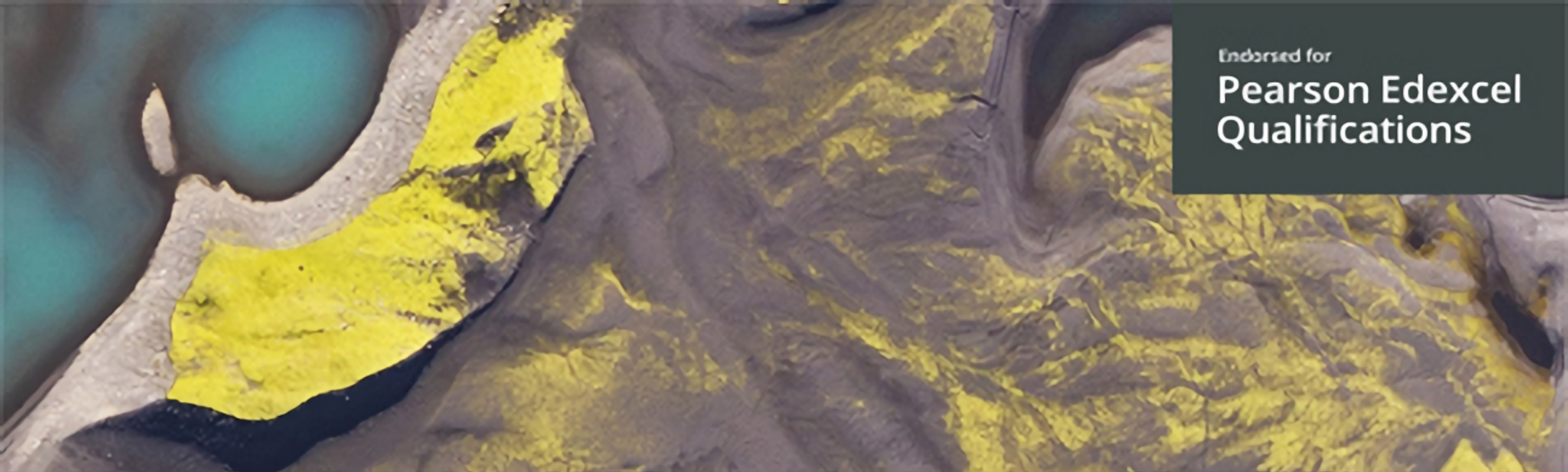

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PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 3

STUDENT BOOK

PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 3

Student Book

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

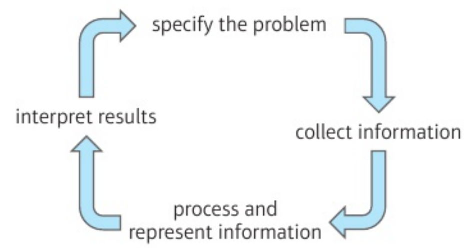
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

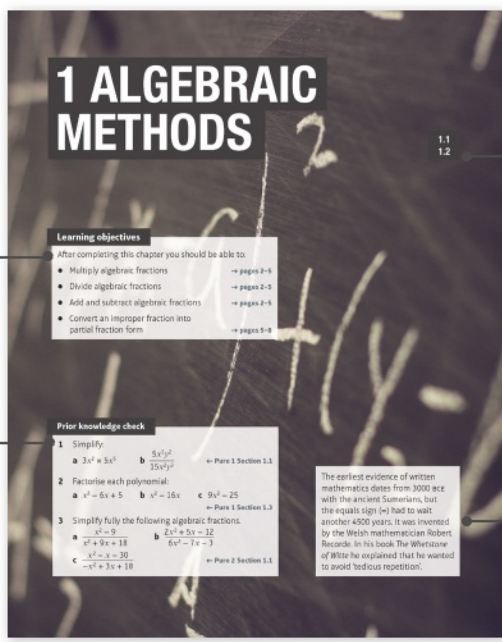
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book



Each chapter is mapped to the specification content for easy reference

Each chapter starts with a list of *Learning objectives*

The *Prior knowledge check* helps make sure you are ready to start the chapter

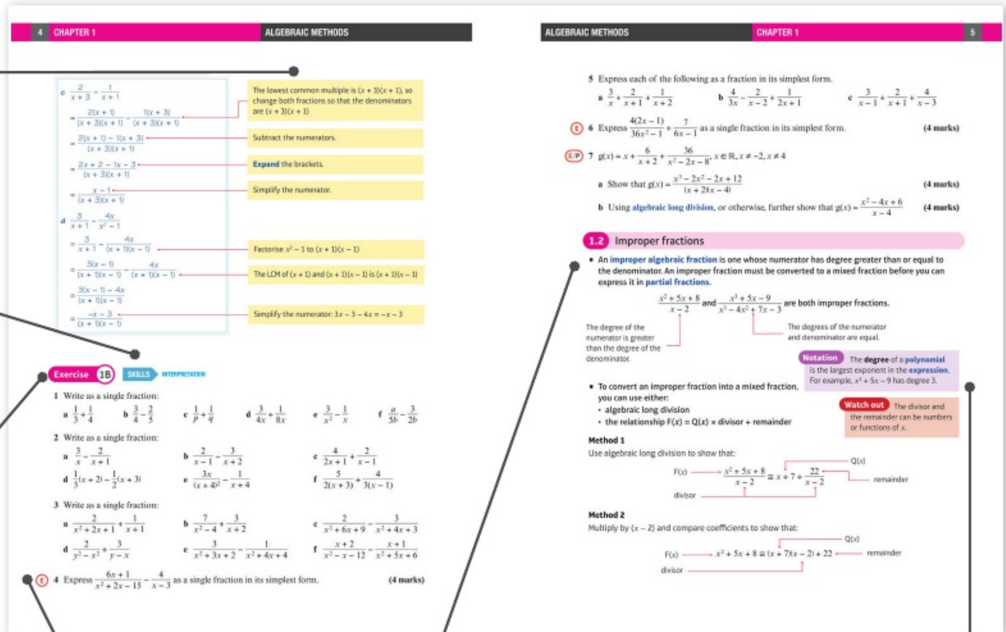
The real world applications of the maths you are about to learn are highlighted at the start of the chapter.

Glossary terms will be identified by bold blue text on their first appearance.

Step-by-step worked examples focus on the key types of questions you'll need to tackle

Transferable skills are signposted where they naturally occur in the exercises and examples

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard



Exercises are packed with exam-style questions to ensure you are ready for the exams

Each section begins with explanation and key learning points

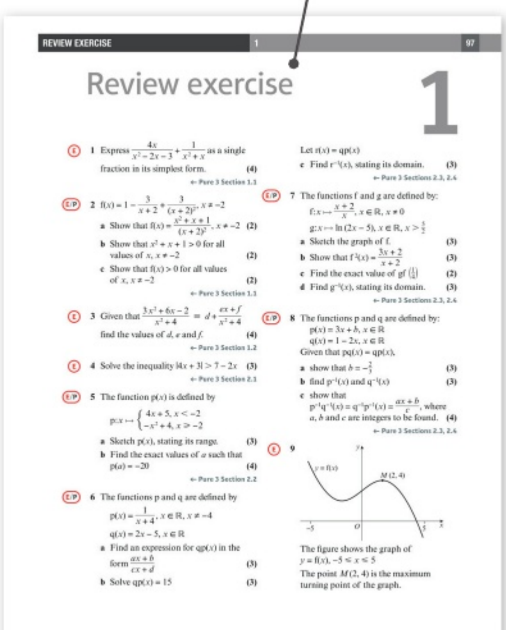
Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

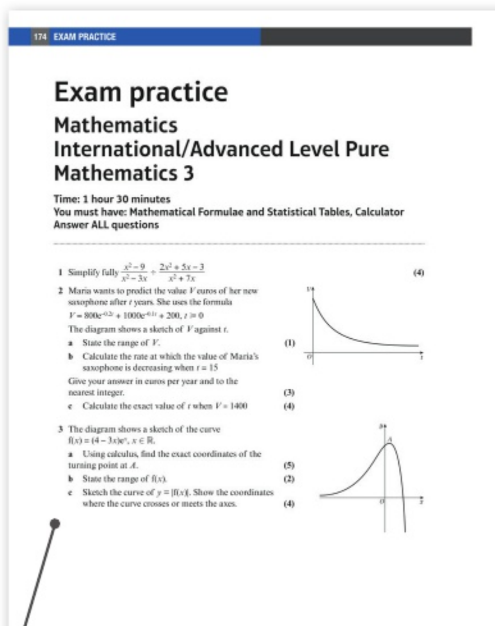
Exam-style questions are flagged with (E)

Problem-solving questions are flagged with (P)

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions





A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Pure Mathematics 3 (P3) is a **compulsory** unit in the following qualifications:

International Advanced Level in Mathematics

International Advanced Level in Pure Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
P3: Pure Mathematics 3 Paper code WMA13/01	$16\frac{2}{3}$% of IAL	75	1 hour 30 min	January, June and October First assessment June 2020

IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

P3	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	5–10	5–10	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, –, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides a full worked solution for questions in the book.
Download all the solutions as a PDF or quickly find the solution you need online.

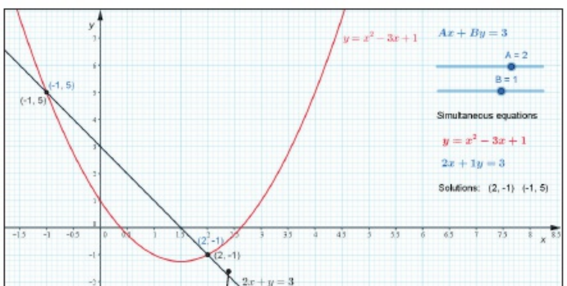
Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

Online Find the point of intersection graphically using technology.

GeoGebra

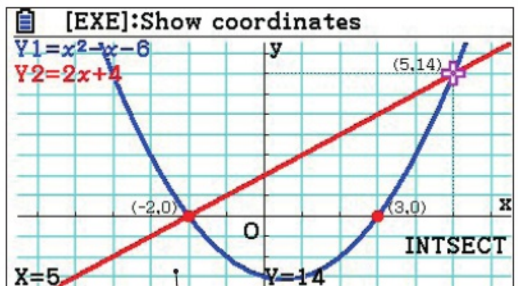
GeoGebra-powered interactives



Interact with the mathematics you are learning using GeoGebra's easy-to-use tools

CASIO

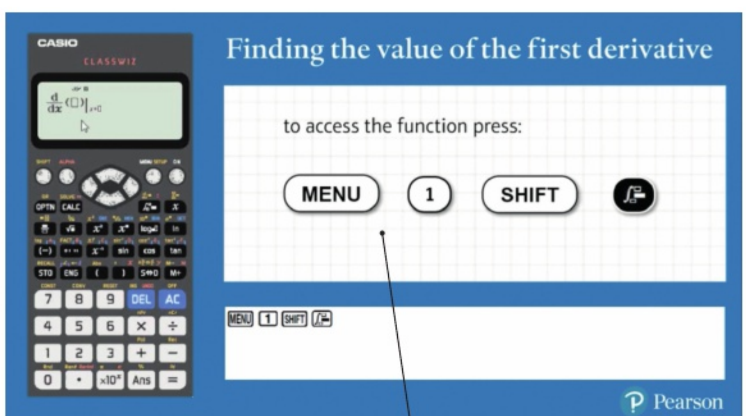
Graphic calculator interactives



Explore the mathematics you are learning and gain confidence in using a graphic calculator

Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.

1 ALGEBRAIC METHODS

1.1
1.2

Learning objectives

After completing this chapter you should be able to:

- Multiply algebraic fractions → pages 2–5
- Divide algebraic fractions → pages 2–5
- Add and subtract algebraic fractions → pages 2–5
- Convert an improper fraction into partial fraction form → pages 5–8

Prior knowledge check

1 Simplify:

a $3x^2 \times 5x^5$ **b** $\dfrac{5x^3y^2}{15x^2y^3}$ ← Pure 1 Section 1.1

2 Factorise each polynomial:

a $x^2 - 6x + 5$ **b** $x^2 - 16x$ **c** $9x^2 - 25$ ← Pure 1 Section 1.3

3 Simplify fully the following algebraic fractions.

a $\dfrac{x^2 - 9}{x^2 + 9x + 18}$ **b** $\dfrac{2x^2 + 5x - 12}{6x^2 - 7x - 3}$

c $\dfrac{x^2 - x - 30}{-x^2 + 3x + 18}$ ← Pure 2 Section 1.1

The earliest evidence of written mathematics dates from 3000 BCE with the ancient Sumerians, but the equals sign (=) had to wait another 4500 years. It was invented by the Welsh mathematician Robert Recorde. In his book *The Whetstone of Witte* he explained that he wanted to avoid 'tedious repetition'.

