^2 = 8n$ which is always divisible by 4.

27 **a** The assumption is that x is positive.
b $x = 0$

Challenge

1 **a** Perimeter of inside square $= 4\left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$
Perimeter of outside square = 4
Circumference of circle $= \pi \times 1^2 = \pi$
therefore $2\sqrt{2} < \pi < 4$.
b Perimeter of inside hexagon = 3
Perimeter of outside hexagon $= 6 \times \frac{\sqrt{3}}{3} = 2\sqrt{3}$,
therefore $3 < \pi < 2\sqrt{3}$

2 $ax^3 + bx^2 + cx + d \div (x - p) = ax^2 + (b + ap)x + (c + bp + ap^2)$ with remainder $d + cp + bp^2 + ap^3$
$f(p) = ap^3 + bp^2 + cp + d = 0$, which matches the remainder, so $(x - p)$ is a factor of $f(x)$.

CHAPTER 2

Prior knowledge check

1 **a** $(x + 5)^2 + 3$ **b** $(x - 3)^2 - 8$
c $(x - 6)^2 - 36$ **d** $(x + \frac{7}{2})^2 - \frac{49}{4}$

2 **a** $y = \frac{9}{4}x - 6$ **b** $y = -\frac{1}{2}x - \frac{3}{2}$
c $y = \frac{4}{3}x + \frac{10}{3}$

3 **a** $b^2 - 4ac = -7$ No real solutions
b $b^2 - 4ac = 89$ Two real solutions
c $b^2 - 4ac = 0$ One real solution

4 $y = -\frac{5}{6}x - \frac{3}{2}$

Exercise 2A

1 **a** (5, 5) **b** (6, 4) **c** (−1, 4) **d** (0, 0)
e (2, 1) **f** $(-8, \frac{3}{2})$ **g** $(4a, 0)$ **h** $\left(-\frac{u}{2}, -v\right)$
i $(2a, a - b)$ **j** $(3\sqrt{2}, 4)$ **k** $(2\sqrt{2}, \sqrt{2} + 3\sqrt{3})$

2 $a = 10, b = 1$

3 $(\frac{3}{2}, 7)$

4 $\left(\frac{3a}{5}, \frac{b}{4}\right)$

5 **a** $(\frac{3}{2}, 3)$ or $(1.5, 3)$ **b** $y = 2x$, $3 = 2 \times 1.5$

6 **a** $\left(\frac{1}{8}, \frac{5}{3}\right)$ **b** $\frac{2}{3}$

7 Centre is $\left(3, -\frac{7}{2}\right)$. $3 - 2\left(-\frac{7}{2}\right) - 10 = 0$

8 (10, 5)

9 $(-7a, 17a)$

10 $p = 8, q = 7$

11 $a = -2, b = 4$

Challenge

a $p = 9, q = -1$
b $y = -x + 13$
c AC: $y = -x + 8$. Lines have the same slope, so they are parallel.

Exercise 2B

1 a $y = 2x + 3$ b $y = -\frac{1}{3}x + \frac{47}{3}$ c $y = \frac{5}{2}x - 25$
d $y = 3$ e $y = -\frac{3}{4}x + \frac{37}{8}$ f $x = 9$

2 $y = -x + 7$

3 $2x - y - 8 = 0$

4 a $y = -\frac{5}{3}x - \frac{13}{3}$ b $y = 3x - 8$ c $\left(\frac{11}{14}, -\frac{79}{14}\right)$

5 $q = -\frac{5}{4}$, $b = -\frac{189}{8}$

Challenge

a PR: $y = -\frac{5}{2}x + \frac{9}{4}$
PQ: $y = -\frac{1}{4}x + \frac{33}{8}$
RQ: $y = 2x + 6$

b $\left(-\frac{5}{6}, \frac{13}{3}\right)$

Exercise 2C

1 a $(x - 3)^2 + (y - 2)^2 = 16$
b $(x + 4)^2 + (y - 5)^2 = 36$
c $(x - 5)^2 + (y + 6)^2 = 12$
d $(x - 2a)^2 + (y - 7a)^2 = 25a^2$
e $(x + 2\sqrt{2})^2 + (y + 3\sqrt{2})^2 = 1$

2 a $(-5, 4)$, 9 b $(7, 1)$, 4
c $(-4, 0)$, 5 d $(-4a, -a)$, $12a$
e $(3\sqrt{5}, -\sqrt{5})$, $3\sqrt{3}$

3 a $(4 - 2)^2 + (8 - 5)^2 = 4 + 9 = 13$
b $(0 + 7)^2 + (-2 - 2)^2 = 49 + 16 = 65$
c $7^2 + (-24)^2 = 49 + 576 = 625 = 25^2$
d $(6a - 2a)^2 + (-3a + 5a)^2 = 16a^2 + 4a^2 = 20a^2$
e $(\sqrt{5} - 3\sqrt{5})^2 + (-\sqrt{5} - \sqrt{5})^2 = (-2\sqrt{5})^2 + (-2\sqrt{5})^2$
$= 20 + 20 = 40 = (2\sqrt{10})^2$

4 $(x - 8)^2 + (y - 1)^2 = 25$

5 $(x - \frac{3}{2})^2 + (y - 4)^2 = \frac{65}{4}$

6 $\sqrt{5}$

7 a $r = 2$
b Distance $PQ = PR = RQ = 2\sqrt{3}$, three equal length sides therefore triangle is equilateral.

8 a $(x - 2)^2 + y^2 = 15$
b Centre (2, 0) and radius = $\sqrt{15}$

9 a $(x - 5)^2 + (y + 2)^2 = 49$
b Centre (5, −2) and radius = 7

10 a Centre (1, −4), radius 5
b Centre (−6, 2), radius 7
c Centre (11, 3), radius $3\sqrt{10}$
d 10 Centre (−2.5, 1.5), radius $\frac{5\sqrt{2}}{2}$
e Centre (2 ,−2), radius $\sqrt{6.5}$

11 a Centre (−6, −1)
b $k > -37$

12 $Q(-13, 28)$

13 $k = -2$ and $k = 8$

Challenge

1 $k = 3$, $(x - 3)^2 + (y - 2)^2 = 50$
$k = 5$, $(x - 5)^2 + (y - 2)^2 = 50$

2 $(x + f)^2 - f^2 + (y + g)^2 - g^2 + c = 0$
So $(x + f)^2 + (y + g)^2 = f^2 + g^2 - c$
Circle with centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$.

Exercise 2D

1 (7, 0), (−5, 0)

2 (0, 2), (0, −8)

3 (6, 10), (−2, 2)

4 (4, −9), (−7, 2)

5 $2x^2 - 24x + 79 = 0$ has no real solutions, therefore lines do not intersect circle.

6 a $b^2 - 4ac = 64 - 4 \times 1 \times 16 = 0$. So there is only one point of intersection.
b (4, 7)

7 a (0, −2), (4, 6) b Midpoint of AB is (2, 2)

8 a 13 b $p = 1$ or 5

9 a $A(5, 0)$ and $B(-3, -8)$ (or vice versa)
b $y = -x - 3$
c (4, −7) is a solution to $y = -x - 3$.
d 20

10 a Substitute $y = kx$ to give
$(k^2 + 1)x^2 - (12k + 10)x + 57 = 0$
$b^2 - 4ac > 0$, $-84k^2 + 240k - 128 > 0$,
$21k^2 - 60k + 32 < 0$
b $0.71 < k < 2.15$
Exact answer is $\frac{10}{7} - \frac{2\sqrt{57}}{21} < k < \frac{10}{7} + \frac{2\sqrt{57}}{21}$

11 $k < \frac{8}{17}$

12 $k = -20 \pm 2\sqrt{105}$

Exercise 2E

1 a $3\sqrt{10}$
b Gradient of radius = 3, gradient of line = $-\frac{1}{3}$, gradients are negative reciprocals and therefore perpendicular.

2 a $(x - 4)^2 + (y - 6)^2 = 73$ b $3x + 8y + 13 = 0$

3 a $y = -2x - 1$
b Centre of circle (1, −3) satisfies $y = -2x - 1$.

4 a $y = \frac{1}{2}x - 3$
b Centre of circle (2, −2) satisfies $y = \frac{1}{2}x - 3$

5 a (−7, −6) satisfies $x^2 + 18x + y^2 - 2y + 29 = 0$
b $y = \frac{2}{7}x - 4$ c $R(0, -4)$ d $\frac{53}{2}$

6 a (0, −17), (17, 0)
b 144.5

7 $y = 2x + 27$ and $y = 2x - 13$

8 a $p = 4$, $p = -6$
b (3, 4) and (3, −6)

9 a $(x - 11)^2 + (y + 5)^2 = 100$
b $y = \frac{3}{4}x - \frac{3}{4}$
c $A(8 - 4\sqrt{3}, -1 - 3\sqrt{3})$ and $B(8 + 4\sqrt{3}, -1 + 3\sqrt{3})$
d $10\sqrt{3}$

10 a $y = 4x - 22$ b $a = 5$ c $(x - 5)^2 + (y + 2)^2 = 34$
d $A(5 + \sqrt{2}, -2 + 4\sqrt{2})$ and $B(5 - \sqrt{2}, -2 - 4\sqrt{2})$

11 a $P(-2, 5)$ and $Q(4, 7)$
b $y = 2x + 9$ and $y = -\frac{1}{2}x + 9$
c $y = -3x + 9$
d (0, 9)

Challenge

1 $y = \frac{1}{2}x - 2$

2 a $\angle CPR = \angle CQR = 90°$ (Angle between tangent and radius)
$CP = CQ = \sqrt{10}$ (Radii of circle)
$CR = \sqrt{(6 - 2)^2 + (-1 - 1)^2} = \sqrt{20}$
So using Pythagoras' Theorem,
$PR = QR = \sqrt{20 - 10} = \sqrt{10}$
4 equal sides and two opposite right–angles, so $CPRQ$ is a square
b $y = \frac{1}{3}x - 3$ and $y = -3x + 17$

Exercise 2F

1 a $WV^2 = WU^2 + UV^2$
  b (2, 3)
  c $(x - 2)^2 + (y - 3)^2 = 41$
2 a $AC^2 = AB^2 + BC^2$
  b $(x - 5)^2 + (y - 2)^2 = 25$
  c 15
3 a i $y = \frac{3}{2}x + \frac{21}{2}$ ii $y = -\frac{2}{3}x + 4$
  b (–3, 6)
  c $(x + 3)^2 + (y - 6)^2 = 169$
4 a i $y = \frac{1}{3}x + \frac{10}{3}$ ii $x = -1$
  b $(x + 1)^2 + (y - 3)^2 = 125$
5 $(x - 3)^2 + (y + 4)^2 = 50$
6 a $AB^2 + BC^2 = AC^2$
  $AB^2 = 400$, $BC^2 = 100$, $AC^2 = 500$
  b $(x + 2)^2 + (y - 5)^2 = 125$
  c $D(8, 0)$ satisfies the equation of the circle.
7 a $AB = BC = CD = DA = \sqrt{50}$
  b 50
  c (3,6)
8 a $DE^2 = b^2 + 6b + 13$
  $EF^2 = b^2 + 10b + 169$
  $DF^2 = 200$
  So $b^2 + 6b + 13 + b^2 + 10b + 169 = 200$
  $(b + 9)(b - 1) = 0$; as $b > 0$, $b = 1$
  b $(x + 5)^2 + (y + 4)^2 = 50$
9 a Centre (–1, 12) and radius = 13
  b Use distance formula to find $AB = 26$. This is twice radius, so AB is the diameter. Other methods possible.
  c $C(-6, 0)$
10 $x + y + 10 = 0$
15 60
16 l_1: $y = -4x + 12$ and l_2: $y = -\frac{8}{19}x + 12$
17 a $y = \frac{1}{3}x + \frac{8}{3}$
  b $(x + 2)^2 + (y - 2)^2 = 50$
  c 20
18 a $P(-3, 1)$ and $Q(9, -7)$
  b $y = \frac{3}{2}x + \frac{11}{2}$ and $y = \frac{3}{2}x - \frac{41}{2}$
19 a $y = -4x + 6$ and $y = \frac{1}{4}x + 6$
  b $P(-4, 5)$ and $Q(1, 2)$
  c 17
20 a $P(5, 16)$ and $Q(13, 8)$
  b l_2: $y = \frac{1}{7}x + \frac{107}{7}$ and l_3: $y = 7x - 83$
  c l_4: $y = x + 3$
  d All 3 equations have solution $x = \frac{43}{3}$, $y = \frac{52}{3}$ so $R(\frac{43}{3}, \frac{52}{3})$
  e $\frac{200}{3}$
21 a (4,0), (0,12)
  b (2,6)
  c $(x - 2)^2 + (y - 6)^2 = 40$
22 a $q = 4$
  b $(x + \frac{5}{2})^2 + (y - 2)^2 = \frac{65}{4}$
23 a $RS^2 + ST^2 = RT^2$
  b $(x - 2)^2 + (y + 2)^2 = 61$
24 $(x - 1)^2 + (y - 3)^2 = 34$
25 a i $y = -4x - 4$ ii $x = -2$
  b $(x + 2)^2 + (y - 4)^2 = 34$

Challenge
a $x + y - 14 = 0$
b $P(7, 7)$ and $Q(9, 5)$
c 10

Chapter review 2

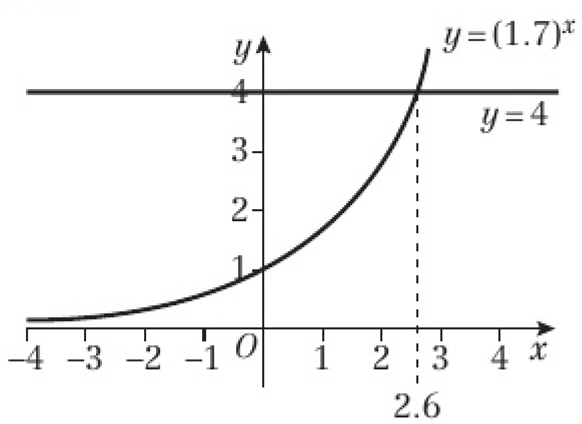
1 a $C(3, 6)$
  b $r = 10$
  c $(x - 3)^2 + (y - 6)^2 = 100$
  d P satisfies the equation of the circle.
2 $(0 - 5)^2 + (0 + 2)^2 = 5^2 + 2^2 = 29 < 30$ therefore point is inside the circle
3 a Centre (0, –4) and radius = 3
  b (0, –1) and (0, –7)
  c Students' own work. Equation $x^2 = -7$ has no real solutions.
4 a $P(8, 8)$, $(8 + 1)^2 + (8 - 3)^2 = 9^2 + 5^2 = 81 + 25 = 106$
  b $\sqrt{106}$
5 a All points satisfy $x^2 + y^2 = 1$, therefore all lie on circle.
  b $AB = BC = CA$
6 a $k = 1$, $k = -\frac{2}{5}$
  b $(x - 1)^2 + (y - 3)^2 = 13$
7 Substitute $y = 3x - 9$ into the equation
  $x^2 + px + y^2 + 4y = 20$
  $x^2 + px + (3x - 9)^2 + 4(3x - 9) = 20$
  $10x^2 + (p - 42)x + 25 = 0$
  Using the discriminant: $(p - 42)^2 - 1000 < 0$
  $42 - 10\sqrt{10} < p < 42 + 10\sqrt{10}$
8 $(x - 2)^2 + (y + 4)^2 = 20$
9 a $2\sqrt{29}$ b 12
10 (–1, 0), (11, 0)
11 The values of m and n are $7 - \sqrt{105}$ and $7 + \sqrt{105}$.
12 a $a = 6$ and $b = 8$ b $y = -\frac{4}{3}x + 8$ c 24
13 a $p = 0$, $q = 24$ b (0, 49), (0, –1)
14 $x + y + 10 = 0$