1.1 Arithmetic operations with algebraic fractions

- To multiply fractions, **cancel** any **common factors**, then multiply the **numerators** and multiply the **denominators**.

Example 1 SKILLS PROBLEM-SOLVING

Simplify the following products:

a $\frac{3}{5} \times \frac{5}{9}$ b $\frac{a}{b} \times \frac{c}{a}$ c $\frac{x+1}{2} \times \frac{3}{x^2-1}$

a $\frac{\cancel{3}^1}{{}_1\cancel{5}} \times \frac{\cancel{5}^1}{\cancel{9}_3} = \frac{1 \times 1}{1 \times 3} = \frac{1}{3}$ — Cancel any common factors and multiply numerators and denominators.

b $\frac{{}^1\cancel{a}}{b} \times \frac{c}{\cancel{a}_1} = \frac{1 \times c}{b \times 1} = \frac{c}{b}$ — Cancel any common factors and multiply numerators and denominators.

c $\frac{x+1}{2} \times \frac{3}{x^2-1} = \frac{x+1}{2} \times \frac{3}{(x+1)(x-1)}$ — **Factorise** $(x^2 - 1)$.

$= \frac{{}^1\cancel{x+1}}{2} \times \frac{3}{{}_1\cancel{(x+1)}(x-1)}$ — Cancel any common factors and multiply numerators and denominators.

$= \frac{3}{2(x-1)}$

- To divide two fractions, multiply the first fraction by the **reciprocal** of the second fraction.

Example 2 SKILLS PROBLEM-SOLVING

Simplify:

a $\frac{a}{b} \div \frac{a}{c}$ b $\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16}$

a $\frac{a}{b} \div \frac{a}{c} = \frac{{}^1\cancel{a}}{b} \times \frac{c}{\cancel{a}_1}$ — Multiply the first fraction by the reciprocal of the second fraction. Cancel the common factor a.

$= \frac{1 \times c}{b \times 1}$ — Multiply numerators and denominators.

$= \frac{c}{b}$

b $\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16}$

$= \frac{x+2}{x+4} \times \frac{x^2-16}{3x+6}$ — Multiply the first fraction by the reciprocal of the second fraction.

$= \frac{x+2}{x+4} \times \frac{(x+4)(x-4)}{3(x+2)}$ — Factorise as much as possible.

$= \frac{\cancel{x+2}^1}{\cancel{x+4}^1} \times \frac{\cancel{(x+4)}^1(x-4)}{3\cancel{(x+2)}^1}$ — Cancel any common factors and multiply numerators and denominators.

$= \frac{x-4}{3}$

Exercise 1A SKILLS PROBLEM-SOLVING

E/P **1** Show that $\dfrac{x^2-64}{x^2-36} \div \dfrac{64-x^2}{x^2-36} = -1$ **(4 marks)**

E/P **2** Show that $\dfrac{2x^2-11x-40}{x^2-4x-32} \times \dfrac{x^2+8x+16}{6x^2-3x-45} \div \dfrac{8x^2+20x-48}{10x^2-45x+45} = \dfrac{a}{b}$ and find the values of the **constants** a and b, where a and b are integers. **(4 marks)**

E/P **3** Simplify fully $\dfrac{x^2+2x-24}{2x^2+10x} \times \dfrac{x^2-3x}{x^2+3x-18}$ **(3 marks)**

E/P **4** $f(x) = \dfrac{2x^2-3x-2}{6x-8} \div \dfrac{x-2}{3x^2+14x-24}$

a Show that $f(x) = \dfrac{2x^2+13x+6}{2}$ **(4 marks)**

b Hence differentiate $f(x)$ and find $f'(4)$. **(3 marks)**

Hint Differentiate each term separately. ← **Pure 1 Section 8.5**

- **To add or subtract two fractions, find a common denominator.**

Example 3

Simplify the following:

a $\dfrac{1}{3}+\dfrac{3}{4}$ **b** $\dfrac{a}{2x}+\dfrac{b}{3x}$ **c** $\dfrac{2}{x+3}-\dfrac{1}{x+1}$ **d** $\dfrac{3}{x+1}-\dfrac{4x}{x^2-1}$

a $\dfrac{1}{3}+\dfrac{3}{4}$

$\times\frac{4}{4}$ $\times\frac{3}{3}$

$= \dfrac{4}{12}+\dfrac{9}{12}$

$= \dfrac{13}{12}$

The lowest **common multiple** of 3 and 4 is 12

b $\dfrac{a}{2x}+\dfrac{b}{3x}$

The lowest common multiple of $2x$ and $3x$ is $6x$

$= \dfrac{3a}{6x}+\dfrac{2b}{6x}$

Multiply the first fraction by $\frac{3}{3}$ and the second fraction by $\frac{2}{2}$

$= \dfrac{3a+2b}{6x}$

c $\frac{2}{x+3} - \frac{1}{x+1}$

$= \frac{2(x+1)}{(x+3)(x+1)} - \frac{1(x+3)}{(x+3)(x+1)}$ — The lowest common multiple is $(x + 3)(x + 1)$, so change both fractions so that the denominators are $(x + 3)(x + 1)$

$= \frac{2(x+1) - 1(x+3)}{(x+3)(x+1)}$ — Subtract the numerators.

$= \frac{2x + 2 - 1x - 3}{(x+3)(x+1)}$ — **Expand** the brackets.

$= \frac{x-1}{(x+3)(x+1)}$ — Simplify the numerator.

d $\frac{3}{x+1} - \frac{4x}{x^2-1}$

$= \frac{3}{x+1} - \frac{4x}{(x+1)(x-1)}$ — Factorise $x^2 - 1$ to $(x + 1)(x - 1)$

$= \frac{3(x-1)}{(x+1)(x-1)} - \frac{4x}{(x+1)(x-1)}$ — The LCM of $(x + 1)$ and $(x + 1)(x - 1)$ is $(x + 1)(x - 1)$

$= \frac{3(x-1) - 4x}{(x+1)(x-1)}$

$= \frac{-x-3}{(x+1)(x-1)}$ — Simplify the numerator: $3x - 3 - 4x = -x - 3$

Exercise 1B

SKILLS INTERPRETATION

1 Write as a single fraction:

a $\frac{1}{3} + \frac{1}{4}$ **b** $\frac{3}{4} - \frac{2}{5}$ **c** $\frac{1}{p} + \frac{1}{q}$ **d** $\frac{3}{4x} + \frac{1}{8x}$ **e** $\frac{3}{x^2} - \frac{1}{x}$ **f** $\frac{a}{5b} - \frac{3}{2b}$

2 Write as a single fraction:

a $\frac{3}{x} - \frac{2}{x+1}$ **b** $\frac{2}{x-1} - \frac{3}{x+2}$ **c** $\frac{4}{2x+1} + \frac{2}{x-1}$

d $\frac{1}{3}(x+2) - \frac{1}{2}(x+3)$ **e** $\frac{3x}{(x+4)^2} - \frac{1}{x+4}$ **f** $\frac{5}{2(x+3)} + \frac{4}{3(x-1)}$

3 Write as a single fraction:

a $\frac{2}{x^2+2x+1} + \frac{1}{x+1}$ **b** $\frac{7}{x^2-4} + \frac{3}{x+2}$ **c** $\frac{2}{x^2+6x+9} - \frac{3}{x^2+4x+3}$

d $\frac{2}{y^2-x^2} + \frac{3}{y-x}$ **e** $\frac{3}{x^2+3x+2} - \frac{1}{x^2+4x+4}$ **f** $\frac{x+2}{x^2-x-12} - \frac{x+1}{x^2+5x+6}$

E **4** Express $\frac{6x+1}{x^2+2x-15} - \frac{4}{x-3}$ as a single fraction in its simplest form. **(4 marks)**

5 Express each of the following as a fraction in its simplest form.

a $\dfrac{3}{x} + \dfrac{2}{x+1} + \dfrac{1}{x+2}$ **b** $\dfrac{4}{3x} - \dfrac{2}{x-2} + \dfrac{1}{2x+1}$ **c** $\dfrac{3}{x-1} + \dfrac{2}{x+1} + \dfrac{4}{x-3}$

E **6** Express $\dfrac{4(2x-1)}{36x^2-1} + \dfrac{7}{6x-1}$ as a single fraction in its simplest form. **(4 marks)**

E/P **7** $g(x) = x + \dfrac{6}{x+2} + \dfrac{36}{x^2-2x-8}, x \in \mathbb{R}, x \neq -2, x \neq 4$

a Show that $g(x) = \dfrac{x^3-2x^2-2x+12}{(x+2)(x-4)}$ **(4 marks)**

b Using **algebraic long division**, or otherwise, further show that $g(x) = \dfrac{x^2-4x+6}{x-4}$ **(4 marks)**

1.2 Improper fractions

- An **improper algebraic fraction** is one whose numerator has degree greater than or equal to the denominator. An improper fraction must be converted to a mixed fraction before you can express it in **partial fractions**.

$\dfrac{x^2+5x+8}{x-2}$ and $\dfrac{x^3+5x-9}{x^3-4x^2+7x-3}$ are both improper fractions.

The degree of the numerator is greater than the degree of the denominator.

The degrees of the numerator and denominator are equal.

Notation The **degree** of a **polynomial** is the largest exponent in the **expression**. For example, $x^3 + 5x - 9$ has degree 3.

- To convert an improper fraction into a mixed fraction, you can use either:
  - algebraic long division
  - the relationship $F(x) = Q(x) \times \text{divisor} + \text{remainder}$

Watch out The divisor and the remainder can be numbers or functions of x.

Method 1

Use algebraic long division to show that:

$$\frac{x^2+5x+8}{x-2} \equiv x + 7 + \frac{22}{x-2}$$

F(x): x^2+5x+8; Q(x): $x + 7$; remainder: 22; divisor: $x - 2$

Method 2

Multiply by $(x - 2)$ and compare coefficients to show that:

$$x^2+5x+8 \equiv (x+7)(x-2) + 22$$

F(x): x^2+5x+8; Q(x): $x + 7$; remainder: 22; divisor: $x - 2$

Example 4

Given that $\frac{x^3 + x^2 - 7}{x - 3} \equiv Ax^2 + Bx + C + \frac{D}{x - 3}$, find the values of A, B, C and D.

Using algebraic long division:

$$\begin{array}{r} x^2 + 4x + 12 \\ x - 3 \overline{) x^3 + x^2 + 0x - 7} \\ \underline{x^3 - 3x^2} \\ 4x^2 + 0x \\ \underline{4x^2 - 12x} \\ 12x - 7 \\ \underline{12x - 36} \\ 29 \end{array}$$

So $\frac{x^3 + x^2 - 7}{x - 3} = x^2 + 4x + 12$ with a remainder of 29.

$\frac{x^3 + x^2 - 7}{x - 3} = x^2 + 4x + 12 + \frac{29}{x - 3}$

So $A = 1$, $B = 4$, $C = 12$ and $D = 29$

Problem-solving

Solving this problem using algebraic long division will give you an answer in the form asked for in the question.

The divisor is $(x - 3)$ so you need to write the remainder as a fraction with denominator $(x - 3)$.

It's always a good idea to list the value of each unknown asked for in the question.

Example 5

SKILLS ANALYSIS

Given that $x^3 + x^2 - 7 \equiv (Ax^2 + Bx + C)(x - 3) + D$, find the values of A, B, C and D.