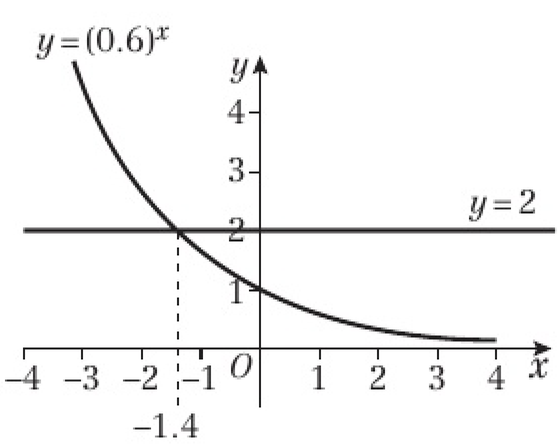
CHAPTER 3

Prior knowledge check

1 a 125 b $\frac{1}{3}$ c 32 d 49 e 1
2 a 6^6 b y^{21} c 2^6 d x^4
3 gradient 1.5, intercept 4.1

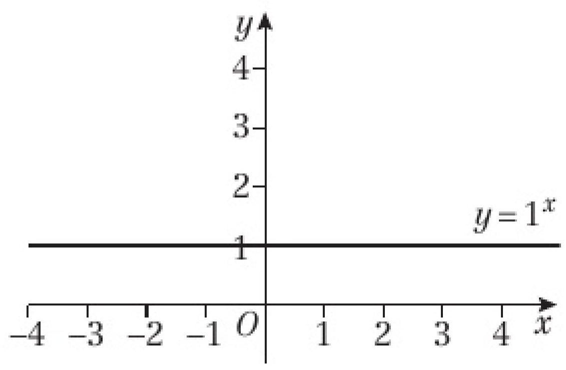
Exercise 3A

1 a


  b $x \approx 2.6$
2 a


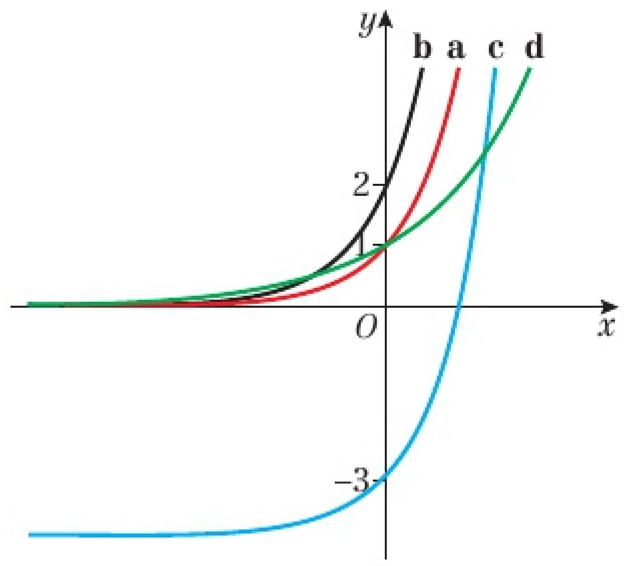
b $x \approx -1.4$

3



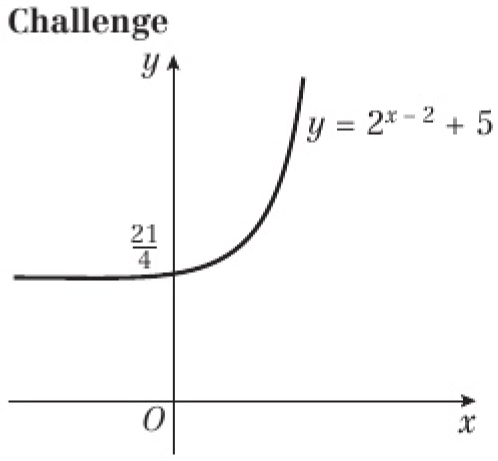
4 **a** True, because $a^0 = 1$ whenever a is positive
b False, for example when $a = \frac{1}{2}$
c True, because when a is positive, $a^x > 0$ for all values of x

5



6 $k = 3, a = 2$
7 **a** As x increases, y decreases
b $p = 1.2, q = 0.2$

Challenge



Exercise 3B

1 **a** $\log_4 256 = 4$ **b** $\log_3 \frac{1}{9} = -2$
c $\log_{10} 1\,000\,000 = 6$ **d** $\log_{11} 11 = 1$
e $\log_{0.2} 0.008 = 3$
2 **a** $2^4 = 16$ **b** $5^2 = 25$
c $9^{\frac{1}{2}} = 3$ **d** $5^{-1} = 0.2$
e $10^5 = 100\,000$
3 **a** 3 **b** 2 **c** 7 **d** 1 **e** 6
f $\frac{1}{2}$ **g** –1 **h** –2 **i** 10 **j** –2
4 **a** 625 **b** 9 **c** 7 **d** 9
e 20 **f** 2
5 **a** 2.475 **b** 2.173 **c** 3.009
6 **a** $5 = \log_2 32 < \log_2 50 < \log_2 64 = 6$
b 5.644
7 **a** **i** 1 **ii** 1 **iii** 1 **b** $a^1 \equiv a$
8 **a** **i** 0 **ii** 0 **iii** 0 **b** $a^0 \equiv 1$

Exercise 3C

1 **a** $\log_2 21$ **b** $\log_2 9$ **c** $\log_5 80$
d $\log_6 \left(\frac{64}{81}\right)$ **e** $\log_{10} 120$
2 **a** $\log_2 8 = 3$ **b** $\log_6 36 = 2$ **c** $\log_{12} 144 = 2$
d $\log_8 2 = \frac{1}{3}$ **e** $\log_{10} \frac{1}{10} = -1$
3 **a** $3\log_a x + 4\log_a y + \log_a z$
b $5\log_a x - 2\log_a y$
c $2 + 2\log_a x$
d $\log_a x - \frac{1}{2}\log_a y - \log_a z$
e $\frac{1}{2} + \frac{1}{2}\log_a x$
4 **a** $\frac{4}{3}$ **b** $\frac{1}{18}$ **c** $\sqrt{30}$ **d** 2
5 **a** $\log_3 (x + 1) - 2\log_3 (x - 1) = 1$

$\log_3 \left(\frac{x + 1}{(x - 1)^2}\right) = 1$

$\frac{x + 1}{(x - 1)^2} = 3$

$x + 1 = 3(x-1)^2$

$x + 1 = 3(x^2 - 2x + 1)$

$3x^2 - 7x + 2 = 0$

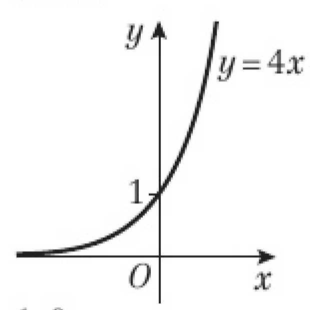
b $x = 2$
6 $a = 9, b = 4$

Challenge
$\log_a x = m$ and $\log_a y = n$
$x = a^m$ and $y = a^n$
$x \div y = a^m \div a^n = a^{m-n}$

$\log_a \left(\frac{x}{y}\right) = m - n = \log_a x - \log_a y$

Exercise 3D

1 **a** 6.23 **b** 2.10 **c** 0.431
d 1.66 **e** –3.22 **f** 1.31
g 1.25 **h** –1.73
2 **a** 0, 2.32 **b** 1.26, 2.18 **c** 1.21
d 0.631 **e** 0.565, 0.712 **f** 0
g 2 **h** –1
3 **a** 5.92 **b** 3.2
4 **a** (0, 1)



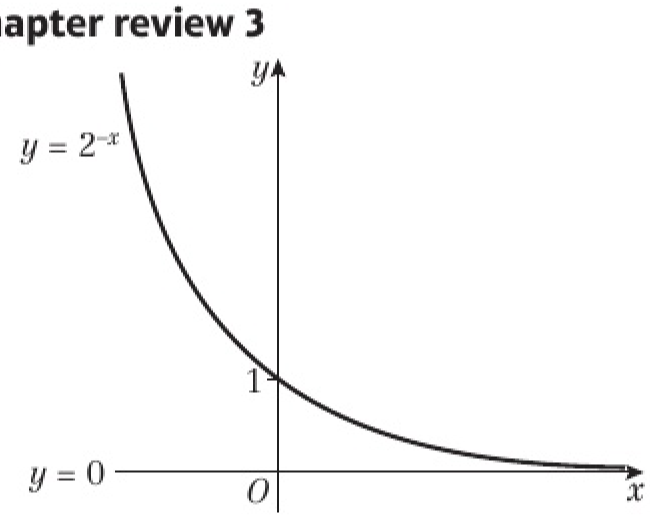
b $\frac{1}{2}, \frac{3}{2}$
5 **a** 0.7565 **b** 7.9248 **c** 0.2966

Exercise 3E

1 **a** 2.460 **b** 3.465 **c** 0.431 **d** 0.458
2 **a** 1.27 **b** 2.09 **c** 0.721
3 **a** $\frac{1}{2}$, 512 **b** $\frac{1}{64}, \frac{1}{4}$ **c** 2.52

Chapter review 3

1



2 a $2\log_a p + \log_a q$ b $\log_a p = 4$, $\log_a q = 1$

3 a $\frac{1}{4}p$ b $\frac{3}{4}p + 1$

4 a 2.26 b 1.27 c 7.02

5 a $4^x - 2^{x+1} - 15 = 0$
$2^{2x} - 2 \times 2^x - 15 = 0$
$(2^x)^2 - 2 \times 2^x - 15 = 0$
$u^2 - 2u - 15 = 0$
b 2.32

6 $x = 6$

7 a Proof b $x = \frac{1}{3}$ or $x = 9$

8 $x = 3$, $x = 6$

9 a 1.43 b $2\frac{1}{9}$ or $\frac{19}{9}$ or $2.\dot{1}$

10 $\frac{4}{5}$ (−1)

11 a 6 b $\frac{3}{2}$, $\left[-\frac{1}{4}\right]$

12 a 0.125 or $\frac{1}{8}$ b $x = 8$, $x = \frac{1}{8}$

13 a Proof b $x = 8$

14 a $x = \frac{4}{11}$ accept $0.\dot{3}\dot{6}$ b $y = \frac{a^5}{8}$

CHAPTER 4

Prior knowledge check

1 a $4x^2 - 12xy + 9y^2$ b $x^3 - 3x^2y + 3xy^2 - y^3$
c $8 + 12x + 6x^2 + x^3$

2 a $-8x^3$ b $\frac{1}{81x^4}$ c $\frac{4}{25}x^2$ d $\frac{27}{x^3}$

3 a $5\sqrt{x}$ b $\frac{1}{16\sqrt[3]{x^2}}$ c $\frac{10}{3\sqrt{x}}$ d $\frac{16\sqrt[3]{x^4}}{81}$

Exercise 4A

1 a 4th row b 16th row
c $(n + 1)$th row d $(n + 5)$th row

2 a $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
b $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$
c $a^3 - 3a^2b + 3ab^2 - b^3$
d $x^3 + 12x^2 + 48x + 64$
e $16x^4 - 96x^3 + 216x^2 - 216x + 81$
f $a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32$
g $81x^4 - 432x^3 + 864x^2 - 768x + 256$
h $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

3 a 16 b −10 c 8 d 1280
e 160 f −2 g 40 h −96

4 $1 + 9x + 30x^2 + 44x^3 + 24x^4$

5 $8 + 12y + 6y^2 + y^3$, $8 + 12x - 6x^2 - 11x^3 + 3x^4 + 3x^5 - x^6$

6 ±3

7 $\frac{5}{2}$, −1

8 $12p$

9 $500 + 25X + \frac{X^2}{2}$

Challenge

$\frac{3}{4}$

Exercise 4B

1 a 24 b 362 880 c 720 d 210

2 a 6 b 15 c 20 d 5
e 45 f 126

3 a 5005 b 120 c 184 756 d 1140
e 2002 f 8568

4 $a = {}^4C_1$, $b = {}^5C_2$, $c = {}^6C_2$, $d = {}^6C_3$

5 330

6 a 120, 210 b 960

7 a 286, 715 b 57 915

8 0.1762 to 4 decimal places. Whilst it seems a low probability, there is more chance of the coin landing on 10 heads than any other number of heads.

9 a $^nC_1 = \frac{n!}{1!(n-1)!}$
$= \frac{1 \times 2 \times \ldots \times (n-2) \times (n-1) \times n}{1 \times 1 \times 2 \times \ldots \times (n-3) \times (n-2) \times (n-1)} = n$
b $^nC_2 = \frac{n!}{2!(n-2)!}$
$= \frac{1 \times 2 \times \ldots \times (n-2) \times (n-1) \times n}{1 \times 2 \times 1 \times 2 \times \ldots \times (n-3) \times (n-2)} = \frac{n(n-1)}{2}$

10 $a = 37$

11 $p = 17$

Challenge

a $^{10}C_3 = \frac{10!}{3!7!} = 120$ and $^{10}C_7 = \frac{10!}{7!3!} = 120$

b $^{14}C_5 = \frac{14!}{5!9!} = 2002$ and $^{14}C_9 = \frac{14!}{9!5!} = 2002$

c The two answers for part **a** are the same and the two answers for part **b** are the same.

d $^nC_r = \frac{n!}{r!(n-r)!}$ and $^nC_{n-r} = \frac{n!}{(n-r)!r!}$, therefore $^nC_r = {}^nC_{n-r}$

Exercise 4C

1 a $1 + 4x + 6x^2 + 4x^3 + x^4$
b $81 + 108x + 54x^2 + 12x^3 + x^4$
c $256 - 256x + 96x^2 - 16x^3 + x^4$
d $x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$
e $1 + 8x + 24x^2 + 32x^3 + 16x^4$
f $1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

2 a $1 + 10x + 45x^2 + 120x^3$
b $1 - 10x + 40x^2 - 80x^3$
c $1 + 18x + 135x^2 + 540x^3$
d $256 - 1024x + 1792x^2 - 1792x^3$
e $1024 - 2560x + 2880x^2 - 1920x^3$
f $2187 - 5103x + 5103x^2 - 2835x^3$

3 a $64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3$
b $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3$
c $p^8 - 8p^7q + 28p^6q^2 - 56p^5q^3$
d $729x^6 - 1458x^5y + 1215x^4y^2 - 540x^3y^3$
e $x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3$
f $512x^9 - 6912x^8y + 41\,472x^7y^2 - 145\,152x^6y^3$

4 a $1 + 8x + 28x^2 + 56x^3$
b $1 - 12x + 60x^2 - 160x^3$
c $1 + 5x + \frac{45}{4}x^2 + 15x^3$
d $1 - 15x + 90x^2 - 270x^3$
e $128 + 448x + 672x^2 + 560x^3$
f $27 - 54x + + 36x^2 - 8x^3$
g $64 - 576x + 2160x^2 - 4320x^3$
h $256 + 256x + 96x^2 + 16x^3$
i $128 + 2240x + 16\,800x^2 + 70\,000x^3$

5 $64 - 192x + 240x^2$

6 $243 - 810x + 1080x^2$

7 $x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

Challenge

a $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$
$(a + b)^4 - (a - b)^4 = 8a^3b + 8ab^3 = 8ab(a^2 + b^2)$

b $82\,896 = 2^4 \times 3 \times 11 \times 157$

Exercise 4D

1 a 90 b 80 c −20
d 1080 e 120 f −4320
g 1140 h −241 920 i −2.5
j 354.375 k −224 l 3.90625

2 $a = \pm\frac{1}{2}$

3 $b = -2$

4 $1, \dfrac{5 \pm \sqrt{105}}{8}$

5 a $p = 5$ b −10 c −80

6 a $5^{30} + 5^{29} \times 30px + 5^{28} \times 435p^2x^2$
b $p = 10$

7 a $1 + 10qx + 45q^2x^2 + 120q^3x^3$
b $q = \pm 3$

8 a $1 + 11px + 55p^2x^2$
b $p = 7, q = 2695$

9 a $1 + 15px + 105p^2x^2$
b $p = -\frac{5}{7}, q = 10\frac{5}{7}$

10 $\dfrac{q}{p} = 2.1$

Challenge

a 314 928 b 43 750

Exercise 4E

1 a $1 - 0.6x + 0.15x^2 - 0.02x^3$
b 0.941 48

2 a $1024 + 1024x + 460.8x^2 + 122.88x^3$
b 1666.56

3 $(1 - 3x)^5 = 1^5 + \binom{5}{1}1^4(-3x)^1 + \binom{5}{2}1^3(-3x)^2 = 1 - 15x + 90x^2$

$(2 + x)(1 - 3x)^5 = (2 + x)(1 - 15x + 90x^2)$
$= 2 - 30x + 180x^2 + x - 15x^2 + 90x^3 \approx 2 - 29x + 165x^2$

4 $a = 162, b = 135, c = 0$

5 a $1 + 16x + 112x^2 + 448x^3$
b $x = 0.01$, $1.02^8 \approx 1.171\,648$

6 a $1 - 150x + 10875x^2 - 507500x^3$
b 0.860 368
c 0.860 384, 0.0019%

7 a $59\,049 - 39\,366x + 11\,809.8x^2$
b Substitute $x = 0.1$ into the expansion.

8 a $1 - 15x + 90x^2 - 270x^3$
b $(1 + x)(1 - 3x)^5 \approx (1 + x)(1 - 15x) \approx 1 - 14x$

9 a So that higher powers of p can be ignored as they tend to 0
b $1 - 200p + 19\,900p^2$
c $p = 0.000417$ (3 s.f.)

Chapter review 4

1 a 455, 1365 b 3640

2 $a = 28$

3 a 0.0148 b 0.000 000 000 034 9 c 0.166

4 a $p = 16$ b 270 c −1890

5 $A = 8192, B = -53\,248, C = 159\,744$

6 a $1 - 20x + 180x^2 - 960x^3$
b 0.817 04, $x = 0.01$

7 a $1024 - 15\,360x + 103\,680x^2 - 414\,720x^3$
b 880.35

8 a $81 + 216x + 216x^2 + 96x^3 + 16x^4$
b $81 - 216x + 216x^2 - 96x^3 + 16x^4$
c 1154

9 a $n = 8$ b $\frac{35}{8}$

10 a $81 + 1080x + 5400x^2 + 12\,000x^3 + 10\,000x^4$
b 1 012 054 108 081, $x = 100$

11 a $1 + 24x + 264x^2 + 1760x^3$ b 1.268 16, $x = 0.01$
c 1.268 241 795 d 0.006 45% (3 sf)

12 $x^5 - 5x^3 + 10x - \dfrac{10}{x} + \dfrac{5}{x^3} - \dfrac{1}{x^5}$

13 a $\binom{n}{2}(2k)^{n-2} = \binom{n}{3}(2k)^{n-3}$

$\dfrac{n!(2k)^{n-2}}{2!(n-2)!} = \dfrac{n!(2k)^{n-3}}{3!(n-3)!}$

$\dfrac{2k}{n-2} = \dfrac{1}{3}$

So $n = 6k + 2$

b $\dfrac{4096}{729} + \dfrac{2048}{81}x + \dfrac{1280}{27}x^2 + \dfrac{1280}{27}x^3$

14 a $64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$
b $k = 1560$

15 a $k = 1.25$ b 3500

16 a $A = 64, B = 160, C = 20$ b $x = \pm\sqrt{\frac{3}{2}}$

17 a $p = 1.5$ b 50.625

18 672

19 a $128 + 448px + 672p^2x^2$
b $p = 5, q = 16\,800$

20 a $1 - 12px + 66p^2x^2$
b $p = -1\frac{1}{11}, q = 13\frac{1}{11}$

21 a $128 + 224x + 168x^2$
b Substitute $x = 0.1$ into the expansion.

22 $k = \frac{1}{2}$

Challenge

1 $540 - 405p = 0, p = \frac{4}{3}$

2 −4704

Review exercise 1

1 $(x + 3)^2 + (y - 8)^2 = 10$

2 a $(x - 3)^2 + (y + 1)^2 = 20$ ($a = 3, b = -1, r = \sqrt{20}$)
b Centre (3, −1), radius $\sqrt{20}$

3 a (3, 5) and (4, 2) b $\sqrt{10}$

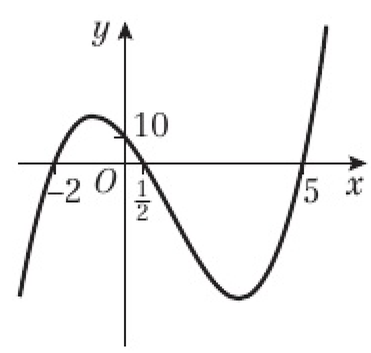
4 $0 < r < \sqrt{\frac{2}{5}}$

5 a $(x - 1)^2 + (y - 5)^2 = 58$ b $7y - 3x + 26 = 0$

6 a $AB = \sqrt{32}$; $BC = \sqrt{8}$; $AC = \sqrt{40}$; $AC^2 = AB^2 + BC^2$
b AC is a diameter of the circle.
c $(x - 5)^2 + (y - 2)^2 = 10$

7 $a = 3, b = -2, c = -8$

8 a $2(\frac{1}{2})^3 - 7(\frac{1}{2})^2 - 17(\frac{1}{2}) + 10 = 0$
b $(2x - 1)(x - 5)(x + 2)$
c



9 a 24
b $(x - 3)(3x - 2)(x + 4)$

10 a $g(3) = 3^3 - 13(3) + 12 = 0$
b $(x - 3)(x + 4)(x - 1)$

11 a $a = -5, b = 4$ b 4

12 a $a = -1, b = -7$ b $x = 1, -2, \frac{3}{2}$
13 a $a = -20$ b $b = -6$
14 a $a = 0, b = 0$
b $a > 0, b > 0$
15 a $5^2 = 24 + 1$; $7^2 = 2(24) + 1$; $11^2 = 5(24) + 1$; $13^2 = 7(24) + 1$; $17^2 = 12(24) + 1$; $19^2 = 15(24) + 1$
b $3(24) + 1 = 73$ which is not a square of a prime number
16 a $(x - 5)^2 + (y - 4)^2 = 3^2$
b $\sqrt{41}$
c Sum of radii $= 3 + 3 < \sqrt{41}$ so circles do not touch
17 a -0.179
b $x = 15$
18 a $x = 1.55$
b $x = \frac{1}{2}, 4$
19 a $\log_p 2$
b $x = \frac{1}{8}$
20 $x = 2.52$ or 4 (both roots required)
21 $x = 64, 2$
22 $t = 3\sqrt{3}$
23 a $1 - 20x + 180x^2 - 960x^3 + \ldots$
b 0.817
24 $a = 2, b = 19, c = 70$
25 4
26 a $1 + 6x + 6x^2 - 4x^3$
b $\left(1 + 4\left(\frac{3}{100}\right)\right)^{\frac{3}{2}} = \left(\frac{112}{100}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{112}{100}}\right)^3 = \left(\frac{\sqrt{112}}{10}\right)^3$
$= \frac{112\sqrt{112}}{1000}$
c 10.58296 d 0.00039%

Challenge
1 a $-5, b = -23$
b $(x + 3)(x - 2)(2x + 1)(x - 4)$
2 $5x - 1$

CHAPTER 5

Prior knowledge check

1 a 22, 27, 32 b $-1, -4, -7$ c 9, 15, 21
d 48, 96, 192 e $\frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ f $-16, 64, -256$
2 a $x = 5.64$ b $x = 3.51$ c $x = 9.00$

Exercise 5A

1 a i 7, 12, 17, 22 ii $a = 7, d = 5$
b i 7, 5, 3, 1 ii $a = 7, d = -2$
c i 7.5, 8, 8.5, 9 ii $a = 7.5, d = 0.5$
d i $-9, -8, -7, -6$ ii $a = -9, d = 1$
2 a $2n + 3$, 23 b $3n + 2$, 32
c $27 - 3n$, -3 d $4n - 5$, 35
e nx, $10x$ f $a + (n - 1)d$, $a + 9d$
3 a 22 b 40 c 39 d 46 e 18 f n
4 $d = 6$ 5 $p = \frac{2}{3}, q = 5$ 6 -1.5
7 24 8 -70 9 $k = \frac{1}{2}, k = 8$
10 $-2 + 3\sqrt{5}$

Challenge
$a = 4, b = 2$

Exercise 5B

1 a 820 b 450 c -1140
d -294 e 1440 f 1425
g -1155 h $231x + 21$
2 a 20 b 25 c 65 d 4 or 14
3 2550 4 20
5 $d = -\frac{1}{2}$, 20th term $= -5.5$ 6 $a = 6, d = -2$
7 $S_{50} = 1 + 2 + 3 + \ldots + 50$
$S_{50} = 50 + 49 + 48 + \ldots + 1$
$2 \times S_{50} = 50(51) \Rightarrow S_{50} = 1275$
8 $S_{2n} = 1 + 2 + 3 + \ldots + 2n$
$S_{2n} = 2n + (2n - 1) + (2n - 2) + \ldots + 1$
$2 \times S_n = 2n(2n + 1) \Rightarrow S_n = n(2n + 1)$
9 $S_n = 1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1)$
$S_n = (2n - 1) + (2n - 3) + \ldots + 5 + 3 + 1$
$2 \times S_n = n(2n) \Rightarrow S_n = n^2$
10 a $a + 4d = 33$, $a + 9d = 68$
$d = 7, a = 5$ so $S_n = \frac{n}{2}[2(5) + (n - 1)7]$
$\Rightarrow 2225 = \frac{n}{2}(7n + 3) \Rightarrow 7n^2 + 3n - 4450 = 0$
b 25
11 a $\frac{304}{k + 2}$
b $S_n = \frac{152}{k + 2}(k + 1 + 303) = \frac{152k + 46\,208}{k + 2}$
c 17
12 a 1683
b i $\frac{100}{p}$
ii $S_{\frac{100}{p}} = \frac{50}{p}\left[8p + \left(\frac{100 - p}{p}\right)4p\right]$
$S_{\frac{100}{p}} = \frac{50}{p}[4p + 400] = 200\left[1 + \frac{100}{p}\right]$
c $161p + 81$
13 a $5n + 1$ b 285
c $S_k = \frac{k}{2}[2(6) + (k - 1)5] = \frac{k}{2}(5k + 7)$
$\frac{k}{2}(5k + 7) \leqslant 1029$
$5k^2 + 7k - 2058 \leqslant 0$
$(5k - 98)(k + 21) \leqslant 0$
d $k = 19$

Challenge
$n = 16$

Exercise 5C

1 a Geometric, $r = 2$ b Not geometric
c Not geometric d Geometric, $r = 3$
e Geometric, $r = \frac{1}{2}$ f Geometric, $r = -1$
g Geometric, $r = 1$ h Geometric, $r = -\frac{1}{4}$
2 a 135, 405, 1215 b $-32, 64, -128$
c 7.5, 3.75, 1.875 d $\frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$
e p^3, p^4, p^5 f $-8x^4, 16x^5, -32x^6$
3 a $x = 3\sqrt{3}$ b $9\sqrt{3}$
4 a $486, 2 \times 3^{n-1}$ b $\frac{25}{8}, 100 \times \left(\frac{1}{2}\right)^{n-1}$
c $-32, (-2)^{n-1}$ d $1.610\,51, (1.1)^{n-1}$
5 10, 6250 6 $a = 1, r = 2$ 7 $\frac{1}{8}, -\frac{1}{8}$

8 **a** $\frac{x^2}{2x} = \frac{2x}{8-x} \Rightarrow x^2(8-x) = 4x^2 \Rightarrow x^3 - 4x^2 = 0$
b 2 097 152
c Yes, 4096 is in sequence as n is integer, $n = 11$

9 **a** $ar^5 = 40 \Rightarrow 200p^5 = 40$
$\Rightarrow p^5 = \frac{1}{5} \Rightarrow \log p^5 = \log\left(\frac{1}{5}\right)$
$\Rightarrow 5\log p = \log 1 - \log 5 \Rightarrow 5\log p + \log 5 = 0$
b $p = 0.725$

10 $k = 12$

11 $n = 8.69$, so not a sequence as n not an integer.

12 No, −49 152 is in sequence.

13 $n = 11$, 3 145 728

Exercise 5D

1 **a** 255 **b** 63.938 **c** 1.110
d −728 **e** $546\frac{2}{3}$ **f** −1.667

2 4.9995 **3** 14.4147 **4** $\frac{5}{4}, -\frac{9}{4}$

5 19 terms **6** 22 terms

7 **a** $\frac{25\left(1-\left(\frac{3}{5}\right)^k\right)}{\left(1-\frac{3}{5}\right)} > 61 \Rightarrow 1 - \left(\frac{3}{5}\right)^k > \frac{122}{125} \Rightarrow \left(\frac{3}{5}\right)^k < \frac{3}{125}$
$\Rightarrow k\log\left(\frac{3}{5}\right) < \log\left(\frac{3}{125}\right) \Rightarrow k > \frac{\log(0.024)}{\log(0.6)}$
b $k = 8$

8 $r = \pm 0.4$

9 $S_{10} = \frac{a\left[(\sqrt{3})^{10} - 1\right]}{\sqrt{3}-1} = \frac{a(243-1)}{\sqrt{3}-1}$
$= \frac{242a(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = 121a(\sqrt{3}+1)$

10 $\frac{a(2^4-1)}{1} = \frac{b(3^4-1)}{2}$
$15a = 40b \Rightarrow a = \frac{15}{8r} = a = \frac{8}{3}b$

11 **a** $\frac{2k+5}{k} = \frac{k}{k-6} \Rightarrow (k-6)(2k+5) = k^2$
$k^2 - 7k - 30 = 0$
b $k = 10$ **c** 2.5 **d** 25 429

Exercise 5E

1 **a** Yes as $|r| < 1$, $\frac{10}{9}$ **b** No as $|r| \geqslant 1$
c Yes as $|r| < 1$, $6\frac{2}{3}$
d No; arithmetic series does not converge.
e No as $|r| \geqslant 1$ **f** Yes as $|r| < 1$, $4\frac{1}{2}$
g No; arithmetic series does not converge.
h Yes as $|r| < 1$, 90

2 $\frac{2}{3}$ **3** $-\frac{2}{3}$ **4** 20 **5** $13\frac{1}{3}$

6 $\frac{23}{99}$ **7** $r = -\frac{1}{2}$, $a = 12$

8 **a** $-\frac{1}{2} < x < \frac{1}{2}$ **b** $S_\infty = \frac{1}{1+2x}$

9 **a** 0.9787 **b** 1.875

10 **a** $\frac{30}{1-r} = 240 \Rightarrow 1 - r = \frac{1}{8} \Rightarrow r = \frac{7}{8}$
b 2.51 **c** 99.3 **d** 11

11 **a** $ar = \frac{15}{8} \Rightarrow$
$\frac{a}{1-r} = 8 \Rightarrow a = 8(1-r)$
$\frac{15}{8r} = 8(1-r) \Rightarrow 15 = 64r - 64r^2$
$\Rightarrow 64r^2 - 64r + 15 = 0$
b $\frac{3}{8}, \frac{5}{8}$ **c** 5, 3 **d** 7