Let $x = 3$:

$27 + 9 - 7 = (9A + 3B + C) \times 0 + D$

$D = 29$

Let $x = 0$:

$0 + 0 - 7 = (A \times 0 + B \times 0 + C) \times (0 - 3) + D$

$-7 = -3C + D$

$-7 = -3C + 29$

$3C = 36$

$C = 12$

Compare the coefficients of x^3 and x^2

Compare coefficients in x^3: $1 = A$

Compare coefficients in x^2: $1 = -3A + B$

$1 = -3 + B$

Therefore $A = 1$, $B = 4$, $C = 12$ and $D = 29$ and we can write

$x^3 + x^2 - 7 \equiv (x^2 + 4x + 12)(x - 3) + 29$

This can also be written as:

$\frac{x^3 + x^2 - 7}{x - 3} \equiv x^2 + 4x + 12 + \frac{29}{x - 3}$

Problem-solving

The **identity** is given in the form $F(x) \equiv Q(x) \times$ divisor + remainder, so solve by equating coefficients.

Set $x = 3$ to find the value of D.

Set $x = 0$ and use your value of D to find the value of C.

You can find the remaining values by equating coefficients of x^3 and x^2.
Remember there are two x^2 terms when you expand the brackets on the **RHS**:
x^3 terms: **LHS** $= x^3$, RHS $= Ax^3$
x^2 terms: LHS $= x^2$, RHS $= (-3A + B)x^2$

Example 6

$$f(x) = \frac{x^4 + x^3 + x - 10}{x^2 + 2x - 3}$$

Show that f(x) can be written as $Ax^2 + Bx + C + \dfrac{Dx + E}{x^2 + 2x - 3}$ and find the values of A, B, C, D and E.

Using algebraic long division:

$$\begin{array}{r|l} & x^2 - x + 5 \\ \hline x^2 + 2x - 3 & x^4 + x^3 + 0x^2 + x - 10 \\ & \underline{x^4 + 2x^3 - 3x^2} \\ & \quad -x^3 + 3x^2 + x \\ & \quad \underline{-x^3 - 2x^2 + 3x} \\ & \qquad 5x^2 - 2x - 10 \\ & \qquad \underline{5x^2 + 10x - 15} \\ & \qquad\quad -12x + 5 \end{array}$$

$$\frac{x^4 + x^3 + x - 10}{x^2 + 2x - 3} \equiv x^2 - x + 5 + \frac{-12x + 5}{x^2 + 2x - 3}$$

So $A = 1$, $B = -1$, $C = 5$, $D = -12$ and $E = 5$

Write the remainder as a fraction over the whole divisor.

Watch out When you are dividing by a quadratic expression, the remainder can be a constant or a **linear** expression. The degree of $(-12x + 5)$ is less than that of $(x^2 + 2x - 3)$ so stop your division here. The remainder is $-12x + 5$.

Exercise 1C

SKILLS ANALYSIS

E **1** $\dfrac{x^3 + 2x^2 + 3x - 4}{x + 1} \equiv Ax^2 + Bx + C + \dfrac{D}{x + 1}$

Find the values of the constants A, B, C and D. **(4 marks)**

E **2** Given that $\dfrac{2x^3 + 3x^2 - 4x + 5}{x + 3} \equiv ax^2 + bx + c + \dfrac{d}{x + 3}$, find the values of a, b, c and d. **(4 marks)**

E **3** $f(x) = \dfrac{x^3 - 8}{x - 2}$

Show that f(x) can be written in the form $px^2 + qx + r$ and find the values of p, q and r. **(4 marks)**

E **4** Given that $\dfrac{2x^2 + 4x + 5}{x^2 - 1} \equiv m + \dfrac{nx + p}{x^2 - 1}$, find the values of m, n and p. **(4 marks)**

E **5** Find the values of the constants A, B, C and D in the following identity:

$8x^3 + 2x^2 + 5 \equiv (Ax + B)(2x^2 + 2) + Cx + D$ **(4 marks)**

E **6** $\dfrac{4x^3 - 5x^2 + 3x - 14}{x^2 + 2x - 1} \equiv Ax + B + \dfrac{Cx + D}{x^2 + 2x - 1}$

Find the values of the constants A, B, C and D. **(4 marks)**

E 7 $g(x) = \frac{x^4 + 3x^2 - 4}{x^2 + 1}$. Show that g(x) can be written in the form $px^2 + qx + r + \frac{sx + t}{x^2 + 1}$ and find the values of p, q, r, s and t. **(4 marks)**

E 8 Given that $\frac{2x^4 + 3x^3 - 2x^2 + 4x - 6}{x^2 + x - 2} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 + x - 2}$, find the values of a, b, c, d and e. **(5 marks)**

E 9 Find the values of the constants A, B, C, D and E in the following identity:

$3x^4 - 4x^3 - 8x^2 + 16x - 2 \equiv (Ax^2 + Bx + C)(x^2 - 3) + Dx + E$ **(5 marks)**

E/P 10 **a** Fully factorise the expression $x^4 - 1$ **(2 marks)**

b Hence, or otherwise, write the **algebraic fraction** $\frac{x^4 - 1}{x + 1}$ in the form $(ax + b)(cx^2 + dx + e)$ and find the values of a, b, c, d and e. **(4 marks)**

Chapter review 1

1 Simplify these fractions as far as possible:

a $\frac{3x^4 - 21x}{3x}$ **b** $\frac{x^2 - 2x - 24}{x^2 - 7x + 6}$ **c** $\frac{2x^2 + 7x - 4}{2x^2 + 9x + 4}$

2 Divide $3x^3 + 12x^2 + 5x + 20$ by $(x + 4)$

3 Simplify $\frac{2x^3 + 3x + 5}{x + 1}$

4 Simplify:

a $\frac{x - 4}{6} \times \frac{2x + 8}{x^2 - 16}$ **b** $\frac{x^2 - 3x - 10}{3x^2 - 21} \times \frac{6x^2 + 24}{x^2 + 6x + 8}$ **c** $\frac{4x^2 + 12x + 9}{x^2 + 6x} \div \frac{4x^2 - 9}{2x^2 + 9x - 18}$

E/P 5 **a** Simplify fully $\frac{4x^2 - 8x}{x^2 - 3x - 4} \times \frac{x^2 + 6x + 5}{2x^2 + 10x}$ **(3 marks)**

b Given that $\ln[(4x^2 - 8x)(x^2 + 6x + 5)] = 6 + \ln[(x^2 - 3x - 4)(2x^2 + 10x)]$ find x in terms of e. **(4 marks)**

E/P 6 $g(x) = \frac{4x^3 - 9x^2 - 9x}{32x + 24} \div \frac{x^2 - 3x}{6x^2 - 13x - 5}$

a Show that g(x) can be written in the form $ax^2 + bx + c$, where a, b and c are constants to be found. **(4 marks)**

b Hence differentiate g(x) and find $g'(-2)$. **(3 marks)**

E **7** Express $\dfrac{6x+1}{x-5} + \dfrac{5x+3}{x^2-3x-10}$ as a single fraction in its simplest form. **(4 marks)**

E **8** $f(x) = x + \dfrac{3}{x-1} - \dfrac{12}{x^2+2x-3}, x \in \mathbb{R}, x > 1$

Show that $f(x) = \dfrac{x^2+3x+3}{x+3}$ **(4 marks)**

9 Find the values of the constants A, B, C and D in the following identity:

$x^3 - 6x^2 + 11x + 2 \equiv (x-2)(Ax^2 + Bx + C) + D$ **(5 marks)**

E **10** Show that $\dfrac{4x^3 - 6x^2 + 8x - 5}{2x+1}$ can be put in the form $Ax^2 + Bx + C + \dfrac{D}{2x+1}$

Find the values of the constants A, B, C and D. **(5 marks)**

E **11** Show that $\dfrac{x^4+2}{x^2-1} \equiv Ax^2 + Bx + C + \dfrac{D}{x^2-1}$

where A, B, C and D are constants to be found. **(5 marks)**

Challenge

SKILLS CREATIVITY

1 Given that $\dfrac{6x^3 - 7x^2 + 3}{3x^2 + x - 10} \equiv Ax + B + \dfrac{C}{3x-5} + \dfrac{D}{x+2}$

find the values of the constants A, B, C and D.

2 Prove that if $f(x) = ax^3 + bx^2 + cx + d$ and $f(p) = 0$, then $(x - p)$ is a **factor** of $f(x)$.

3 Given that $f(x) = 2x^3 + 9x^2 + 10x + 3$:

a show that -3 is a root of $f(x)$

b express $\dfrac{10}{f(x)}$ as partial fractions.

Summary of key points

1 To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators.

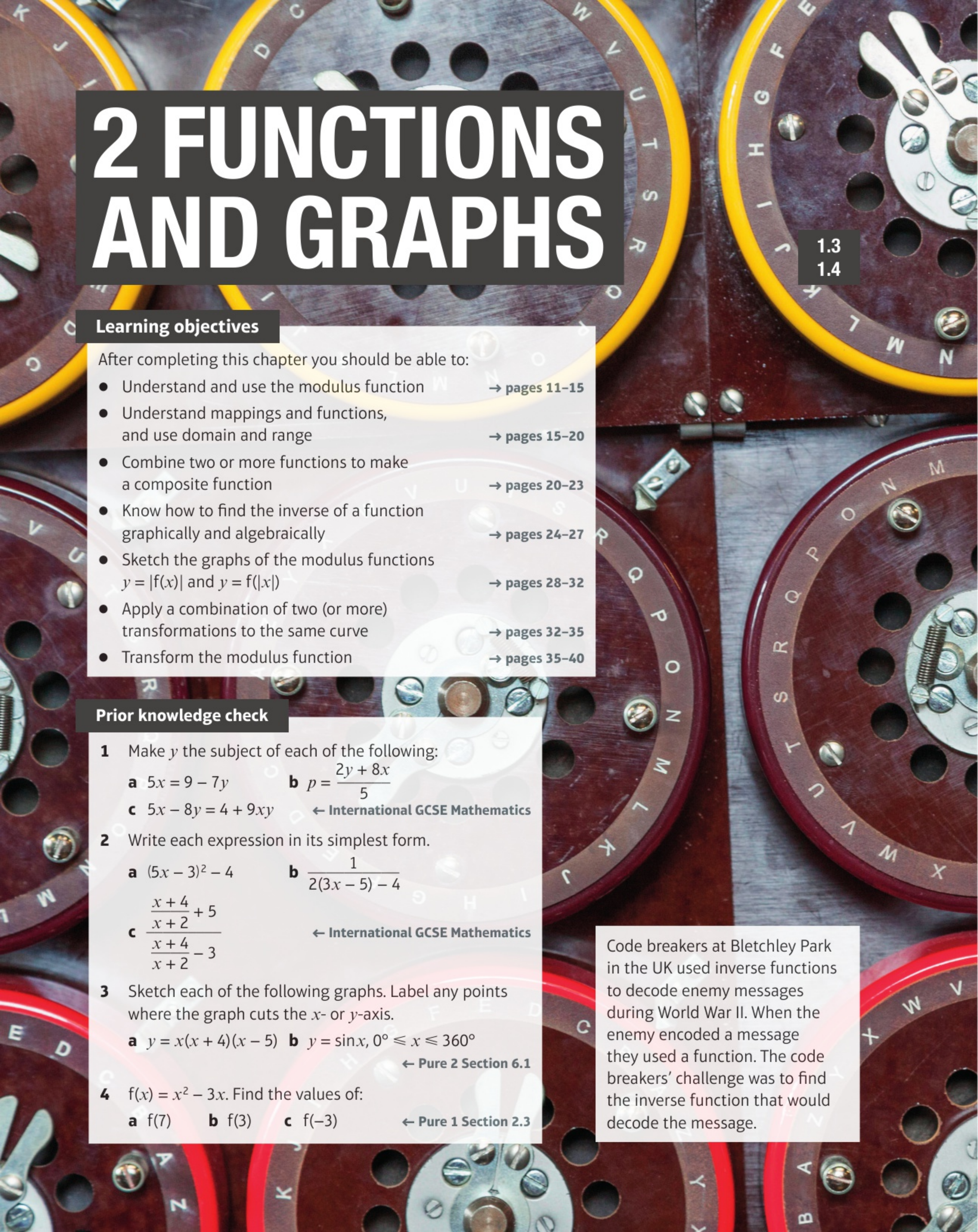
2 To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

3 To add or subtract two fractions, find a common denominator.

4 An improper algebraic fraction is one whose numerator has degree greater than or equal to the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions.

5 To convert an improper fraction into a mixed fraction, you can use either:

- algebraic long division
- the relationship $F(x) = Q(x) \times \text{divisor} + \text{remainder}$



2 FUNCTIONS AND GRAPHS

1.3
1.4

Learning objectives

After completing this chapter you should be able to:

Prior knowledge check

1 Make y the subject of each of the following:

a $5x = 9 - 7y$ **b** $p = \dfrac{2y + 8x}{5}$

c $5x - 8y = 4 + 9xy$ ← International GCSE Mathematics

2 Write each expression in its simplest form.

a $(5x - 3)^2 - 4$ **b** $\dfrac{1}{2(3x - 5) - 4}$

c $\dfrac{\dfrac{x+4}{x+2} + 5}{\dfrac{x+4}{x+2} - 3}$ ← International GCSE Mathematics

3 Sketch each of the following graphs. Label any points where the graph cuts the x- or y-axis.

a $y = x(x + 4)(x - 5)$ **b** $y = \sin x$, $0° \leqslant x \leqslant 360°$

← Pure 2 Section 6.1

4 $f(x) = x^2 - 3x$. Find the values of:

a f(7) **b** f(3) **c** f(–3) ← Pure 1 Section 2.3

Code breakers at Bletchley Park in the UK used inverse functions to decode enemy messages during World War II. When the enemy encoded a message they used a function. The code breakers' challenge was to find the inverse function that would decode the message.

2.1 The modulus function

The **modulus** of a number a, written as $|a|$, is its **non-negative** numerical value.

So, for example, $|5| = 5$ and also $|-5| = 5$.

- A modulus function is, in general, a function of the type $y = |f(x)|$
  - When $f(x) \geqslant 0$, $|f(x)| = f(x)$
  - When $f(x) < 0$, $|f(x)| = -f(x)$

Notation The modulus function is also known as the **absolute value** function. On a calculator, the button is often labelled 'Abs'.

Example 1

Write down the values of:

a $|-2|$ **b** $|6.5|$ **c** $\left|\frac{1}{3} - \frac{4}{5}\right|$

a $|-2| = 2$ — The positive numerical value of −2 is 2.

b $|6.5| = 6.5$ — 6.5 is a positive number.

c $\left|\frac{1}{3} - \frac{4}{5}\right| = \left|\frac{5}{15} - \frac{12}{15}\right| = \left|-\frac{7}{15}\right| = \frac{7}{15}$ — Work out the value inside the modulus.

Example 2

$f(x) = |2x - 3| + 1$

Write down the values of:

a $f(5)$ **b** $f(-2)$ **c** $f(1)$

a $f(5) = |2 \times 5 - 3| + 1$
$= |7| + 1 = 7 + 1 = 8$

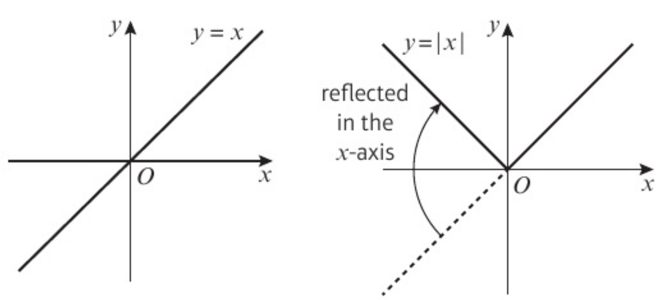
b $f(-2) = |2(-2) - 3| + 1$
$= |-7| + 1 = 7 + 1 = 8$

c $f(1) = |2 \times 1 - 3| + 1$
$= |-1| + 1 = 1 + 1 = 2$

Watch out The modulus function acts like a pair of brackets. Work out the value inside the modulus function first.

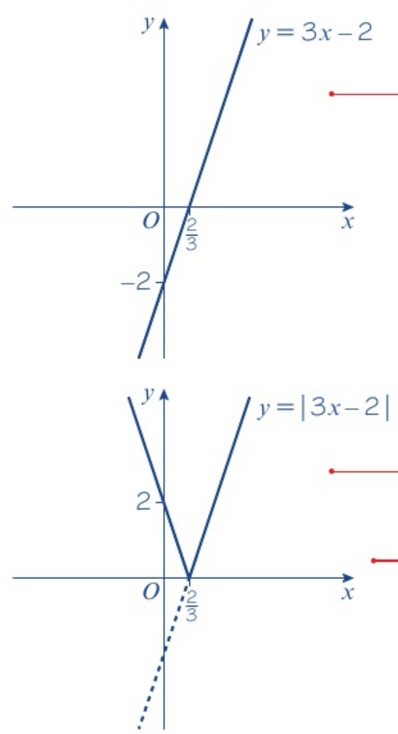
Online Use your calculator to work out values of modulus functions.

- To **sketch** the graph of $y = |ax + b|$, sketch $y = ax + b$ then reflect the section of the graph below the x-axis in the x-axis.



Example 3 SKILLS INTERPRETATION

Sketch the graph of $y = |3x - 2|$



Online Explore graphs of f(x) and |f(x)| using technology.

Step 1
Sketch the graph of $y = 3x - 2$
(Ignore the modulus for the moment.)

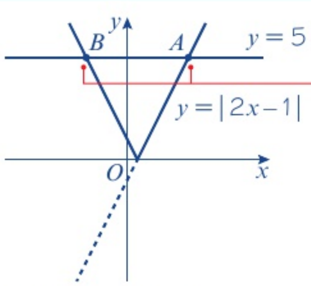
Step 2
For the part of the line below the x-axis (the negative values of y), reflect in the x-axis. For example, this will change the y-value –2 into the y-value 2.

You could check your answer using a table of values:

x	–1	0	1	2
$y = \|3x - 2\|$	5	2	1	4

Example 4 SKILLS INTERPRETATION

Solve the equation $|2x - 1| = 5$



Start by sketching the graphs of $y = |2x - 1|$ and $y = 5$.

The graphs intersect at two **points**, A and B, so there will be two solutions to the equation.

At A: $2x - 1 = 5$
$2x = 6$
$x = 3$

A is the point of **intersection** on the original part of the graph.

At B: $-(2x - 1) = 5$
$-2x + 1 = 5$
$2x = -4$
$x = -2$

B is the point of intersection on the reflected part of the graph.

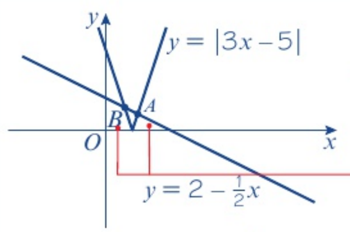
The solutions are $x = 3$ and $x = -2$

Notation The function inside the modulus is called the **argument** of the modulus. You can solve modulus equations algebraically by considering the positive argument and the negative argument separately.

Example 5

Solve the equation $|3x - 5| = 2 - \frac{1}{2}x$

Online Explore intersections of straight lines and modulus graphs using technology.



Start by sketching the graphs of $y = |3x - 5|$ and $y = 2 - \frac{1}{2}x$

The sketch shows there are two solutions, at A and B, the points of intersection.

At A: $3x - 5 = 2 - \frac{1}{2}x$

$\frac{7}{2}x = 7$

$x = 2$

This is the solution on the original part of the graph.

At B: $-(3x - 5) = 2 - \frac{1}{2}x$

When $f(x) < 0$, $|f(x)| = -f(x)$, so $-(3x - 5) = 2 - \frac{1}{2}x$ gives you the second solution.

$-3x + 5 = 2 - \frac{1}{2}x$

$-\frac{5}{2}x = -3$

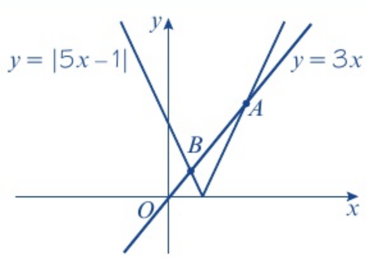
$x = \frac{6}{5}$

This is the solution on the reflected part of the graph.

The solutions are $x = 2$ and $x = \frac{6}{5}$

Example 6

Solve the inequality $|5x - 1| > 3x$



First draw a sketch of $y = |5x - 1|$ and $y = 3x$

Solve the equation $|5x - 1| = 3x$ to find the x-coordinates of the points of intersection, A and B.

At A: $5x - 1 = 3x$

$2x = 1$

$x = \frac{1}{2}$

This is the intersection on the original part of the graph.

At B: $-(5x - 1) = 3x$

$-5x + 1 = 3x$

$8x = 1$

$x = \frac{1}{8}$

Consider the negative argument to find the point of intersection on the reflected part of the graph.

The points of intersection are

$x = \frac{1}{2}$ and $x = \frac{1}{8}$

So the solution to $|5x - 1| > 3x$ is

$x < \frac{1}{8}$ or $x > \frac{1}{2}$

Problem-solving

Look at the sketch to work out which values of x satisfy the inequality. $y = |5x - 1|$ is above $y = 3x$ when $x > \frac{1}{2}$ or $x < \frac{1}{8}$. You could write the solution in set notation as $\left\{x : x > \frac{1}{2}\right\} \cup \left\{x : x < \frac{1}{8}\right\}$

Exercise 2A

SKILLS INTERPRETATION

1 Write down the values of:

a $\left|\frac{3}{4}\right|$ b $|-0.28|$ c $|3 - 11|$ d $\left|\frac{5}{7} - \frac{3}{8}\right|$ e $|20 - 6 \times 4|$ f $|4^2 \times 2 - 3 \times 7|$

2 $f(x) = |7 - 5x| + 3$. Write down the values of:

a $f(1)$ b $f(10)$ c $f(-6)$

3 $g(x) = |x^2 - 8x|$. Write down the values of:

a $g(4)$ b $g(-5)$ c $g(8)$

4 Sketch the graph of each of the following. In each case, write down the **coordinates** of any points at which the graph meets the **coordinate axes**.

a $y = |x - 1|$ b $y = |2x + 3|$ c $y = |4x - 7|$ d $y = \left|\frac{1}{2}x - 5\right|$

e $y = |7 - x|$ f $y = |6 - 4x|$

g $y = -|x|$ h $y = -|3x - 1|$

Hint $y = -|x|$ is a **reflection** of $y = |x|$ in the x-axis. ← Pure 1 Section 4.5

5 $g(x) = \left|4 - \frac{3}{2}x\right|$ and $h(x) = 5$

a On the same axes, sketch the graphs of $y = g(x)$ and $y = h(x)$.

b Hence solve the equation $\left|4 - \frac{3}{2}x\right| = 5$

6 Solve:

a $|3x - 1| = 5$ b $\left|\frac{x - 5}{2}\right| = 1$ c $|4x + 3| = -2$

d $|7x - 3| = 4$ e $\left|\frac{4 - 5x}{3}\right| = 2$ f $\left|\frac{x}{6} - 1\right| = 3$

7 a On the same diagram, sketch the graphs $y = -2x$ and $y = \left|\frac{1}{2}x - 2\right|$

b Solve the equation $-2x = \left|\frac{1}{2}x - 2\right|$

E 8 Solve $|3x - 5| = 11 - x$ **(4 marks)**

9 a On the same set of axes, sketch $y = |6 - x|$ and $y = \frac{1}{2}x - 5$

b State with a reason whether there are any solutions to the equation $|6 - x| = \frac{1}{2}x - 5$

(P) **10** A student attempts to solve the equation $|3x + 4| = x$. The student writes the following working:

$3x + 4 = x$		$-(3x + 4) = x$
$4 = -2x$	or	$-3x - 4 = x$
$x = -2$		$-4 = 4x$
		$x = -1$
Solutions are $x = -2$ and $x = -1$.		

Explain the error made by the student.

11 **a** On the same diagram, sketch the graphs of $y = -|3x + 4|$ and $y = 2x - 9$

b Solve the inequality $-|3x + 4| < 2x - 9$

(E) **12** Solve the inequality $|2x + 9| < 14 - x$ **(4 marks)**

(E/P) **13** The equation $|6 - x| = \frac{1}{2}x + k$ has exactly one solution.

a Find the value of k. **(2 marks)**

b State the solution to the equation. **(2 marks)**

Problem-solving

The solution must be at the **vertex** of the graph of the modulus function.