Challenge

a First series: $a + ar + ar^2 + ar^3 + \ldots$
Second series: $a^2 + a^2r^2 + a^2r^4 + a^2r^6 + \ldots$
Second series is geometric with common ratio is r^2 and first term a^2.

b $\frac{a}{1-r} = 7 \Rightarrow a = 7(1-r) \Rightarrow a^2 = 49(1-r)(1-r)$
$\frac{a^2}{1-r^2} = 35 \Rightarrow \frac{49(1-r)(1-r)}{(1-r)(1+r)} = 35$
$49(1-r) = 35(1+r) \Rightarrow 49 - 49r = 35 + 35r \Rightarrow r = \frac{1}{6}$

Exercise 5F

1 **a** **i** $4 + 7 + 10 + 13 + 16$ **ii** 50
b **i** $3 + 12 + 27 + 48 + 75 + 108$ **ii** 273
c **i** $1 + 0 + (-1) + 0 + 1$ **ii** 1
d **i** $-\frac{2}{243} + \frac{2}{729} - \frac{2}{2187} + \frac{2}{6561}$ **ii** $-\frac{40}{6561}$

2 **a** **i** $\sum_{r=1}^{4} 2r$ **ii** 20
b **i** $\sum_{r=1}^{5} (2 \times 3^{r-1})$ **ii** 242
c **i** $\sum_{r=1}^{6}\left(-\frac{3}{2}r + \frac{15}{2}\right)$ **ii** 13.5

3 **a** **i** 26 **ii** $\sum_{r=1}^{26}(6r+1)$
b **i** 7 **ii** $\sum_{r=1}^{7}\left(\frac{1}{3} \times \left(\frac{2}{5}\right)^{r-1}\right)$
c **i** 16 **ii** $\sum_{r=1}^{16}(17-9r)$

4 **a** −280 **b** 4 194 300
c 9300 **d** $-\frac{7}{4}$

5 **a** 2134 **b** 45 854 **c** $\frac{3}{16}$ **d** 96

6 $\sum_{r=1}^{n} 2r = 2 + 4 + 6 + \ldots + 2n$; $a = 2$, $d = 2$
$S_n = \frac{n}{2}(4 + (n-1)2) = \frac{n}{2}(2 + 2n) = n + n^2$

7 $\sum_{r=1}^{n} 2r = n + n^2$
$\sum_{r=1}^{n}(2r-1) = \frac{n}{2}(2 + (n-1)2) = \frac{n}{2}(2n) = n^2$
$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n}(2r-1) = n + n^2 - n^2 = n$

8 **a** $\frac{8}{3}((-2)^k - 1)$ **b** $99k - k^2$
c $6k - k^2 + 27$

9 $\frac{25}{98\,304}$

10 **a** $a = 11$, $d = 3$
$377 = \frac{k}{2}(2(11) + (k-1)(3)) = \frac{k}{2}(19 + 3k)$
$3k^2 + 19k - 754 = 0 \Rightarrow (3k + 58)(k - 13) = 0$
b $k = 13$

11 **a** $a = 6$, $d = 3$; $S_k = \frac{6(3^k - 1)}{3 - 1} = 3(3^k - 1)$
$\Rightarrow 3(3^k - 1) = 59046 \Rightarrow 3^k = 19683$

$\Rightarrow k \log 3 = \log 19683 \Rightarrow k = \dfrac{\log 19683}{\log 3}$

b 4 723 920

12 **a** $|x| < \frac{1}{3}$ **b** $\frac{1}{6}$

Challenge

$\sum_{r=1}^{10}[a + (r - 1)d]$

$S_{10} = 5(2a + 9d)$

$\sum_{r=11}^{14}[a + (r - 1)d] = \sum_{r=1}^{14}[a + (r - 1)d] - \sum_{r=1}^{10}[a + (r - 1)d]$

$= [7(2a + 13d) - 5(2a + 9d)] = 4a + 46d$

$4a + 46d = 10a + 45d \Rightarrow 6a = d$

Exercise 5G

1 **a** 1, 4, 7, 10 **b** 9, 4, −1, −6
c 3, 6, 12, 24 **d** 2, 5, 11, 23
e 10, 5, 2.5, 1.25 **f** 2, 3, 8, 63

2 **a** $u_{n+1} = u_n + 2, u_1 = 3$ **b** $u_{n+1} = u_n - 3, u_1 = 20$
c $u_{n+1} = 2u_n, u_1 = 1$ **d** $u_{n+1} = \dfrac{u_n}{4}, u_1 = 100$
e $u_{n+1} = -1 \times u_n, u_1 = 1$ **f** $u_{n+1} = 2u_n + 1, u_1 = 3$
g $u_{n+1} = (u_n)^2 + 1, u_1 = 0$ **h** $u_{n+1} = \dfrac{u_n + 2}{2}, u_1 = 26$

3 **a** $u_{n+1} = u_n + 2, u_1 = 1$ **b** $u_{n+1} = u_n + 3, u_1 = 5$
c $u_{n+1} = u_n + 1, u_1 = 3$ **d** $u_{n+1} = u_n + \frac{1}{2}, u_1 = 1$
e $u_{n+1} = u_n + 2n + 1, u_1 = 1$ **f** $u_{n+1} = 3u_n + 2, u_1 = 2$

4 **a** $3k + 2$ **b** $3k^2 + 2k + 2$ **c** $\frac{10}{3}, -4$

5 $p = -4, q = 7$

6 **a** $x_2 = x_1(p - 3x_1) = 2(p - 3(2)) = 2p - 12$
$x_3 = (2p - 12)(p - 3(2p - 12)) = (2p - 12)(-5p + 36)$
$= -10p^2 + 132p - 432$
b 12 **c** −252 288

7 **a** $16k + 25$
b $a_4 = 4(16k + 25) + 5 = 64k + 105$
$\sum_{r=1}^{4} a_r = k + 4k + 5 + 16k + 25 + 64k + 105$
$= 85k + 135 = 5(17k + 27)$

Exercise 5H

1 **a** **i** increasing
b **i** decreasing
c **i** increasing
d **i** periodic **ii** 2

2 **a** **i** 17, 14, 11, 8, 5 **ii** decreasing
b **i** 1, 2, 4, 8, 16 **ii** increasing
c **i** −1, 1, −1, 1, −1 **ii** periodic
iii 2
d **i** −1, 1, −1, 1, −1 **ii** periodic
iii 2
e **i** 20, 15, 10, 5, 0 **ii** decreasing
f **i** 20, −15, 20, −15, 20 **ii** periodic
iii 2
g **i** $k, \dfrac{2k}{3}, \dfrac{4k}{9}, \dfrac{8k}{27}, \dfrac{16k}{81}$
ii dependent on value of k

3 $0 < k < 1$ **4** $p = -1$

5 **a** 4 **b** 0

Challenge

$u_3 = \dfrac{1 + b}{a}, u_4 = \dfrac{a + b + 1}{ab}, u_5 = \dfrac{a + 1}{b}, u_6 = a, u_7 = b$

Order is 5 as $u_6 = u_1$ and $u_7 = u_2$

Exercise 5I

1 **a** \$5800 **b** \$(3800 + 200m)

2 **a** €222 500 **b** €347 500
c It is unlikely their salary will rise by the same amount each year.

3 **a** \$9.03 **b** 141 days

4 **a** 220 **b** 242 **c** 266 **d** 519

5 57.7, 83.2

6 **a** €18 000 **b** after 7.88 years

7 **a** \$13 780
b Let a denote term of first year and u denote term of second year
$a_{52} = 10 + 51(10) = 520$
$u_1 = 520 + 11$
$u_2 = 531 + 11 = 542$
c \$42 198

8 **a** 500 m is 10 terms,
$S_{10} = \frac{10}{2}(1000 + 9(140)) = 11\,300$
b 1500 m

9 **a** €2450 **b** €59 000 **c** $d = 30$

10 59 days **11** 20.15 years

12 11.2 years **13** $2^{64} - 1 = 1.84 \times 10^{19}$

14 **a** 2.401 m **b** 48.8234 m

15 **a** 26 days **b** 98.5miles on 25th day

16 25 years

Chapter review 5

1 **a** $ar^2 = 27, ar^5 = 8 \Rightarrow r^3 = \frac{8}{27} \Rightarrow r = \frac{2}{3}$
b 60.75 **c** 182.25 **d** 3.16

2 **a** $ar = 80, ar^4 = 5.12$
$\Rightarrow r^3 = \frac{8}{125} \Rightarrow r = \frac{2}{5} = 0.4$
b 200 **c** $333\frac{1}{3}$ **d** 8.95×10^{-4}

3 **a** 76, 60.8 **b** 0.876 **c** 367 **d** 380

4 **a** $1, \frac{1}{3}, -\frac{1}{9}$
b $\sum_{n=1}^{15}\left(3\left(\frac{2}{3}\right)^n - 1\right) = \sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n - \sum_{r=1}^{15} 1$

$\sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n = \dfrac{2\left(1 - \left(\frac{2}{3}\right)^{15}\right)}{\frac{1}{3}} = 5.9863$

$\sum_{r=1}^{15} 1 = 15$

$5.9863 - 15 = -9.014$

c $u_{n+1} = 3\left(\frac{2}{3}\right)^{n+1} - 1 = 3 \times \frac{2}{3}\left(\frac{2}{3}\right)^n - 1 = \frac{1}{3}\left(2 \times 3\left(\frac{2}{3}\right)^n - 3\right)$
$= \dfrac{2_{un} - 1}{3}$

5 **a** 0.8 **b** 10 **c** 50 **d** 0.189

6 **a** \$8874.11 **b** after 9.9 years

7 **a** $\dfrac{p(2q + 2)}{p(3q + 1)} = \dfrac{p(2q - 1)}{p(2q + 2)}$
$(2q + 2)^2 = (2q - 1)(3q + 1)$
$4q^2 + 8q + 4 = 6q^2 - q - 1$
$0 = 2q^2 - 9q - 5 = (q - 5)(2q + 1) \Rightarrow q = 5$ or $-\frac{1}{2}$
b 867.62

8 **a** $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 2)d)$
$+ (a + (n - 1)d)$ (1)
$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \ldots (a + 2d)$
$+ (a + d) + a$ (2)
Adding (1) and (2):
$2 \times S_n = n(2a + (n - 1)d) \Rightarrow S_n = \frac{n}{2}(2a + (n - 1)d)$

b 5050

9 32

10 a $a = 25, d = -3$ **b** -3810

11 a 26 733 **b** 53 467

12 45 cm

13 $S_{2n} = \frac{2n}{2}(2(4) + (2n - 1)4) = n(4 + 8n) = 4n(2n + 1)$

14 a $u_2 = 2k - 4, u_3 = 2k^2 - 4k - 4$ **b** 5, −3

15 a $a + 4d = 14, \frac{3}{2}(2a + 2d) = -3$

$3a + 3d = -3, 3a + 12d = 42$

$9d = 45 \Rightarrow d = 5 \Rightarrow a = -6$

b 59

16 a $a + 3d = 3k, 3(2a + 5d) = 7k + 9 \Rightarrow$

$6a + 15d = 7k + 9$

$6a + 15\left(\frac{3k - a}{3}\right) = 7k + 9$

$6a + 15k - 5a = 7k + 9 \Rightarrow a = 9 - 8k$

b $\frac{11k - 9}{3}$ **c** 1.5 **d** 415

17 a $a_1 = p, a_2 = \frac{1}{p}, a_3 = \frac{1}{\frac{1}{p}} = 1 \times \frac{p}{1} = p$

$a_1 = a_3 \Rightarrow$ Sequence is periodic, order 2

b $500\left(p + \frac{1}{p}\right)$

18 a $a_1 = k, a_2 = 2k + 6, a_3 = 2(2k + 6) + 6 = 4k + 18$

$a_1 < a_2 < a_3 \Rightarrow k < 2k + 6 < 4k + 18 \Rightarrow k > -6$

b $a_4 = 8k + 42$

c $a_4 = 8k + 42$

$\sum_{r=1}^{4} a_r = k + 2k + 6 + 4k + 18 + 8k + 42$

$= 15k + 66 = 3(5k + 22)$

therefore divisible by 3

19 a $a = 130$

$S_\infty = \frac{130}{1 - r} = 650 \Rightarrow 130 = 650 - 650r$

$-520 = -650r \Rightarrow r = \frac{-520}{-650} = \frac{4}{5}$

b 6.82 **c** 513.69 (2 d.p.)

d $\frac{130(1 - (0.8)^n)}{0.2} > 600 \Rightarrow 1 - (0.8)^n > \frac{12}{13}$

$(0.8)^n < \frac{1}{13} \Rightarrow n \log(0.8) < -\log 13 \Rightarrow n > \frac{-\log 13}{\log 0.8}$

20 a $25\,000 \times 1.02^2 = 26\,010$

b $25\,000 \times 1.02^n > 50\,000$

$1.02^n > 2 \Rightarrow n \log 1.02 > \log 2 \Rightarrow n > \frac{\log 2}{\log 1.02}$

c 2053 **d** 214 574

e People may visit the doctor more frequently than once a year, some may not visit at all, depends on state of health.

21 a $2n + 1$ **b** 150

c i $S_q = \frac{q}{2}(2(3) + (q - 1)2) = 4q + q^2$

$S_q = p \Rightarrow q^2 + 2q - p = 0$

ii 39

22 a $ar = -3, \frac{a}{1 - r} = 6.75$

$\Rightarrow -\frac{3}{r} \times \frac{1}{1 - r} = 6.75 \Rightarrow \frac{-3}{r - r^2} = 6.75$

$6.75r - 6.75r^2 + 3 = 0$

$27r^2 - 27r - 12 = 0$

b $-\frac{1}{3}$ series is convergent so $|r| < 1$

c 6.78

Challenge

a $u_{n+2} = 5u_{n+1} - 6u_n.$

$= 5[p(3^{n+1}) + q(2^{n+1})] - 6[p(3^n) + q(2^n)]$

$= 5\left[p\left(\frac{1}{3}\right)(3^{n+2}) + q\left(\frac{1}{2}\right)(2^{n+2})\right]$

$-6\left[p\left(\frac{1}{3}\right)^2(3^{n+2}) + q\left(\frac{1}{2}\right)^2(2^{n+2})\right]$

$= \left(\frac{5}{3}p - \frac{6}{9}p\right)(3^{n+2}) + \left(\frac{5}{2}q - \frac{6}{4}q\right)(2^{n+2})$

$= p(3^{n+2}) + q(2^{n+2})$

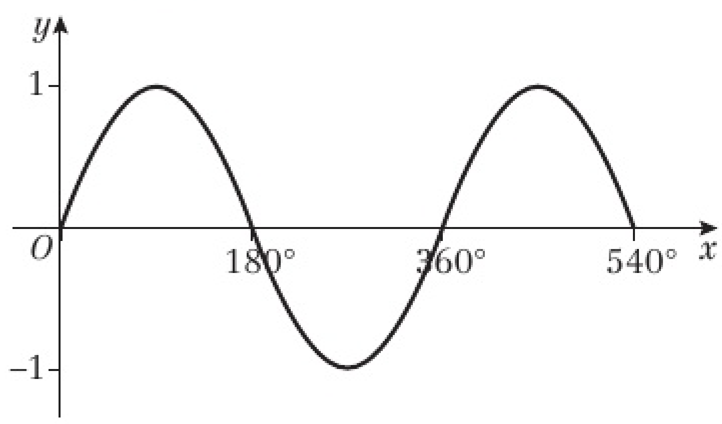
b $u_n = \left(\frac{2}{3}\right)(3^n) + \left(\frac{3}{2}\right)(2^n)$ or e.g. $u_n = 2(3^{n-1}) + 3(2^{n-1})$

c $u_{100} = 3.436 \times 10^{47}$ (4 s.f.) so contains 48 digits.