Challenge

SKILLS INTERPRETATION

$f(x) = |x^2 + 9x + 8|$ and $g(x) = 1 - x$

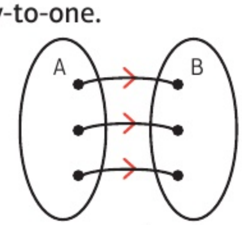
a On the same axes, sketch graphs of $y = f(x)$ and $y = g(x)$

b Use your sketch to find all the solutions to $|x^2 + 9x + 8| = 1 - x$

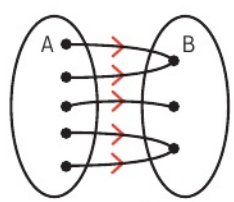
2.2 Functions and mappings

A **mapping** transforms one set of numbers into a different set of numbers. The mapping can be described in words or through an algebraic equation. It can also be represented by a graph.

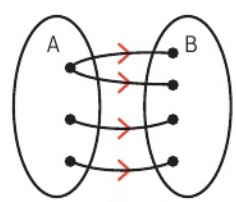
- **A mapping is a function if every input has a distinct output. Functions can either be one-to-one or many-to-one.**



one-to-one function

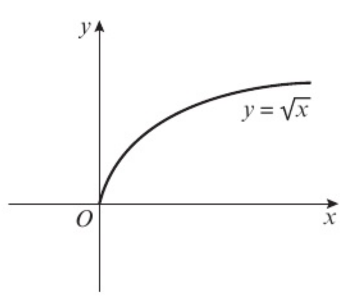


many-to-one function



not a function

Many mappings can be made into functions by changing the domain. Consider $y = \sqrt{x}$:



Notation The **domain** is the set of all possible inputs for a mapping. The **range** is the set of all possible outputs for the mapping.

If the domain were all of the **real** numbers, ℝ, then $y = \sqrt{x}$ would not be a function because values of x less than 0 would not be mapped anywhere.

However, if we restrict the domain to $x \geqslant 0$, then every element in the domain is mapped to exactly one element in the range.

We can write this function together with its domain as $f(x) = \sqrt{x}, x \in \mathbb{R}, x \geqslant 0$.

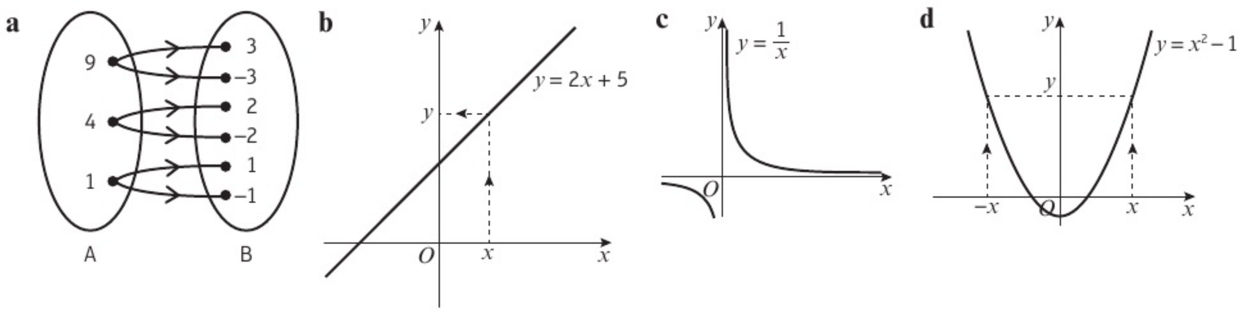
Notation You can also write this function as: $f: x \mapsto \sqrt{x}, x \in \mathbb{R}, x \geqslant 0$

Example 7 SKILLS ANALYSIS

For each of the following mappings:

i state whether the mapping is one-to-one, many-to-one or one-to-many

ii state whether or not the mapping is a function.



a **i** Every element in set A gets mapped to two elements in set B, so the mapping is **one-to-many**.

ii The mapping is not a function.

b **i** Every value of x gets mapped to one value of y, so the mapping is **one-to-one**.

ii The mapping is a function.

c **i** The mapping is **one-to-one**.

ii $x = 0$ does not get mapped to a value of y so the mapping is not a function.

d **i** On the graph, you can see that x and $-x$ both get mapped to the same value of y. Therefore, this is a **many-to-one** mapping.

ii The mapping is a function.

You couldn't write down a single value for f(9).

For a mapping to be a function, every input in the domain must map onto exactly one output.

The mapping in part **c** could be a function if $x = 0$ were omitted from the domain. You could write this as a function as $f(x) = \frac{1}{x}, x \in \mathbb{R}, x \neq 0$.

Watch out Normally the domain is all the reals ($x \in \mathbb{R}$), unless otherwise stated.

Example 8

Find the range of each of the following functions:

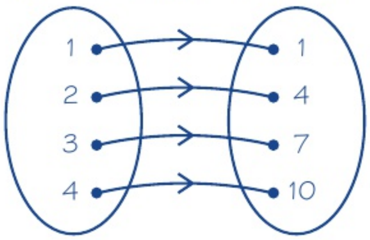
a $f(x) = 3x - 2$, domain $\{x = 1, 2, 3, 4\}$

b $g(x) = x^2$, domain $\{x \in \mathbb{R}, -5 \leqslant x \leqslant 5\}$

c $h(x) = \frac{1}{x}$, domain $\{x \in \mathbb{R}, 0 < x \leqslant 3\}$

State whether the functions are one-to-one or many-to-one.

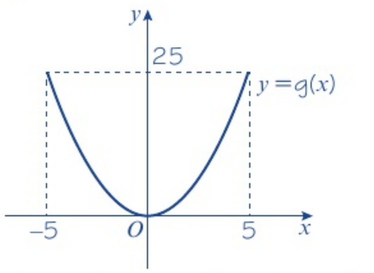
a $f(x) = 3x - 2, \{x = 1, 2, 3, 4\}$



Range of f(x) is {1, 4, 7, 10}.
f(x) is one-to-one.

> The domain contains a finite (non-infinite) number of elements, so you can draw a mapping diagram showing the whole function.

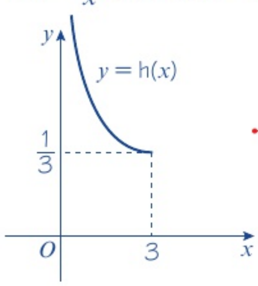
b $g(x) = x^2, \{-5 \leqslant x \leqslant 5\}$



Range of g(x) is $0 \leqslant g(x) \leqslant 25$
g(x) is many-to-one.

> The domain is the set of all the x-values that correspond to points on the graph. The range is the set of y-values that correspond to points on the graph.

c $h(x) = \frac{1}{x}, \{x \in \mathbb{R}, 0 < x \leqslant 3\}$



Range of h(x) is $h(x) \geqslant \frac{1}{3}$
h(x) is one-to-one.

> Calculate $h(3) = \frac{1}{3}$ to find the minimum value in the range of h. As x approaches 0, $\frac{1}{x}$ approaches ∞, so there is no maximum value in the range of h.

Example 9

The function f(x) is defined by

$$f: x \mapsto \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geqslant 1 \end{cases}$$

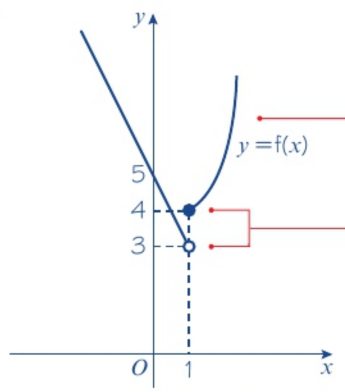
a Sketch $y = f(x)$, and state the range of f(x).

b Solve $f(x) = 19$.

> **Notation** This is an example of a **piecewise-defined function**, that is, a function defined by more than one equation. Here one part is linear (for $x < 1$) and one quadratic (for $x \geqslant 1$).

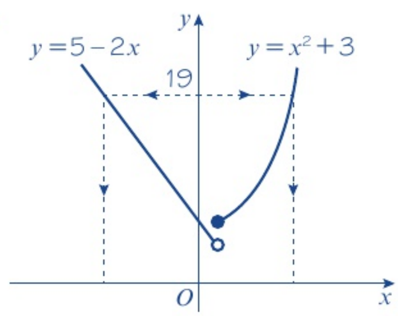
> **Online** Explore graphs of functions on a given domain using technology.

a



The range is the set of values that y takes and therefore $f(x) > 3$

b



The positive solution is where

$x^2 + 3 = 19$

$x^2 = 16$

$x = \pm 4$

$x = 4$

The negative solution is where

$5 - 2x = 19$

$-2x = 14$

$x = -7$

The solutions are $x = 4$ and $x = -7$

Watch out Although the graph jumps at $x = 1$, the function is still defined for all real values of x:

$f(0.9) = 5 - 2(0.9) = 3.2$

$f(1) = (1)^2 + 3 = 4$

Sketch the graph of $y = 5 - 2x$ for $x < 1$, and the graph of $y = x^2 + 3$ for $x \geqslant 1$

f(1) lies on the quadratic curve, so use a solid dot on the quadratic curve, and an open dot on the line.

Note that $f(x) \neq 3$ at $x = 1$

so $f(x) > 3$

not $f(x) \geqslant 3$

There are two values of x such that $f(x) = 19$

Problem-solving

Use $x^2 + 3 = 19$ to find the solution in the range $x \geqslant 1$ and use $5 - 2x = 19$ to find the solution in the range $x < 1$

Ignore $x = -4$ because the function is only equal to $x^2 + 3$ for $x \geqslant 1$

Exercise 2B SKILLS INTERPRETATION

1 For each of the following functions:

i draw the mapping diagram

ii state if the function is one-to-one or many-to-one

iii find the range of the function.

a $f(x) = 5x - 3$, domain $\{x = 3, 4, 5, 6\}$

b $g(x) = x^2 - 3$, domain $\{x = -3, -2, -1, 0, 1, 2, 3\}$

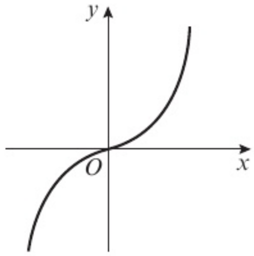
c $h(x) = \dfrac{7}{4 - 3x}$, domain $\{x = -1, 0, 1\}$

2 For each of the following mappings:

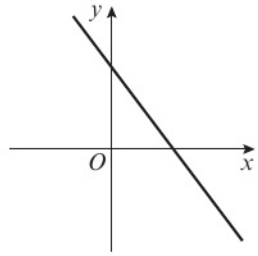
i state whether the mapping is one-to-one, many-to-one or one-to-many

ii state whether or not the mapping could represent a function.

a



b



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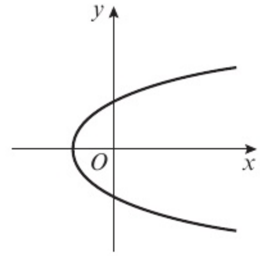

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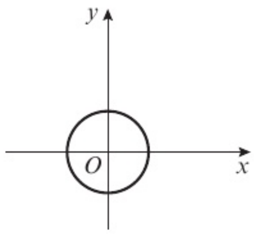

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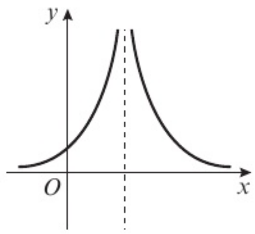

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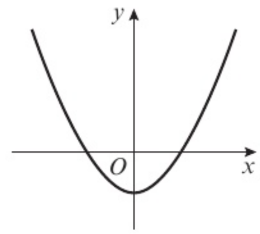


3 Calculate the value(s) of a, b, c and d given that:

a $p(a) = 16$ where $p: x \mapsto 3x - 2, x \in \mathbb{R}$

b $q(b) = 17$ where $q: x \mapsto x^2 - 3, x \in \mathbb{R}$

c $r(c) = 34$ where $r: x \mapsto 2(2^x) + 2, x \in \mathbb{R}$

d $s(d) = 0$ where $s: x \mapsto x^2 + x - 6, x \in \mathbb{R}$

4 For each function below:

i represent the function on a mapping diagram, writing down the elements in the range

ii state whether the function is one-to-one or many-to-one.

a $f(x) = 2x + 1$ for the domain $\{x = 1, 2, 3, 4, 5\}$

b $g: x \mapsto \sqrt{x}$ for the domain $\{x = 1, 4, 9, 16, 25, 36\}$

c $h(x) = x^2$ for the domain $\{x = -2, -1, 0, 1, 2\}$

d $j: x \mapsto \frac{2}{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$

e $k(x) = e^x + 3$ for the domain $\{x = -2, -1, 0, 1, 2\}$

Notation Remember, $\sqrt{x}$ means the positive square root of x.

5 For each function:

i sketch the graph of $y = f(x)$

ii state the range of $f(x)$

iii state whether $f(x)$ is one-to-one or many-to-one.

a $f: x \mapsto 3x + 2$ for the domain $\{x \geqslant 0\}$

b $f(x) = x^2 + 5$ for the domain $\{x \geqslant 2\}$

c $f: x \mapsto 2\sin x$ for the domain $\{0 \leqslant x \leqslant 180\}$

d $f: x \mapsto \sqrt{x+2}$ for the domain $\{x \geqslant -2\}$

e $f(x) = e^x$ for the domain $\{x \geqslant 0\}$

f $f(x) = 7\log x$, for the domain $\{x \in \mathbb{R}, x > 0\}$

6 The following mappings f and g are defined on all the real numbers by

$$f(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x \geqslant 4 \end{cases} \qquad g(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x > 4 \end{cases}$$

a Explain why $f(x)$ is a function and $g(x)$ is not.

b Sketch $y = f(x)$

c Find the values of: **i** $f(3)$ **ii** $f(10)$

d Find the solution of $f(a) = 90$

(P) **7** The function s is defined by

$$s(x) = \begin{cases} x^2 - 6, & x < 0 \\ 10 - x, & x \geqslant 0 \end{cases}$$

a Sketch $y = s(x)$

b Find the value(s) of a such that $s(a) = 43$

c Solve $s(x) = x$

Problem-solving The solutions of $s(x) = x$ are the values in the domain that get mapped to themselves in the range.

(E/P) **8** The function p is defined by

$$p(x) = \begin{cases} e^{-x}, & -5 \leqslant x < 0 \\ x^3 + 4, & 0 \leqslant x \leqslant 4 \end{cases}$$

a Sketch $y = p(x)$ **(3 marks)**

b Find the values of a, to 2 decimal places, such that $p(a) = 50$ **(4 marks)**

(E/P) **9** The function h has domain $-10 \leqslant x \leqslant 6$, and is linear from $(-10, 14)$ to $(-4, 2)$ and from $(-4, 2)$ to $(6, 27)$.

a Sketch $y = h(x)$ **(2 marks)**

b Write down the range of $h(x)$ **(1 mark)**

c Find the values of a, such that $h(a) = 12$ **(4 marks)**

Problem-solving The graph of $y = h(x)$ will consist of two line segments which meet at $(-4, 2)$.

(P) **10** The function g is defined by $g(x) = cx + d$ where c and d are constants to be found. Given $g(3) = 10$ and $g(8) = 12$, find the values of c and d.

(P) **11** The function f is defined by $f(x) = ax^3 + bx - 5$ where a and b are constants to be found. Given that $f(1) = -4$ and $f(2) = 9$, find the values of the constants a and b.

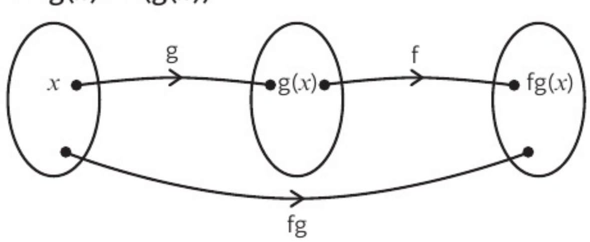
(E/P) **12** The function h is defined by $h(x) = x^2 - 6x + 20$ and has domain $x \geqslant a$. Given that $h(x)$ is a one-to-one function, find the smallest possible value of the constant a. **(6 marks)**

Hint First complete the square for $h(x)$.

2.3 Composite functions

Two or more functions can be combined to make a new function. The new function is called a **composite function**.

- fg(x) means apply g first, then apply f.
- $fg(x) = f(g(x))$



Watch out The order in which the functions are combined is important: fg(x) is not normally the same as gf(x).

Example 10 SKILLS INTERPRETATION

Given $f(x) = x^2$ and $g(x) = x + 1$, find:

a fg(1) **b** gf(3) **c** ff(−2)

a $fg(1) = f(1 + 1)$
$= 2^2$
$= 4$

b $gf(3) = g(3^2)$
$= g(9)$
$= 9 + 1$
$= 10$

c $ff(-2) = f((-2)^2)$
$= f(4)$
$= 4^2$
$= 16$

$g(1) = 1 + 1$

$f(2) = 2^2$

$f(3) = 3^2$

$g(9) = 9 + 1$

$f(-2) = (-2)^2$

$f(4) = 4^2$

Example 11

The functions f and g are defined by $f(x) = 3x + 2$ and $g(x) = x^2 + 4$. Find:

a the function $fg(x)$

b the function $gf(x)$

c the function $f^2(x)$

d the values of b such that $fg(b) = 62$.

Notation $f^2(x)$ is $ff(x)$

a $fg(x) = f(x^2 + 4)$
$= 3(x^2 + 4) + 2$
$= 3x^2 + 14$

b $gf(x) = g(3x + 2)$
$= (3x + 2)^2 + 4$
$= 9x^2 + 12x + 8$

c $f^2(x) = f(3x + 2)$
$= 3(3x + 2) + 2$
$= 9x + 8$

d $fg(x) = 3x^2 + 14$
If $fg(b) = 62$
then $3b^2 + 14 = 62$
$b^2 = 16$
$b = \pm 4$

g acts on x first, mapping it to $x^2 + 4$

f acts on the result.

Simplify answer.

f acts on x first, mapping it to $3x + 2$

g acts on the result.

f maps x to $3x + 2$

f acts on the result.

Set up and solve an equation in b.

Example 12

The functions f and g are defined by:

$$f: x \mapsto |2x - 8|$$

$$g: x \mapsto \frac{x + 1}{2}$$

a Find $fg(3)$. **b** Solve $fg(x) = x$.

a $fg(3) = f\left(\frac{3+1}{2}\right)$ — $g(3) = \left(\frac{3+1}{2}\right)$

$= f(2)$

$= |2 \times 2 - 8|$ — $f(2) = |2 \times 2 - 8|$

$= |-4|$

$= 4$

b First find $fg(x)$:

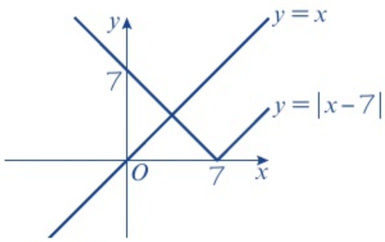
$fg(x) = f\left(\frac{x+1}{2}\right)$ — g acts on x first, mapping it to $\frac{x+1}{2}$

$= \left|2\left(\frac{x+1}{2}\right) - 8\right|$ — f acts on the result.

$= |x - 7|$ — Simplify the answer.

$fg(x) = x$

$|x - 7| = x$



Draw a sketch of $y = |x - 7|$ and $y = x$

The sketch shows there is only one solution to the equation $|x - 7| = x$ and that it occurs on the reflected part of the graph.

$-(x - 7) = x$ — When $f(x) < 0$, $|f(x)| = -f(x)$. The solution is on the reflected part of the graph so use $-(x - 7)$

$-x + 7 = x$

$2x = 7$

$x = 3.5$ — This is the x-coordinate at the point of intersection marked on the graph.

Exercise 2C

SKILLS PROBLEM-SOLVING

1 Given the functions $p(x) = 1 - 3x$, $q(x) = \frac{x}{4}$ and $r(x) = (x - 2)^2$, find:

a $pq(-8)$ **b** $qr(5)$ **c** $rq(6)$ **d** $p^2(-5)$ **e** $pqr(8)$

2 Given the functions $f(x) = 4x + 1$, $g(x) = x^2 - 4$ and $h(x) = \frac{1}{x}$, find expressions for the functions:

a $fg(x)$ **b** $gf(x)$ **c** $gh(x)$ **d** $fh(x)$ **e** $f^2(x)$

(E) **3** The functions f and g are defined by:

$f(x) = 3x - 2,\ x \in \mathbb{R}$

$g(x) = x^2,\ x \in \mathbb{R}$

a Find an expression for $fg(x)$. **(2 marks)**

b Solve $fg(x) = gf(x)$. **(4 marks)**

(E) **4** The functions p and q are defined by:

$p(x) = \frac{1}{x - 2},\ x \in \mathbb{R},\ x \neq 2$

$q(x) = 3x + 4,\ x \in \mathbb{R}$

a Find an expression for $qp(x)$ in the form $\frac{ax + b}{cx + d}$ **(3 marks)**

b Solve $qp(x) = 16$. **(3 marks)**

E **5** The functions f and g are defined by:

f: $x \mapsto |9 - 4x|$

g: $x \mapsto \frac{3x - 2}{2}$

a Find fg(6). **(2 marks)**

b Solve fg(x) = x. **(5 marks)**

P **6** Given $f(x) = \frac{1}{x + 1}$, $x \neq -1$

a Prove that $f^2(x) = \frac{x + 1}{x + 2}$

b Find an expression for $f^3(x)$.

7 The functions s and t are defined by

$s(x) = 2^x, x \in \mathbb{R}$

$t(x) = x + 3, x \in \mathbb{R}$

a Find an expression for st(x).

b Find an expression for ts(x).

E **8** Given $f(x) = e^{5x}$ and $g(x) = 4 \ln x$, find in its simplest form:

a gf(x) **(2 marks)**

b fg(x) **(2 marks)**

E/P **9** The functions p and q are defined by

p: $x \mapsto \ln(x + 3), x \in \mathbb{R}, x > -3$

q: $x \mapsto e^{3x} - 1, x \in \mathbb{R}$

Hint The range of p will be the set of possible inputs for q in the function qp.

a Find qp(x) and state its range. **(3 marks)**

b Find the value of qp(7). **(1 mark)**

c Solve qp(x) = 124 **(3 marks)**

E/P **10** The function t is defined by:

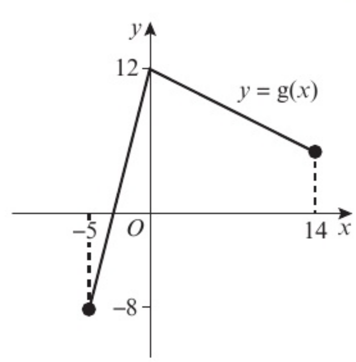
t: $x \mapsto 5 - 2x$

Solve the equation $t^2(x) - (t(x))^2 = 0$ **(5 marks)**

Problem-solving

You need to work out the intermediate steps for this problem yourself, so plan your answer before you start. You could start by finding an expression for tt(x).

E **11** The function g has domain $-5 \leqslant x \leqslant 14$ and is linear from $(-5, -8)$ to $(0, 12)$ and from $(0, 12)$ to $(14, 5)$. A sketch of the graph of $y = g(x)$ is shown in the diagram.



a Write down the range of g. **(1 mark)**

b Find gg(0). **(2 marks)**

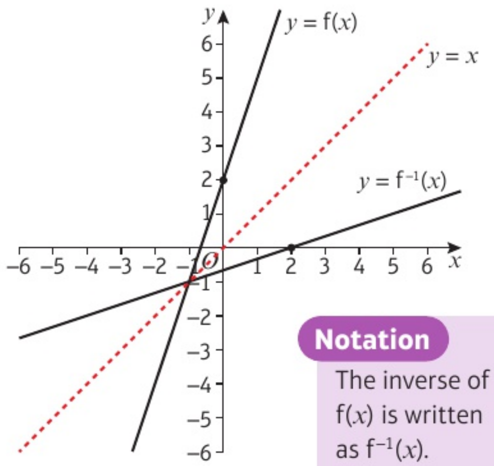
The function h is defined by h: $x \mapsto \frac{2x - 5}{10 - x}$

c Find gh(7). **(2 marks)**

2.4 Inverse functions

The **inverse** of a function performs the opposite operation to the original function. It takes the elements in the range of the original function and maps them back into elements of the domain of the original function. For this reason, inverse functions exist only for one-to-one functions.