CHAPTER 6

Prior knowledge check

1 a



b 4

c 143.1°, 396.9°, 503.1°

2 a 57.7° **b** 73.0°

3 a $x = 11$ **b** $x = \frac{9}{4}$ **c** $x = -44.4°$

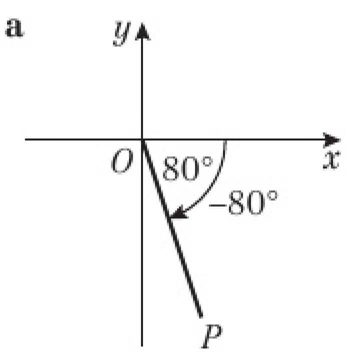
4 a $x = 1$ or $x = 3$

b $x = 1$ or $x = -9$

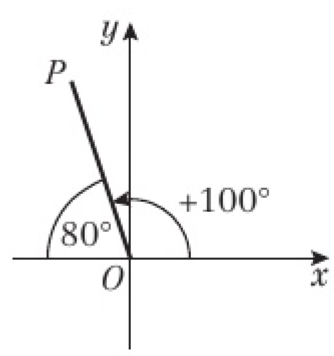
c $x = \frac{3 \pm \sqrt{65}}{4}$

Exercise 6A

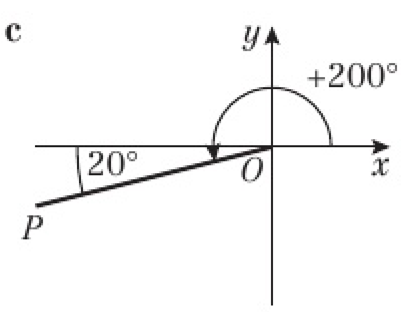
1 a



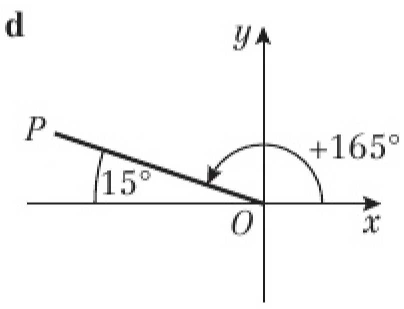
b



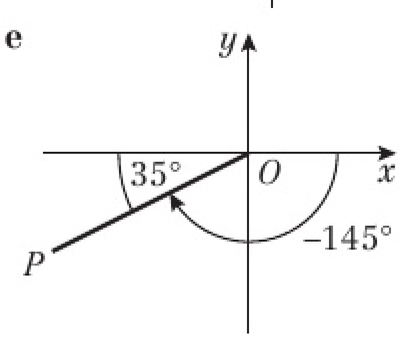
c



d



e



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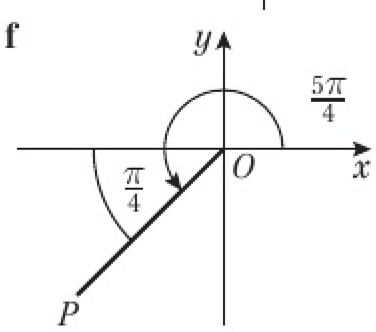
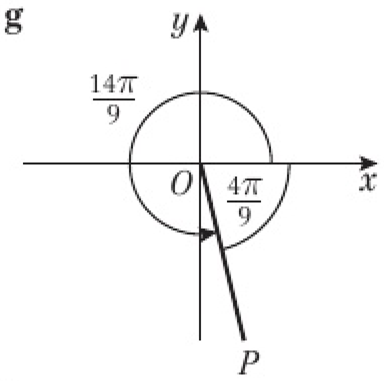


g



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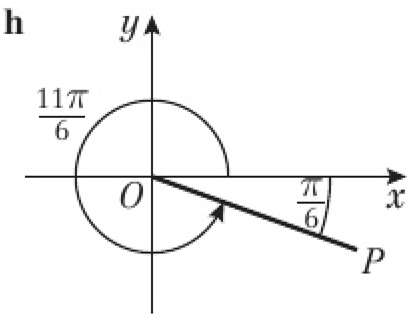
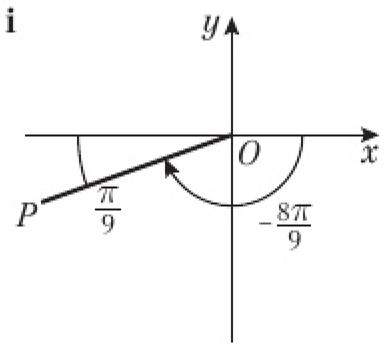
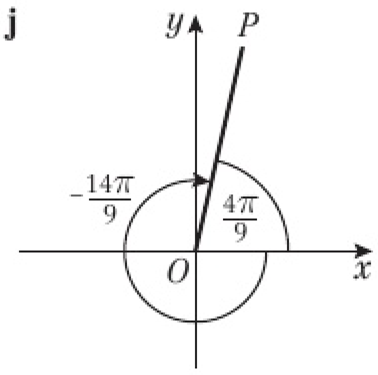


i



j



2 **a** First **b** Second **c** Second
d Third **e** Third

3 **a** −1 **b** 1 **c** 0 **d** −1 **e** −1
f 0 **g** 0 **h** 0 **i** 0 **j** 0

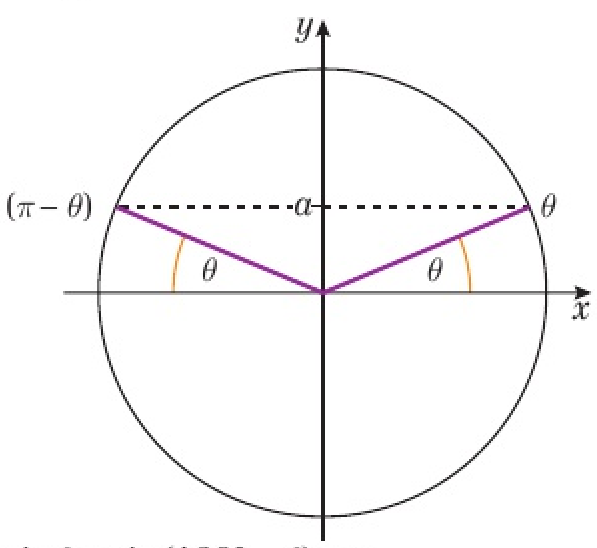
4 **a** $-\sin 60°$ **b** $-\sin 80°$ **c** $\sin\left(\frac{\pi}{9}\right)$
d $-\sin\left(\frac{\pi}{3}\right)$ **e** $-\cos 70°$ **f** $-\cos 80°$
g $-\cos\left(\frac{\pi}{9}\right)$ **h** $-\cos\left(\frac{\pi}{36}\right)$ **i** $-\tan 80°$
j $-\tan 35°$ **k** $-\tan\left(\frac{\pi}{6}\right)$ **l** $\tan\left(\frac{\pi}{3}\right)$

5 **a** $-\sin\theta$ **b** $-\sin\theta$ **c** $-\sin\theta$
d $\sin\theta$ **e** $-\sin\theta$ **f** $\sin\theta$
g $-\sin\theta$ **h** $-\sin\theta$ **i** $\sin\theta$

6 **a** $-\cos\theta$ **b** $-\cos\theta$ **c** $\cos\theta$
d $-\cos\theta$ **e** $\cos\theta$ **f** $-\cos\theta$
g $-\tan\theta$ **h** $-\tan\theta$ **i** $\tan\theta$
j $\tan\theta$ **k** $-\tan\theta$ **l** $\tan\theta$

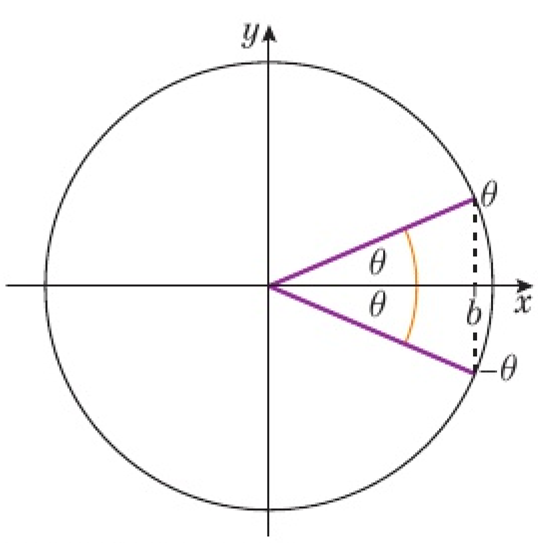
Challenge

a



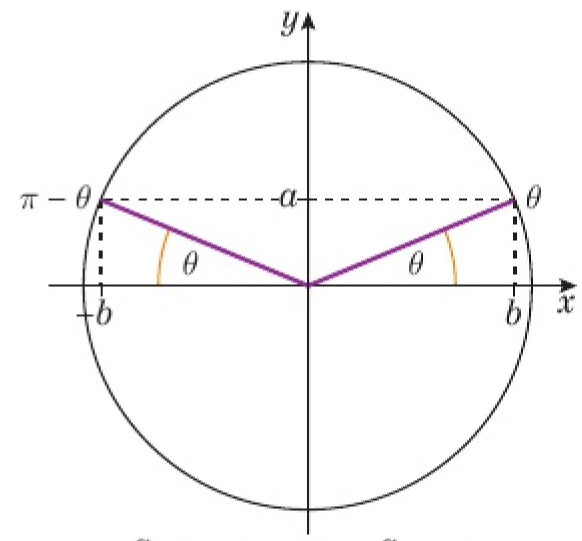
$\sin\theta = \sin(180° - \theta) = a$

b



$\cos\theta = \cos(-\theta) = b$

c



$\tan\theta = \frac{a}{b}$; $\tan(\pi - \theta) = \frac{a}{-b} = -\tan\theta$

Exercise 6B

1 **a** $\frac{\sqrt{2}}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $-\frac{1}{2}$ **d** $\frac{\sqrt{3}}{2}$
e $\frac{\sqrt{3}}{2}$ **f** $-\frac{1}{2}$ **g** $\frac{1}{2}$ **h** $-\frac{\sqrt{2}}{2}$
i $-\frac{\sqrt{3}}{2}$ **j** $-\frac{\sqrt{2}}{2}$ **k** −1 **l** −1
m $\frac{\sqrt{3}}{3}$ **n** $-\sqrt{3}$ **o** $\sqrt{3}$

Challenge

a **i** $\sqrt{3}$ **ii** 2 **iii** $\sqrt{2+\sqrt{3}}$ **iv** $\sqrt{2+\sqrt{3}} - \sqrt{2}$
b 15°
c **i** $\frac{\sqrt{2+\sqrt{3}} - \sqrt{2}}{2}$ **ii** $\frac{\sqrt{2+\sqrt{3}}}{2}$

Exercise 6C

1 **a** $\sin^2\frac{\theta}{2}$ **b** 5 **c** $-\cos^2 A$
d $\cos\theta$ **e** $\tan x$ **f** $\tan 3A$
g 4 **h** $\sin^2\theta$ **i** 1

2 $1\frac{1}{2}$

3 $3\tan y$

4 **a** $1 - \sin^2\theta$ **b** $\frac{\sin^2\theta}{1-\sin^2\theta}$ **c** $\sin\theta$
d $\frac{1-\sin^2\theta}{\sin\theta}$ **e** $1 - 2\sin^2\theta$

5 (One outline example of a proof is given)

a $\text{LHS} = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$
$= 1 + 2\sin\theta\cos\theta$
$= \text{RHS}$

b $\text{LHS} = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta} = \sin\theta \times \frac{\sin\theta}{\cos\theta}$
$= \sin\theta\tan\theta = \text{RHS}$

c $\text{LHS} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x\cos x}$
$= \frac{1}{\sin x\cos x} = \text{RHS}$

d $\text{LHS} = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$
$= 2(1 - \sin^2 A) - 1 = 1 - 2\sin^2 A = \text{RHS}$

e $\text{LHS} = (4\sin^2\theta - 4\sin\theta\cos\theta + \cos^2\theta)$
$+ (\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta)$
$= 5(\sin^2\theta + \cos^2\theta) = 5 = \text{RHS}$

f $\text{LHS} = 2 - (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)$
$= 2(\sin^2\theta + \cos^2\theta) - (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)$
$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$
$= (\sin\theta + \cos\theta)^2 = \text{RHS}$

g LHS = $\sin^2 x\,(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y$
$= \sin^2 x - \sin^2 y =$ RHS

6 **a** $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$
b $\sin\theta = \frac{4}{5}$, $\tan\theta = -\frac{4}{3}$
c $\cos\theta = \frac{24}{25}$, $\tan\theta = -\frac{7}{24}$

7 **a** $-\frac{\sqrt{5}}{3}$ **b** $-\frac{2\sqrt{5}}{5}$

8 **a** $-\frac{\sqrt{3}}{2}$ **b** $\frac{1}{2}$

9 **a** $-\frac{\sqrt{7}}{4}$ **b** $-\frac{\sqrt{7}}{3}$

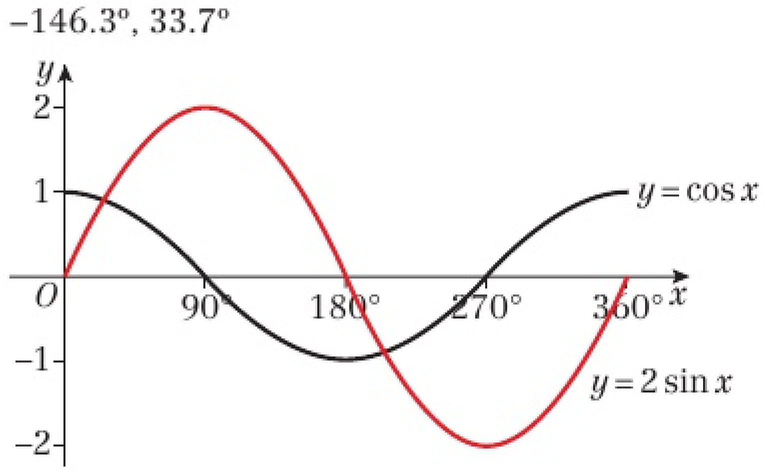
10 **a** $x^2 + y^2 = 1$
b $4x^2 + y^2 = 4$ $\left(\text{or } x^2 + \frac{y^2}{4} = 1\right)$
c $x^2 + y = 1$
d $x^2 = y^2(1 - x^2)$ $\left(\text{or } x^2 + \frac{x^2}{y^2} = 1\right)$
e $x^2 + y^2 = 2$ $\left(\text{or } \frac{(x+y)^2}{4} + \frac{(x-y)^2}{4} = 1\right)$

11 **a** Using cosine rule: $\cos B = \frac{8^2 + 12^2 - 10^2}{2 \times 8 \times 12} = \frac{9}{16}$
b $\frac{\sqrt{175}}{16}$

12 **a** Using sine rule: $\sin Q = \frac{\sin 30}{6} \times 8 = \frac{2}{3}$
b $-\frac{\sqrt{5}}{3}$