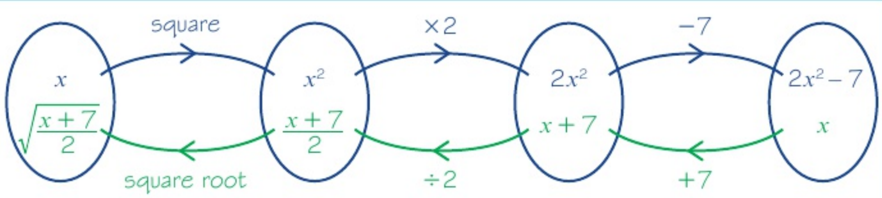
- **Functions f(x) and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = f^{-1}f(x) = x$**
- **The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$**
- **The domain of f(x) is the range of $f^{-1}(x)$**
- **The range of f(x) is the domain of $f^{-1}(x)$**



Notation
The inverse of f(x) is written as $f^{-1}(x)$.

Example 13

Find the inverse of the function $h(x) = 2x^2 - 7, x \geqslant 0$



Range of h(x) is $h(x) \geqslant -7$, so domain of $h^{-1}(x)$ is $x \geqslant -7$

Therefore, $h^{-1}(x) = \sqrt{\frac{x+7}{2}}, x \geqslant -7$

An inverse function can often be found using a flow diagram.

The range of h(x) is the domain of $h^{-1}(x)$.

Example 14

SKILLS ANALYSIS

Find the inverse of the function $f(x) = \frac{3}{x-1}, x \in \mathbb{R}, x \neq 1$, by changing the subject of the formula.

Let $y = f(x)$

$$y = \frac{3}{x-1}$$
$$y(x-1) = 3$$
$$yx - y = 3$$
$$yx = 3 + y$$
$$x = \frac{3+y}{y}$$

Range of f(x) is $f(x) \neq 0$, so domain of $f^{-1}(x)$ is $x \neq 0$

Therefore $f^{-1}(x) = \frac{3+x}{x}, x \neq 0$

$f(4) = \frac{3}{4-1} = \frac{3}{3} = 1$

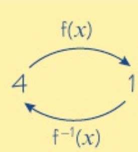
$f^{-1}(1) = \frac{3+1}{1} = \frac{4}{1} = 4$

You can **rearrange** to find an inverse function. Start by letting $y = f(x)$

Rearrange to make x the subject of the formula.

Define $f^{-1}(x)$ in terms of x.

Check to see that at least one element works. Try 4. Note that $f^{-1}f(4) = 4$



Example 15

The function f(x) is defined by $f(x) = \sqrt{x-2}$, $x \in \mathbb{R}$, $x \geqslant 2$

a State the range of f(x).

b Find the function $f^{-1}(x)$ and state its domain and range.

c Sketch $y = f(x)$ and $y = f^{-1}(x)$ and the line $y = x$

a The range of f(x) is $y \in \mathbb{R}$, $y \geqslant 0$

> f(2) = 0. As x increases from 2, f(x) also increases without **limit**, so the range is $f(x) \geqslant 0$, or $y \geqslant 0$

b
$$y = \sqrt{x-2}$$
$$y^2 = x - 2$$
$$x^2 = y - 2$$
$$y = x^2 + 2$$

> Rearrange to make y the subject of the equation.

The inverse function is $f^{-1}(x) = x^2 + 2$

> Always write your function in terms of x.

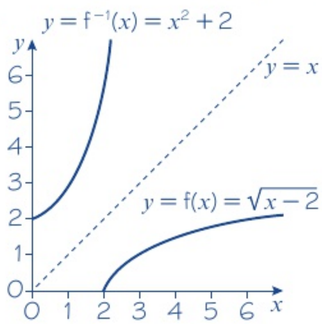
The domain of $f^{-1}(x)$ is $x \in \mathbb{R}$, $x \geqslant 0$

> The range of f(x) is the same as the domain of $f^{-1}(x)$.

The range of $f^{-1}(x)$ is $y \in \mathbb{R}$, $y \geqslant 2$

> The range of $f^{-1}(x)$ is the same as the domain of f(x).

c



> The graph of $f^{-1}(x)$ is a reflection of f(x) in the line $y = x$. This is because the reflection transforms y to x and x to y.

Example 16

The function f(x) is defined by $f(x) = x^2 - 3$, $x \in \mathbb{R}$, $x \geqslant 0$

a Find $f^{-1}(x)$.

b Sketch $y = f^{-1}(x)$ and state its domain.

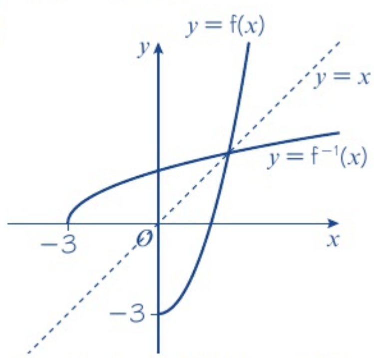
c Solve the equation $f(x) = f^{-1}(x)$

a Let $y = f(x)$
$$y = x^2 - 3$$
$$y + 3 = x^2$$
$$x = \sqrt{y+3}$$
$$f^{-1}(x) = \sqrt{x+3}$$

> Change the subject of the formula.

> **Online** Explore functions and their inverses using technology.

b



> First sketch f(x). Then reflect f(x) in the line $y = x$

The domain of $f^{-1}(x)$ is $x \in \mathbb{R}$, $x \geqslant -3$.

> The range of the original function is $f(x) \geqslant -3$

c When $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$\mathrm{f}(x) = x$

$x^2 - 3 = x$

$x^2 - x - 3 = 0$

So $x = \dfrac{1 + \sqrt{13}}{2}$

Problem-solving

$y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ intersect on the line $y = x$. This means that the solution to $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ is the same as the solution to $\mathrm{f}(x) = x$

From the graph you can see that the solution must be positive, so ignore the negative solution to the equation.

Exercise 2D

SKILLS ANALYSIS

1 For each of the following functions $\mathrm{f}(x)$:

i state the range of $\mathrm{f}(x)$

ii determine the equation of the inverse function $\mathrm{f}^{-1}(x)$

iii state the domain and range of $\mathrm{f}^{-1}(x)$

iv sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same set of axes.

a $\mathrm{f}: x \mapsto 2x + 3, x \in \mathbb{R}$

b $\mathrm{f}: x \mapsto \dfrac{x+5}{2}, x \in \mathbb{R}$

c $\mathrm{f}: x \mapsto 4 - 3x, x \in \mathbb{R}$

d $\mathrm{f}: x \mapsto x^3 - 7, x \in \mathbb{R}$

2 Find the inverse of each function:

a $\mathrm{f}(x) = 10 - x, x \in \mathbb{R}$

b $\mathrm{g}(x) = \dfrac{x}{5}, x \in \mathbb{R}$

c $\mathrm{h}(x) = \dfrac{3}{x}, x \neq 0, x \in \mathbb{R}$

d $\mathrm{k}(x) = x - 8, x \in \mathbb{R}$

Notation Two of these functions are **self-inverse**. A function is self-inverse if $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. In this case $\mathrm{ff}(x) = x$

(P) **3** Explain why the function $\mathrm{g}: x \mapsto 4 - x, x \in \mathbb{R}, x > 0$, is not identical to its inverse.

4 For each of the following functions $\mathrm{g}(x)$ with a restricted domain:

i state the range of $\mathrm{g}(x)$

ii determine the equation of the inverse function $\mathrm{g}^{-1}(x)$

iii state the domain and range of $\mathrm{g}^{-1}(x)$

iv sketch the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$ on the same set of axes.

a $\mathrm{g}(x) = \dfrac{1}{x}, x \in \mathbb{R}, x \geqslant 3$

b $\mathrm{g}(x) = 2x - 1, x \in \mathbb{R}, x \geqslant 0$

c $\mathrm{g}(x) = \dfrac{3}{x-2}, x \in \mathbb{R}, x > 2$

d $\mathrm{g}(x) = \sqrt{x-3}, x \in \mathbb{R}, x \geqslant 7$

e $\mathrm{g}(x) = x^2 + 2, x \in \mathbb{R}, x > 2$

f $\mathrm{g}(x) = x^3 - 8, x \in \mathbb{R}, x \geqslant 2$

(E) **5** The function $\mathrm{t}(x)$ is defined by

$$\mathrm{t}(x) = x^2 - 6x + 5, x \in \mathbb{R}, x \geqslant 5$$

Find $\mathrm{t}^{-1}(x)$. **(5 marks)**

Hint First complete the square for the function $\mathrm{t}(x)$.

(E/P) **6** The function $\mathrm{m}(x)$ is defined by $\mathrm{m}(x) = x^2 + 4x + 9, x \in \mathbb{R}, x > a$, for some constant a.

a State the least value of a for which $\mathrm{m}^{-1}(x)$ exists. **(4 marks)**

b Determine the equation of $\mathrm{m}^{-1}(x)$. **(3 marks)**

c State the domain of $m^{-1}(x)$. **(1 mark)**

7 The function h(x) is defined by $h(x) = \frac{2x+1}{x-2}$, $x \in \mathbb{R}$, $x \neq 2$

a What happens to the function as x approaches 2?

b Find $h^{-1}(3)$.

c Find $h^{-1}(x)$, stating clearly its domain.

d Find the elements of the domain that get mapped to themselves by the function.

8 The functions m and n are defined by:

$m: x \mapsto 2x + 3, x \in \mathbb{R}$

$n: x \mapsto \frac{x-3}{2}, x \in \mathbb{R}$

a Find $nm(x)$.

b What can you say about the functions m and n?

(P) **9** The functions s and t are defined by:

$s(x) = \frac{3}{x+1}, x \neq -1$

$t(x) = \frac{3-x}{x}, x \neq 0$

Show that the functions are inverses of each other.

(E/P) **10** The function f(x) is defined by $f(x) = 2x^2 - 3$, $x \in \mathbb{R}$, $x < 0$

Determine:

a $f^{-1}(x)$, clearly stating its domain **(4 marks)**

b the values of a for which $f(a) = f^{-1}(a)$. **(4 marks)**

(E) **11** The functions f and g are defined by:

$f: x \mapsto e^x - 5, x \in \mathbb{R}$

$g: x \mapsto \ln(x - 4), x > 4$

a State the range of f. **(1 mark)**

b Find f^{-1}, the inverse function of f, stating its domain. **(3 marks)**

c On the same axes, sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes. **(4 marks)**

d Find g^{-1}, the inverse function of g, stating its domain. **(3 marks)**

e Solve the equation $g^{-1}(x) = 11$, giving your answer to 2 decimal places. **(3 marks)**

(E/P) **12** The function f is defined by:

$f: x \mapsto \frac{3(x+2)}{x^2 + x - 20} - \frac{2}{x-4}, x > 4$

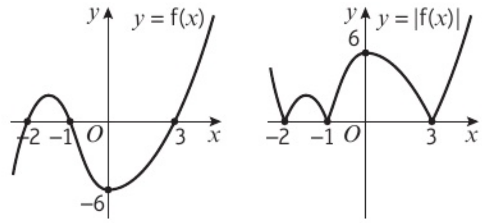
a Show that $f: x \mapsto \frac{1}{x+5}, x > 4$ **(4 marks)**

b Find the range of f. **(2 marks)**

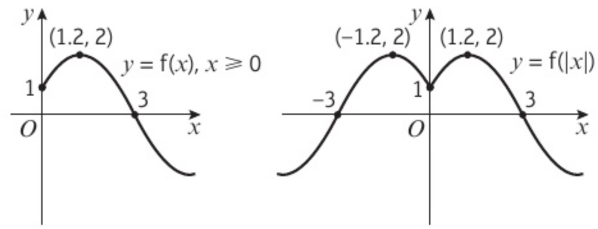
c Find $f^{-1}(x)$. State the domain of this inverse function. **(4 marks)**

2.5 $y = |f(x)|$ and $y = f(|x|)$

- To sketch the graph of $y = |f(x)|$:
  - sketch the graph of $y = f(x)$
  - reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x-axis
  - delete the parts below the x-axis.



- To sketch the graph of $y = f(|x|)$:
  - sketch the graph of $y = f(x)$ for $x \geqslant 0$
  - reflect this in the y-axis.