Exercise 6D

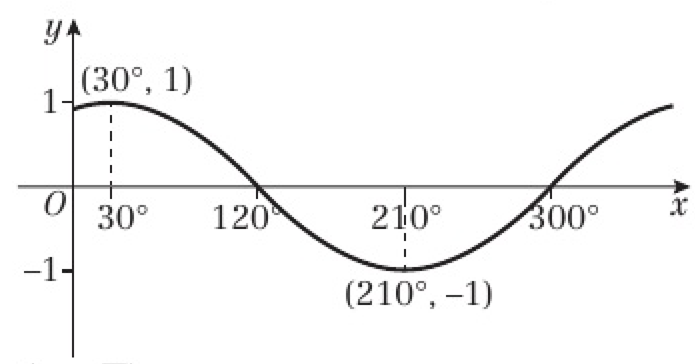
1 **a** −63.4° **b** 116.6°, 296.6°
2 **a** −1.16 **b** 1.16, 1.98, 4.30, 5.12
3 **a** 270° **b** 60°, 240°
c 60°, 300° **d** 15°, 165°
e 140°, 220° **f** 135°, 315°
g 90°, 270° **h** 230°, 310°
4 **a** 0.796, 2.35 **b** $\frac{3\pi}{4}, \frac{5\pi}{4}$
c 2.30, 3.98 **d** 4.00, 5.43
5 **a** 8.13°, 188° **b** 61.9°, 242°
c 105°, 285° **d** 41.8°, 318°
6 **a** $\frac{\pi}{6}, \frac{7\pi}{6}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$
c 0.927, 4.07
7 **a** 56.3°, 236° **b** 54.7°, 235°
c 148°, 328°
8 **a** −120°, −60°, 240°, 300° **b** −2.98, −0.151
c −144°, 144° **d** −5.71, −0.574
e $\frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$ **f** 251°, 431°
9 **a** tan x should be 2/3
b Squaring both sides creates extra solutions
c −146.3°, 33.7°
10 **a**



b 2 **c** 26.6°, 206.6°
11 71.6°, 108.4°, 251.6°, 288.4°
12 **a** $4\sin^2 x - 3(1 - \sin^2 x) = 2$
Rearrange to get $7\sin^2 x = 5$
13 **a** $2\sin^2 x - 5(1 - \sin^2 x) = 1$
Rearrange to get $3\sin^2 x = 4$
b $\sin x > 1$

Exercise 6E

1 **a** 0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°, 360°
b 60°, 180°, 300°
c $22\frac{1}{2}°, 112\frac{1}{2}°, 202\frac{1}{2}°, 292\frac{1}{2}°$
2 **a** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
b $\frac{5\pi}{3}$
c $\frac{5\pi}{4}, \frac{7\pi}{4}$
3 **a** 90°, 270° **b** $\frac{5\pi}{18}, \frac{17\pi}{18}$ or 0.982, 2.97
c 165°, 345° **d** $\frac{25\pi}{18}, \frac{31\pi}{18}$ or 4.36, 5.41
e 16.9°, 123°
4 **a** 11.2°, 71.2°, 131.2° **b** 0.110, 3.25, 6.39
c 37.0°, 127.0° **d** $-\frac{5\pi}{6}, \frac{\pi}{6}$
5 **a** 10°, 130° **b** 71.6°, 108.4°
6 **a**



b $\left(0°, \frac{\sqrt{3}}{2}\right)$, (120°, 0), (300°, 0)
c 86.6°, 333.4°
7 **a** 0.75
b 18.4°, 108.4°, 198.4°, 288.4°
8 **a** 2.5
b No: increasing k will bring another 'branch' of the tan graph into place.

Challenge
25°, 65°, 145°

Exercise 6F

1 **a** 60°, 120°, 240°, 300°
b 45°, 135°, 225°, 315°
c 0°, 180°, 199°, 341°, 360°
d 77.0°, 113°, 257°, 293°
e 60°, 300°
f 204°, 336°
g 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°
2 **a** $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ **b** $-\pi$, 2.04, 0, 0.935, π
c ±1.99 **d** 0, ±1.32, $\pm\pi$
3 **a** $\frac{2\pi}{5}, \frac{4\pi}{5}$ or 1.26, 2.51
b $0, \frac{\pi}{3}$
c No solutions in range
4 **a** ±41.8°, ±138° **b** 38.2°, 142°
5 60°, 75.5°, 284.5°, 300°
6 0.841, 2.30, 3.98, 5.44
7 $2\cos^2 x + \cos x - 6 = (2\cos x - 3)(\cos x + 2)$
There are no solutions to $\cos x = -2$ or to $\cos x = \frac{3}{2}$

8 a $1 - \sin^2 x = 2 - \sin x$
Rearrange to get $\sin^2 x - \sin x + 1 = 0$
b The equation has no real roots as $b^2 - 4ac < 0$

9 a $p = 1, q = 5$
b 72.8°, 129.0°, 252.8°, 309.0°, 432.8°, 489.0°

Challenge

1 –180°, –60°, 60°, 180°

2 0°, 90°, 180°, 270°, 360°

Chapter review 6

1 a $-\cos 57°$ b $-\sin 48°$ c $\tan 10°$
d $\cos\left(\frac{\pi}{6}\right)$ e $-\sin\left(\frac{\pi}{6}\right)$ f $-\tan\left(\frac{\pi}{4}\right)$

2 a 0 b $-\frac{\sqrt{2}}{2}$ c $\sqrt{3}$
d –1 e 1 f –1

3 Using $\sin^2 A = 1 - \cos^2 A$, $\sin^2 A = 1 - \left(-\sqrt{\frac{7}{11}}\right)^2 = \frac{4}{11}$.
Since angle A is obtuse, it is in the second quadrant and sin is positive, so $\sin A = \frac{2}{\sqrt{11}}$.
Then $\tan A = \frac{\sin A}{\cos A} = \frac{2}{\sqrt{11}} \times \left(-\sqrt{\frac{11}{7}}\right) = -\frac{2}{\sqrt{7}} = -\frac{2}{7}\sqrt{7}$.

4 a $-\frac{\sqrt{21}}{5}$ b $-\frac{2}{5}$

5 a $\cos^2\theta - \sin^2\theta$ b $\sin^4 3\theta$ c 1

6 a 1 b $\tan y = \frac{4 + \tan x}{2\tan x - 3}$

7 a LHS $= (1 + 2\sin\theta + \sin^2\theta) + \cos^2\theta$
$= 1 + 2\sin\theta + 1$
$= 2 + 2\sin\theta$
$= 2(1 + \sin\theta) =$ RHS
b LHS $= \cos^4\theta + \sin^2\theta$
$= (1 - \sin^2\theta)^2 + \sin^2\theta$
$= 1 - 2\sin^2\theta + \sin^4\theta + \sin^2\theta$
$= (1 - \sin^2\theta) + \sin^4\theta$
$= \cos^2\theta + \sin^4\theta =$ RHS

8 a No solutions: $-1 \leqslant \sin\theta \leqslant 1$
b 2 solutions: $\tan\theta = -1$ has two solutions in the interval.
c No solutions: $2\sin\theta + 3\cos\theta > -5$ so $2\sin\theta + 3\cos\theta + 6$ can never be equal to 0.
d No solutions: $\tan^2\theta = -1$ has no real solutions.

9 a $(4x - y)(y + 1)$ b 0.244, π, 3.39

10 a $3\cos 3\theta$
b 0.281, 1.82, 2.37, 3.91, 4.47, 6.00

11 a $2\sin 2\theta = \cos 2\theta \Rightarrow \frac{2\sin 2\theta}{\cos 2\theta} = 1$
$\Rightarrow 2\tan 2\theta = 1 \Rightarrow \tan 2\theta = 0.5$
b 0.232, 1.80, 3.37, 4.94

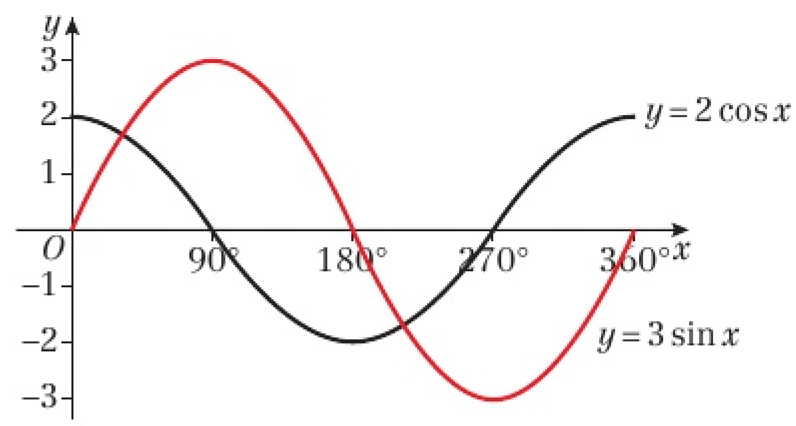
12 a 225°, 345° b 22.2°, 67.8°, 202.2°, 247.8°

13 30°, 150°, 210°

14 0, 2.30, 3.98, π

15 a Found additional solutions after dividing by three rather than before. Not applied the full interval for solutions.
b –350°, –310°, –230°, –190°, –110°, –70°, 10°, 50°, 130°, 170°, 250°, 290°

16 a



b 2 c 33.7°, 213.7°

17 a $\frac{9}{11}$ b $\frac{\sqrt{40}}{11}$

18 a Using sine rule: $\sin Q = \sin 45° \times \frac{6}{5} = \frac{\sqrt{2}}{2} \times \frac{6}{5} = \frac{3\sqrt{2}}{5}$
b $-\frac{\sqrt{7}}{5}$

19 a $3\sin^2 x - 3(1 - \sin^2 x) = 2$
Rearrange to give $4\sin^2 x = 3$
b $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

20 –318.2°, –221.8°, 41.8°, 138.2°

Challenge

45°, 54.7°, 125.3°, 135°, 225°, 234.7°, 305.3°, 315°

CHAPTER 7

Prior knowledge check

1 a $20x^4$ b $-21x^{-4}$ c $6x^{\frac{1}{2}}$ d $-6x^{-\frac{3}{2}}$

2 a $6x^2 + 1$ b $8x^3 + \frac{x^{-\frac{1}{2}}}{2} - \frac{5x^{-\frac{3}{2}}}{2}$
c $-\frac{5}{3}x^{-\frac{4}{3}} - \frac{4}{3}x^{-\frac{1}{3}}$

3 a $168x^6$ b $8x^{-3}$

Exercise 7A

1 a $x \geqslant -\frac{4}{3}$ b $x \leqslant \frac{2}{3}$ c $x \leqslant -2$
d $x \leqslant 2, x \geqslant 3$ e $x \in \mathbb{R}$ f $x \in \mathbb{R}$
g $x \geqslant 0$ h $x \geqslant 6$

2 a $x \leqslant 4.5$ b $x \geqslant 2.5$
c $x \geqslant -1$ d $-1 \leqslant x \leqslant 2$
e $-3 \leqslant x \leqslant 3$ f $-5 \leqslant x < 0, 0 < x \leqslant 5$
g $0 < x \leqslant 9$ h $-2 \leqslant x \leqslant 0$

3 $f'(x) = -6x^2 - 3$
$x^2 \geqslant 0$ for all $x \in \mathbb{R}$, so $-6x^2 - 3 \leqslant 0$ for all $x \in \mathbb{R}$.
$\therefore$ f(x) is decreasing for all $x \in \mathbb{R}$.

4 a Any $p \geqslant 2$
b No. Can be any $p \geqslant 2$.

Exercise 7B

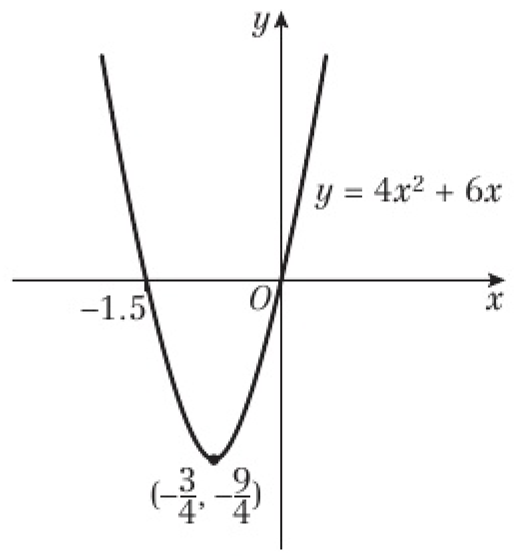
1 a –28 b –17 c $-\frac{1}{5}$

2 a 10 b 4 c 12.25

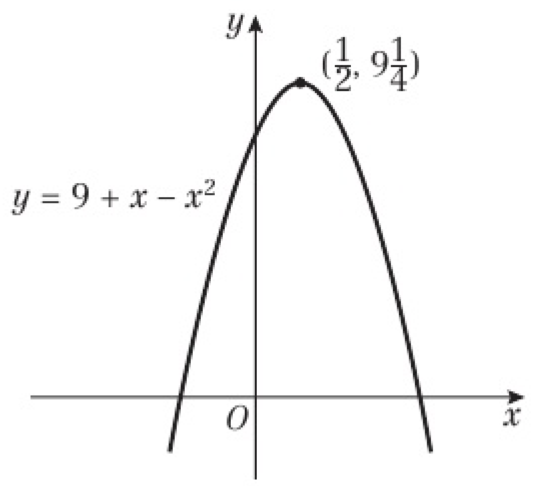
3 a $\left(-\frac{3}{4}, -\frac{9}{4}\right)$ minimum
b $\left(\frac{1}{2}, 9\frac{1}{4}\right)$ maximum
c $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$ maximum, (1, 0) minimum
d (3, –18) minimum, $\left(-\frac{1}{3}, \frac{14}{27}\right)$ maximum
e (1, 2) minimum, (–1, –2) maximum
f (3, 27) minimum
g $\left(\frac{9}{4}, -\frac{9}{4}\right)$ minimum
h $(2, -4\sqrt{2})$ minimum

i $(\sqrt{6}, -36)$ minimum, $(-\sqrt{6}, -36)$ minimum, $(0, 0)$ maximum

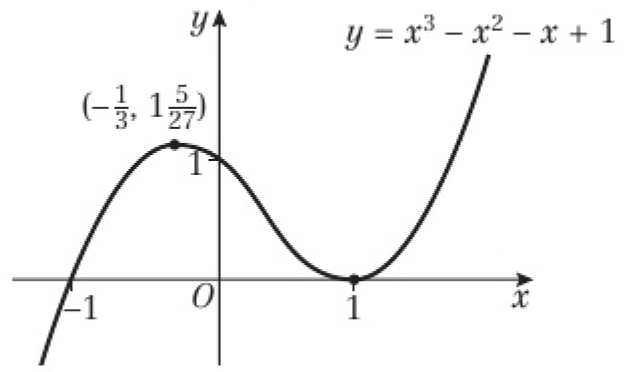
4 a



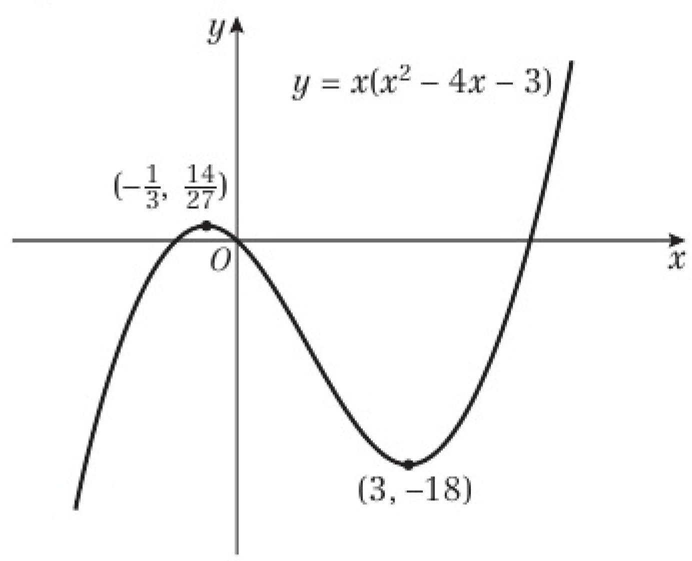
b



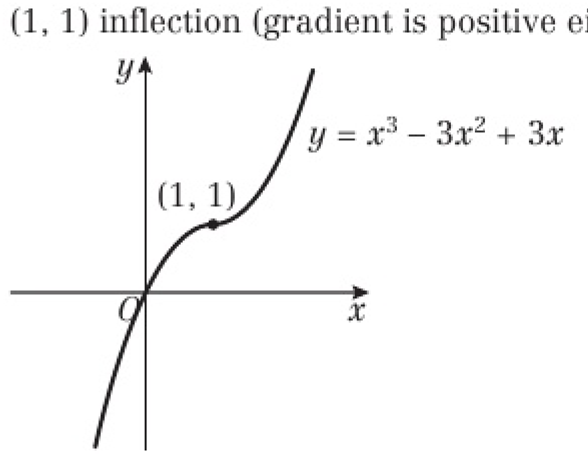
c



d



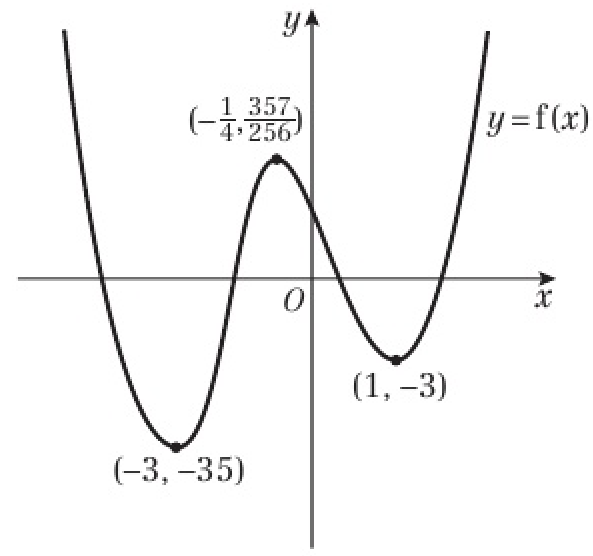
5 (1, 1) inflection (gradient is positive either side of point)



6 Maximum value is 27; $f(x) \leqslant 27$

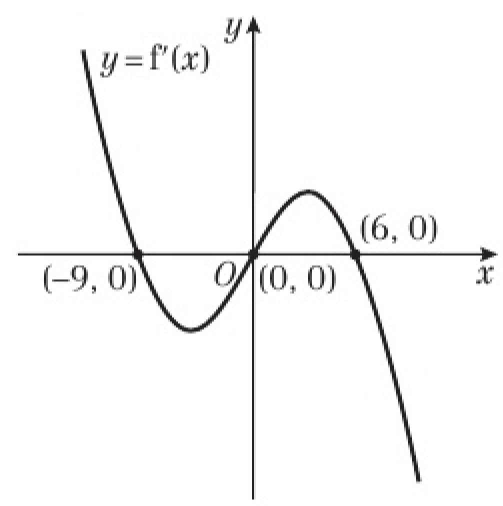
7 a $(1, -3)$: minimum, $(-3, -35)$: minimum, $\left(-\frac{1}{4}, \frac{357}{256}\right)$: maximum

b

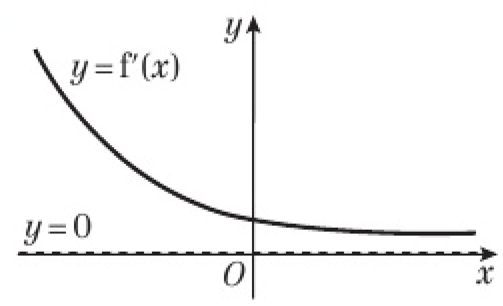


Exercise 7C

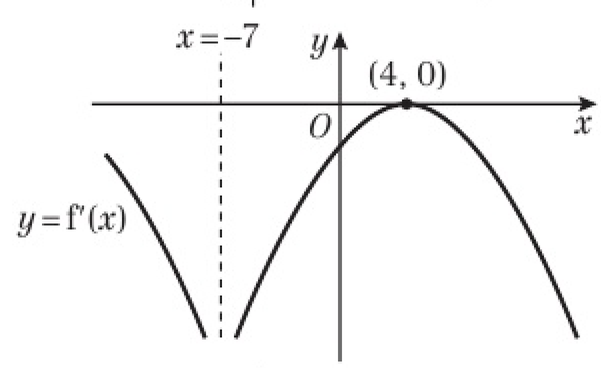
1 a



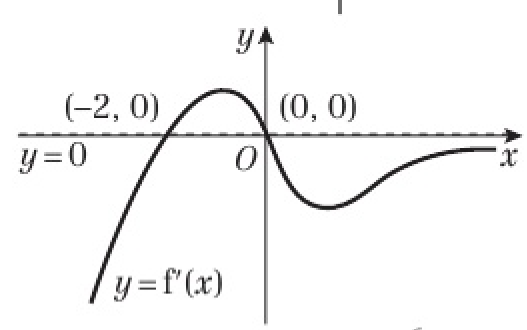
b



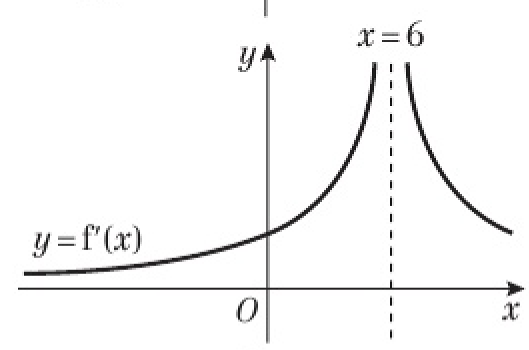
c



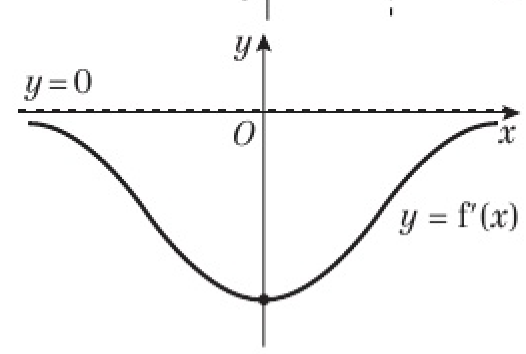
d



e

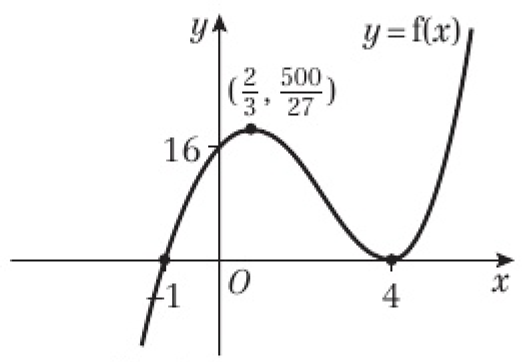


f



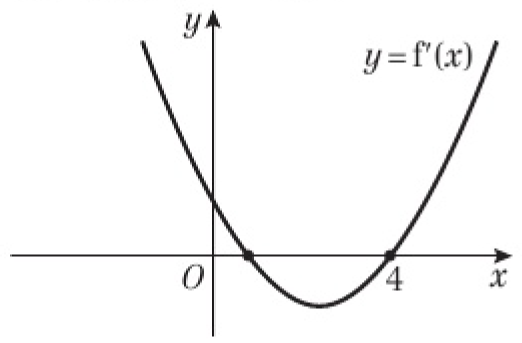
2 **a** $f(x) = x^3 - 7x^2 + 8x + 16$
$f'(x) = 3x^2 - 14x + 8 = (3x - 2)(x - 4)$

b



c $(4, 0)$, $\left(\frac{2}{3}, 0\right)$ and $(0, 8)$

d



Exercise 7D

1 $2t - 3$ **2** 2π **3** $-\frac{4}{3}$

4 48π cm³ per cm

5 $18\,\text{m s}^{-1}$

6 **a** Let x = width of garden.
$x + 2y = 80 \Rightarrow x = 80 - 2y$
Area $A = xy = y(80 - 2y)$

b $20\,\text{m} \times 40\,\text{m}$, $800\,\text{m}^2$

7 **a** $2\pi r^2 + 2\pi rh = 600\pi \Rightarrow h = \dfrac{300 - r^2}{r}$
$V = \pi r^2 h = \pi r(300 - r^2) = 300\pi r - \pi r^3$

b $2000\pi\,\text{cm}^3$

8 **a** Let θ = angle of sector.
$\pi r^2 \times \dfrac{\theta}{360} = 100 \Rightarrow \theta = \dfrac{36\,000}{\pi r^2}$
$P = 2r + 2\pi r \times \dfrac{\theta}{360} = 2r + \dfrac{200\pi r}{\pi r^2}$
$= 2r + \dfrac{200}{r}$
$\theta < 2\pi \Rightarrow$ Area $< \pi r^2$, so $\pi r^2 > 100$
$\therefore r > \sqrt{\dfrac{100}{\pi}}$

b 40 cm

9 **a** Let h = height of rectangle.
$P = \pi r + 2r + 2h = 40 \Rightarrow 2h = 40 - 2r - \pi r$
$A = \dfrac{\pi}{2}r^2 + 2rh = \dfrac{\pi}{2}r^2 + r(40 - 2r - \pi r)$
$= 40r - 2r^2 - \dfrac{\pi}{2}r^2$

b $\dfrac{800}{4 + \pi}\,\text{cm}^2$

10 **a** $18x + 14y = 1512 \Rightarrow y = \dfrac{1512 - 18x}{14}$
$A = 12xy = 12x\left(\dfrac{1512 - 18x}{14}\right)$
$= 1296x - \dfrac{108x^2}{7}$

b $27\,216\,\text{mm}^2$

Chapter review 7

1 **a** $x = 4$, $y = 20$

b $\dfrac{d^2y}{dx^2} = \dfrac{3}{4}x^{-\frac{1}{2}} + 96x^{-3}$
At $x = 4$, $\dfrac{d^2y}{dx^2} = \dfrac{15}{8} > 0$
$(4, 20)$ is a local minimum.

2 $(1, -11)$ and $\left(\frac{7}{3}, -\frac{329}{27}\right)$

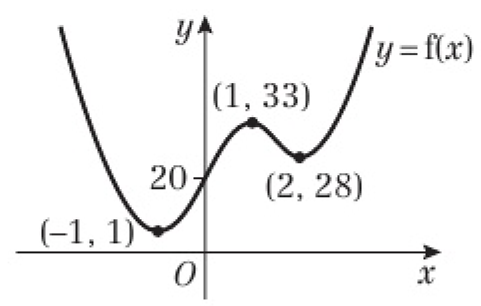
3 **a** $7\frac{31}{32}$

b $f'(x) = \left(x - \dfrac{1}{x}\right)^2 \geqslant 0$ for all values of x

4 $(1, 4)$

5 **a** $(1, 33)$ maximum, $(2, 28)$ and $(-1, 1)$ minimum

b



6 **a** $\dfrac{250}{x^2} - 2x$ **b** $(5, 125)$

7 **a** $P\left(x, 5 - \frac{1}{2}x^2\right)$
$OP^2 = (x - 0)^2 + \left(5 - \frac{1}{2}x^2 - 0\right)^2$
$= \frac{1}{4}x^4 - 4x^2 + 25$

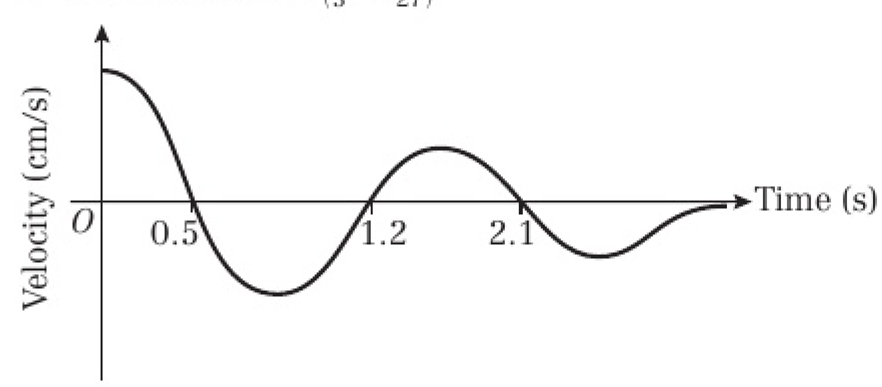
b $x = \pm 2\sqrt{2}$ or $x = 0$

c When $x = \pm 2\sqrt{2}$, $f''(x)$ so minimum
When $x = \pm 2\sqrt{2}$, $y = 9$ so $OP = 3$

8 **a** $3 + 5(3) + 3^2 - 3^3 = 0$ therefore C on curve

b A is $(-1, 0)$; B is $\left(\frac{5}{3}, 9\frac{13}{27}\right)$

9



10 $\dfrac{10}{3}, \dfrac{2300\pi}{27}$

11 $\dfrac{dA}{dx} = 4\pi x - \dfrac{2000}{x^2}$
$\dfrac{dA}{dx} = 0$: $4\pi x = \dfrac{2000}{x^2} \rightarrow x^3 = \dfrac{2000}{4\pi} = \dfrac{500}{\pi}$

12 **a** $y = 1 - \dfrac{x}{2} - \dfrac{\pi x}{4}$

b $R = xy + \dfrac{\pi}{2}\left(\dfrac{x}{2}\right)^2$
$= x\left(1 - \dfrac{x}{2} - \dfrac{\pi x}{4}\right) + \dfrac{\pi x^2}{8}$
$= x - \dfrac{x^2}{2} - \dfrac{\pi x^2}{4} + \dfrac{\pi x^2}{8}$
$= \dfrac{x}{8}(8 - 4x - \pi x)$

c $\dfrac{2}{4 + \pi}\,\text{m}^2$ $(0.280\,\text{m}^2)$

13 a $SA = \pi x^2 + 2\pi x + \pi x^2 + 2\pi xh = 80\pi$

$$h = \frac{40 - x - x^2}{x}$$

$$V = \pi x^2 h = \pi x^2\left(\frac{40 - x - x^2}{x}\right)$$

$$= \pi(40x - x^2 - x^3)$$

b $\frac{10}{3}$ **c** $\frac{d^2V}{dx^2} < 0 \therefore$ maximum

d $\frac{2300\pi}{27}$ **e** $22\frac{2}{9}\%$

14 a Length of short sides $= \frac{x}{\sqrt{2}}$

Area $= \frac{1}{2} \times$ base $\times$ height

$$= \frac{1}{2}\left(\frac{x^2}{2}\right) = \frac{1}{4}x^2\ \text{m}^2$$

b Let l be length of EF.

$$\frac{1}{4}x^2 l = 4000 \Rightarrow l = \frac{16\,000}{x^2}$$

$$S = 2\left(\frac{1}{4}x^2\right) + \frac{2xl}{\sqrt{2}}$$

$$= \frac{1}{2}x^2 + \frac{32\,000x}{\sqrt{2}x^2} = \frac{x^2}{2} + \frac{16\,000\sqrt{2}}{x}$$

c $x = 20\sqrt{2}$, $S = 1200\ \text{m}^2$ **d** $\frac{d^2S}{dx^2} > 0$

Challenge

a Any constant function

b For example $(1 - x^2)$, or any suitably defined piecewise function.

c For example the piecewise function $f(x) = x$ for $0 < x < 1$ and 0 otherwise.

d For example the piecewise function $f(x) = x$ if x is rational and $-x$ if x is irrational.

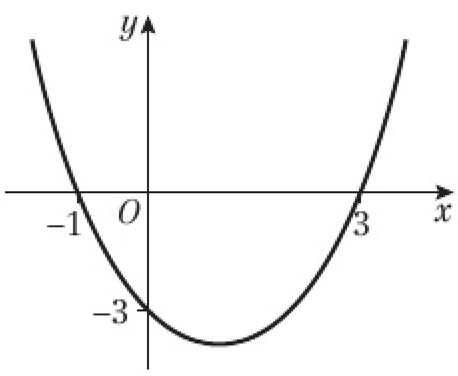
CHAPTER 8

Prior knowledge check

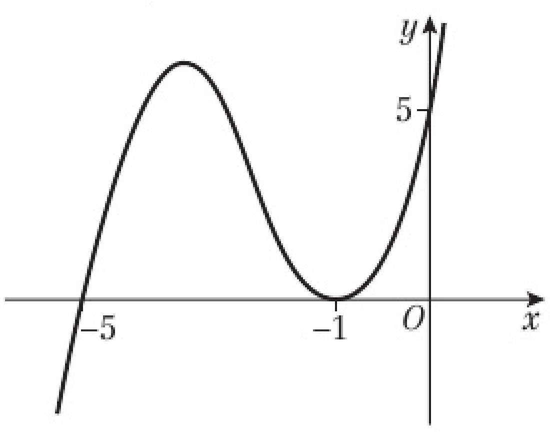
1 a $x^{\frac{5}{2}}$ **b** $2x^{\frac{3}{2}}$ **c** $x^{\frac{5}{2}} - \sqrt{x}$ **d** $x^{-\frac{3}{2}} + 4x$

2 a $6x^2 + 3$ **b** $x - 1$ **c** $3x^2 + 2x$ **d** $-\frac{1}{x^2} - 3x^2$

3 a



b



4 a $\frac{2}{5}x^5 - x^3 + 6x + c$ **b** $2x^{\frac{3}{2}} - \frac{1}{2x^2} + c$

c $2x^{\frac{1}{2}} + 3x^2 + c$

Exercise 8A

1 a $152\frac{1}{4}$ **b** $48\frac{2}{5}$ **c** $5\frac{1}{3}$ **d** 2

2 a $5\frac{1}{4}$ **b** 10 **c** $11\frac{5}{6}$ **d** $60\frac{1}{2}$

3 a $16\frac{2}{3}$ **b** $46\frac{1}{2}$ **c** $\frac{11}{14}$ **d** $2\frac{1}{2}$

4 $A = -7$ or 4

5 28

6 $-8 + 8\sqrt{3}$

7 $k = \frac{25}{4}$

8 450 m

Challenge

$k = 2$

Exercise 8B

1 a 22 **b** $36\frac{2}{3}$ **c** $48\frac{8}{15}$ **d** 6

2 4 **3** 6 **4** $10\frac{2}{3}$

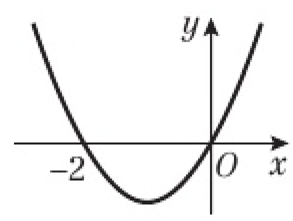
5 $21\frac{1}{3}$ **6** $\frac{4}{81}$ **7** $k = 2$

8 a (–1, 0) and (3, 0) **b** $10\frac{2}{3}$

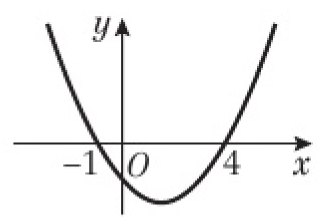
9 $1\frac{1}{3}$

Exercise 8C

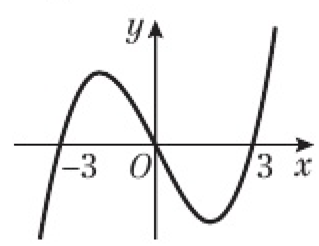
1 a $1\frac{1}{3}$



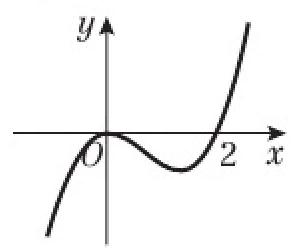
b $20\frac{5}{6}$



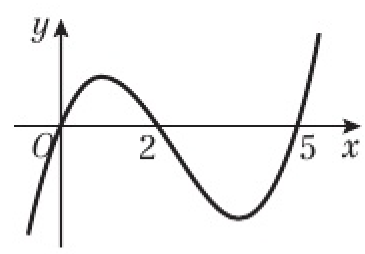
c $40\frac{1}{2}$



d $1\frac{1}{3}$



e $21\frac{1}{12}$



2 a (–3, 0) and (2, 0) **b** $21\frac{1}{12}$

3 a $f(-3) = 0$

b $f(x) = (x + 3)(-x^2 + 7x - 10)$

c $f(x) = (x + 3)(x - 5)(2 - x)$

d (–3, 0), (2, 0) and (5, 0)

e $143\frac{5}{6}$

Challenge

1 **a** $4\frac{1}{2}$ **b** 9 **c** $\frac{9a}{2}$ **d** $4\frac{1}{2}$ **e** $\frac{9}{2a}$

2 **a** B has x-coordinate 1

$\int_0^1 (x^3 + x^2 - 2x)\mathrm{d}x = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_0^1$

$= \frac{1}{4} + \frac{1}{3} - 1 = -\frac{5}{12}$

So area under x-axis is $\frac{5}{12}$

Area above x-axis is

$\left(\frac{1}{4}0^4 + \frac{1}{3}0^3 - 0^2\right) - \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right) = \frac{5}{12}$

So the x-coordinate of a satisfies

$3x^4 + 4x^3 - 12x^2 + 5 = 0$

Then use the factor theorem twice to get

$(x - 1)^2(3x^2 + 10x + 5) = 0$

b A has coordinates $\left(\frac{-5 + \sqrt{10}}{3}, \frac{-80 + 37\sqrt{10}}{27}\right)$

The roots at 1 correspond to point B.

The root $\frac{-5 - \sqrt{10}}{3}$ gives a point on the curve to the left of -2 below the x-axis, so cannot be A.

Exercise 8D

1 **a** $A(-2, 6)$, $B(2, 6)$ **b** $10\frac{2}{3}$

2 **a** $A(1, 3)$, $B(3, 3)$ **b** $1\frac{1}{3}$

3 $6\frac{2}{3}$

4 4.5

5 **a** (2, 12) **b** $13\frac{1}{3}$

6 **a** $20\frac{5}{6}$ **b** $17\frac{1}{6}$

7 **a, b** Substitute into equation for y

c $y = x - 4$ **d** $8\frac{3}{5}$

8 $3\frac{3}{8}$

9 **a** Substitute $x = 4$ into both equations

b 7.2

10 **a** $21\frac{1}{3}$ **b** $2\frac{5}{9}$

11 **a** (−1, 11) and (3, 7) **b** $21\frac{1}{3}$

Exercise 8E

1 11

2 $166\frac{2}{3}$

3 $42\frac{2}{3}$

4 3

5 $10\frac{5}{12}$

Challenge

1 $\frac{29}{6}$

Exercise 8F

1 $x = 2, y = 0.2, x = 3$ $y = 0.1$ Area = 0.464

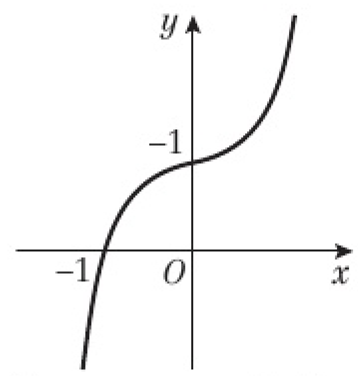
2 $x = 1.5$ $y = 1.414$, $x = 1.75$ $y = 1.581$, $x = 2.25$ $y = 1.871$ Area = 2.33

3 $x = 1.5$ $y = 2.09$, $x = 2$ $y = 3$, Area = 3.24

4 Area = 1.818

5 $x = -0.2$ $y = 0.556$, $x = 0.2$ $y = 0.455$, $x = 1$ $y = 0.333$ Area = 1.09

6 **a**



b 2 **c** 2

d Same; the trapezium rule gives an underestimate of the area between $x = -1$ and $x = 0$, and an overestimate between $x = 0$ and $x = 1$, and these cancel out.

7 Area = 2.908

8 **a** 1.61

b Underestimate; curve bends outwards (convex)

9 **a** **i** 1.8195 **ii** 1.8489

b $\frac{4\sqrt{2}}{3}$ **i** 3.51% **ii** 1.95%

10 **a** Area = 4.339

b Underestimate because the curve is concave.

Chapter review 8

1 **a** −1,3 **b** $10\frac{2}{3}$

2 **a** $-\frac{2x^{\frac{3}{2}}}{3} + 5x - 8\sqrt{x} + c$ **b** $\frac{7}{3}$

3 **a** (3, 0) **b** (1, 4) **c** $6\frac{3}{4}$

4 **a** $\frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$ **b** $2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c$ **c** $A = 6, B = -2$

5 **a** $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$

b (4,16) **c** 133 (3 s.f.)

6 **a** (6,12) **b** $13\frac{1}{3}$

7 **a** $A(1, 0)$, $B(5, 0)$, $C(6, 5)$ **b** $10\frac{1}{6}$

8 **a** $q = -2$ **b** $C(6,17)$ **c** $1\frac{1}{3}$

9 $A = -6$ or 1

10 **a** $\mathrm{f}'(x) = \frac{(2 - x^2)(4 - 4x^2 + x^4)}{x^2} = 8x^{-2} - 12 + 6x^2 - x^4$

b $\mathrm{f}''(x) = -16x^{-3} + 12x - 4x^3$

c $\mathrm{f}(x) = -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} - \frac{47}{5}$

11 **a** (−3, 0) and $(\frac{1}{2}, 0)$ **b** $14\frac{7}{24}$

12 **a** $(-\frac{3}{2}, 0)$ and (4, 0) **b** $55\frac{11}{24}$

13 **a** −2 and 3 **b** $21\frac{1}{12}$

14 **a** 1.538, 0.690 **b** 6.24 **c** 18.24

15 **a** 1.494, 1.741 **b** 1.50

16 $21\frac{1}{3}$

Challenge

$10\frac{5}{12}$

Review exercise 2

1 **a** $k = 0.6$, $k = -4$ **b** $a = 16$, $d = 8$

2 **a** $19p - 18$, $10 - 2p$, 30th term = $272 - 39p$

b $p = 12$

3 a $r^6 = \frac{225}{64} \Rightarrow \ln r^6 = \ln\left(\frac{225}{64}\right) \Rightarrow 6\ln r - \ln\left(\frac{225}{64}\right) = 0$

$\Rightarrow 6\ln r + \ln\left(\frac{64}{225}\right) = 0$

b $r = 1.23$

4 a $4 + 4r + 4r^2 = 7 \Rightarrow 4r^2 + 4r - 3 = 0$

b $r = \frac{1}{2}$ or $r = -\frac{3}{2}$ **c** 8

5 a $x = 1, r = 3$ and $x = -9, r = -\frac{1}{3}$

b 243 **c** 182.25

6 a $a_2 = 3k + 5$

b $a_3 = 3a_2 + 5 = 9k + 20$

c i $40k + 90$ **ii** $10(4k + 9)$

7 a $|x| < \frac{1}{4}$

b $\frac{6}{1+4x} = \frac{24}{5} \Rightarrow x = \frac{1}{16}$

8 a 0.776 **b** 1.2%

8 a 60

b $a = 10, r = \frac{5}{6}$

$\frac{10\left(1 - \left(\frac{5}{6}\right)^k\right)}{1 - \frac{5}{6}} > 55 \Rightarrow 1 - \left(\frac{5}{6}\right)^k > \frac{11}{12}$

$\Rightarrow \frac{1}{12} > \left(\frac{5}{6}\right)^k \Rightarrow \log\left(\frac{1}{12}\right) > \log\left(\left(\frac{5}{6}\right)^k\right)$

$\Rightarrow \log\left(\frac{1}{12}\right) > k\log\left(\frac{5}{6}\right) \Rightarrow \frac{\log\left(\frac{1}{12}\right)}{\log\left(\frac{5}{6}\right)} < k$

c $k = 14$

9 a 2860

b $2400 \times 1.06^{N-1} > 6000 \Rightarrow 1.06^{N-1} > 2.5$

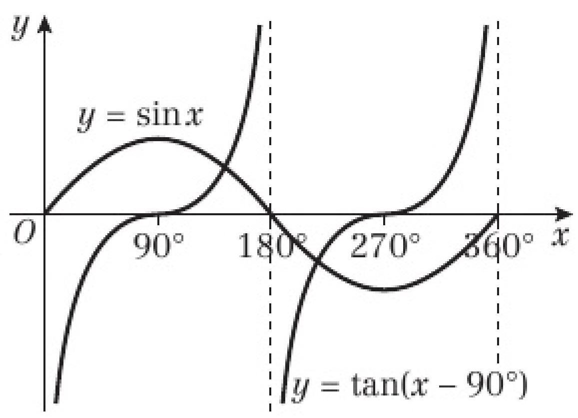
$\Rightarrow \log 1.06^{N-1} > \log 2.5 \Rightarrow (N-1)\log 1.06 > \log 2.5$

c $N = 16.7\ldots$, therefore $N = 17$

d $S_n = \frac{2400(1.06^{10} - 1)}{1.06 - 1} = 31\,633.90\ldots$ employees.

Total donations at 5 times this, so £158 000 over the 10-year period.

10 a



b 2

11 a 1

b $\frac{\pi}{4}, \frac{5\pi}{4}$

12 a 1

b 45°, 225°

13 $\frac{\pi}{2}, \frac{5\pi}{6}$

14 90°, 150°

15 1.26, 2.57, 4.40, 5.72

16 72.3°, 147.5°, 252.3°, 327.5°

17 0°, 78.5°, 281.5°, 360°

18 $\cos^2 x(\tan^2 x + 1)$

$= \cos^2 x\left(\frac{\sin^2 x}{\cos^2 x} + 1\right)$

$= \sin^2 x + \cos^2 x = 1$

19 $f'(x) = 3x^2 - 24x + 48 = 3(x-4)^2 > 0$

20 a $A(1,0)$ and $B(2,0)$

b $(\sqrt{2}, 2\sqrt{2} - 3)$

21 a $V = \pi r^2 h = 128\pi$, so $h = \frac{128}{r^2}$

$S = 2\pi rh + 2\pi r^2 = \frac{256\pi}{r} + 2\pi r^2$

b 96π cm²

22 a $\frac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$

b $\frac{d^2y}{dx^2} = 6 - x^{-\frac{3}{2}}$

c $x^3 + \frac{8}{3}x^{\frac{3}{2}} + c$

23 $6\frac{3}{4}$

24 4

25 a $-x^4 + 3x^2 + 4 = (-x^2 + 4)(x^2 + 1)$; $x^2 + 1 = 0$ has no real solutions; so solutions are $A(-2, 0)$, $B(2, 0)$

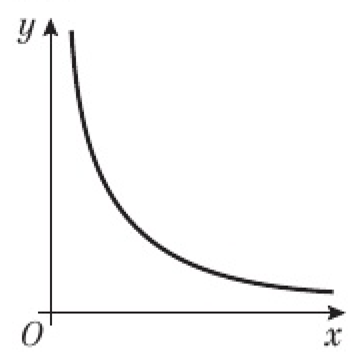
b 19.2

26 $4\frac{1}{2}$

27 a $P(-1, 4)$, $Q(2, 1)$

b 4.5

28 a



b

1.2	1.4	1.6	1.8
0.833	0.714	0.625	0.556

c 0.70 **d** Overestimate – concave shape

29 a 0.776

b 1.2%

30 $\frac{1}{3}$

Challenge

1 $a_1 = m, a_2 = m + k, a_3 = m + 2k, \ldots$

$6m + 45k = 4m + 50k \Rightarrow 2m = 5k \Rightarrow m\frac{5}{2}k$

2 0°, 30°, 150°, 180°, 270°, 360°

INDEX