Example 17 SKILLS INTERPRETATION

$f(x) = x^2 - 3x - 10$

a Sketch the graph of $y = f(x)$

b Sketch the graph of $y = |f(x)|$

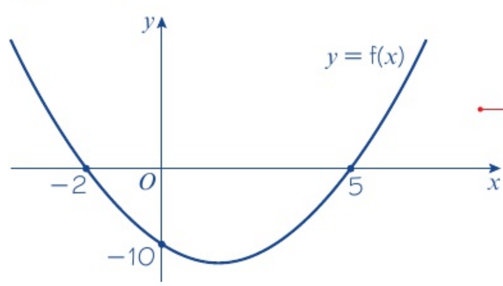
c Sketch the graph of $y = f(|x|)$

a $f(x) = x^2 - 3x - 10 = (x - 5)(x + 2)$

$f(x) = 0$ implies $(x - 5)(x + 2) = 0$

So $x = 5$ or $x = -2$

$f(0) = -10$



The graph of $y = x^2 - 3x - 10$ cuts the x-axis at $x = -2$ and $x = 5$.

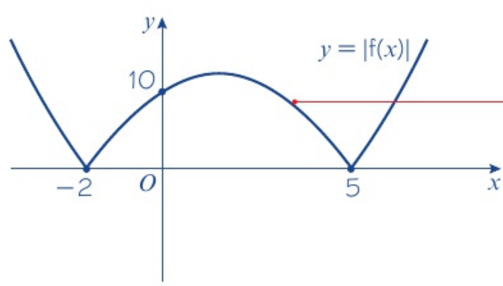
The graph cuts the y-axis at $(0, -10)$.

This is the sketch of $y = x^2 - 3x - 10$

The sketch includes the points where the graph **intercepts** the coordinate axes.

A sketch does not have to be to scale.

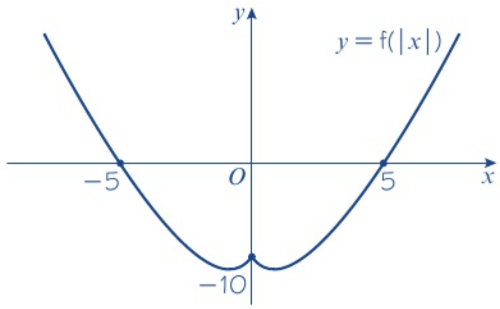
b $y = |f(x)| = |x^2 - 3x - 10|$



Reflect the part of the curve where $y = f(x) < 0$ (the negative values of y) in the x-axis.

Online Explore graphs of modulus functions using technology.

c $y = f(|x|) = |x|^2 - 3|x| - 10$



Reflect the part of the curve where $x \geqslant 0$ (the positive values of x) in the y-axis.

Example 18

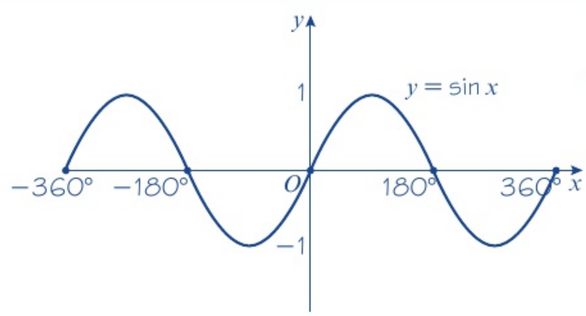
$g(x) = \sin x, -360° \leqslant x \leqslant 360°$

a Sketch the graph of $y = g(x)$

b Sketch the graph of $y = |g(x)|$

c Sketch the graph of $y = g(|x|)$

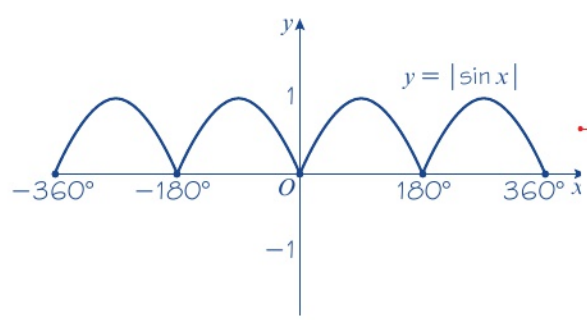
a



The graph is periodic and passes through the **origin**, (±180°, 0) and (±360°, 0).

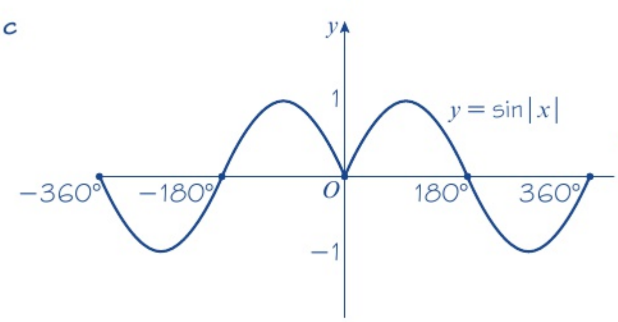
← Pure 1 Section 6.5

b



Reflect the part of the curve below the x-axis in the x-axis.

c



Reflect the part of the curve where $x \geqslant 0$ in the y-axis.

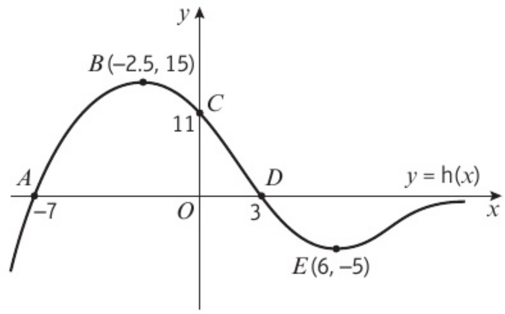
Example 19

The diagram shows the graph of $y = \mathrm{h}(x)$, with five points labelled.

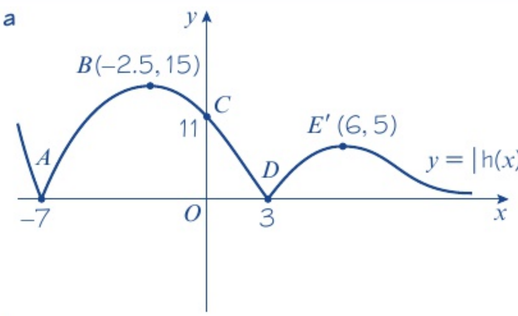
Sketch each of the following graphs, labelling the points corresponding to A, B, C, D and E, and any points of intersection with the coordinate axes.

a $y = |\mathrm{h}(x)|$

b $y = \mathrm{h}(|x|)$



a

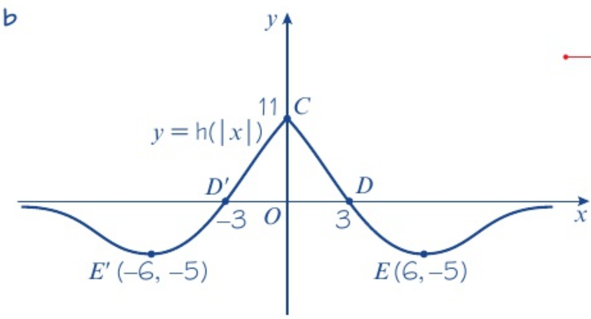


The parts of the curve below the x-axis are reflected in the x-axis.

The points A, B, C and D are unchanged.

The point E was reflected, so the new coordinates are $E'(6, 5)$.

b



The part of the curve to the right of the y-axis is reflected in the y-axis.

The old points A and B had negative x-values so they are no longer part of the graph.

The points C, D and E are unchanged.

There is a new point of intersection with the x-axis at $(-3, 0)$.

The point E was reflected, so the new coordinates are $E'(-6, -5)$.

Exercise 2E

SKILLS INTERPRETATION

1 $\mathrm{f}(x) = x^2 - 7x - 8$

a Sketch the graph of $y = \mathrm{f}(x)$

b Sketch the graph of $y = |\mathrm{f}(x)|$

c Sketch the graph of $y = \mathrm{f}(|x|)$

2 $\mathrm{g}: x \mapsto \cos x, -360° \leqslant x \leqslant 360°$

a Sketch the graph of $y = \mathrm{g}(x)$

b Sketch the graph of $y = |\mathrm{g}(x)|$

c Sketch the graph of $y = \mathrm{g}(|x|)$

3 $\mathrm{h}: x \mapsto (x - 1)(x - 2)(x + 3)$

a Sketch the graph of $y = \mathrm{h}(x)$

b Sketch the graph of $y = |\mathrm{h}(x)|$

c Sketch the graph of $y = \mathrm{h}(|x|)$

P **4** The function k is defined by $k(x) = \frac{a}{x^2}$, $a > 0$, $x \in \mathbb{R}$, $x \neq 0$

a Sketch the graph of $y = k(x)$

b Explain why it is not necessary to sketch $y = |k(x)|$ and $y = k(|x|)$

The function m is defined by $m(x) = \frac{a}{x^2}$, $a < 0$, $x \in \mathbb{R}$, $x \neq 0$

c Sketch the graph of $y = m(x)$

d State with a reason whether the following statements are true or false:

i $|k(x)| = |m(x)|$ **ii** $k(|x|) = m(|x|)$ **iii** $m(x) = m(|x|)$

E **5** The diagram shows the graph of $y = p(x)$ with five points labelled.

Sketch each of the following graphs, labelling the points corresponding to A, B, C, D and E, and any points of intersection with the coordinate axes.

a $y = |p(x)|$ **(3 marks)**

b $y = p(|x|)$ **(3 marks)**

E **6** The diagram shows the graph of $y = q(x)$ with seven points labelled.

Sketch each of the following graphs, labelling the points corresponding to A, B, C, D, E, F and G, and any points of intersection with the coordinate axes.

a $y = |q(x)|$ **(4 marks)**

b $y = q(|x|)$ **(3 marks)**

7 $k(x) = \frac{a}{x}$, $a > 0$, $x \neq 0$

a Sketch the graph of $y = k(x)$

b Sketch the graph of $y = |k(x)|$

c Sketch the graph of $y = k(|x|)$

8 $m(x) = \frac{a}{x}$, $a < 0$, $x \neq 0$

a Sketch the graph of $y = m(x)$

b Describe the relationship between $y = |m(x)|$ and $y = m(|x|)$

9 $f(x) = 2^x$ and $g(x) = 2^{-x}$

a Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes.

b Explain why it is not necessary to sketch $y = |f(x)|$ and $y = |g(x)|$

c Sketch the graphs of $y = f(|x|)$ and $y = g(|x|)$ on the same axes.

(E/P) **10** The function f(x) is defined by:

$$f(x) = \begin{cases} -2x - 6, & -5 \leqslant x < -1 \\ (x + 1)^2, & -1 \leqslant x \leqslant 2 \end{cases}$$

a Sketch f(x), stating its range. **(5 marks)**

b Sketch the graph of $y = |f(x)|$ **(3 marks)**

c Sketch the graph of $y = f(|x|)$ **(3 marks)**

Problem-solving

A piecewise function like this does not have to be continuous. Work out the value of both expressions when $x = -1$ to help you with your sketch.

2.6 Combining transformations

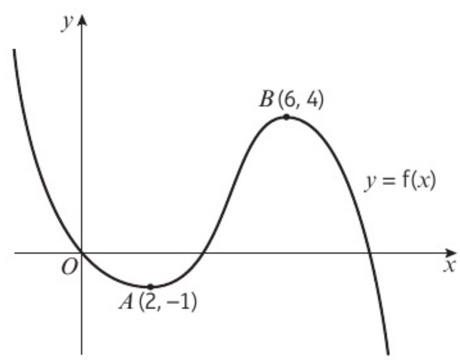
You can use combinations of the following transformations of a function to sketch graphs of more complicated transformations.

- $f(x + a)$ is a **translation** by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
- $f(x) + a$ is a translation by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$
- $f(-x)$ reflects $f(x)$ in the y-axis
- $-f(x)$ reflects $f(x)$ in the x-axis
- $f(ax)$ is a horizontal **stretch** of scale factor $\frac{1}{a}$
- $af(x)$ is a vertical stretch of scale factor a

Links You can think of $f(-x)$ and $-f(x)$ as stretches with scale factor -1. ← Pure 1 Sections 4.5, 4.6

Example 20

The diagram shows a sketch of the graph of $y = f(x)$. The curve passes through the origin O, the point $A(2, -1)$ and the point $B(6, 4)$.



Sketch the graphs of:

a $y = 2f(x) - 1$

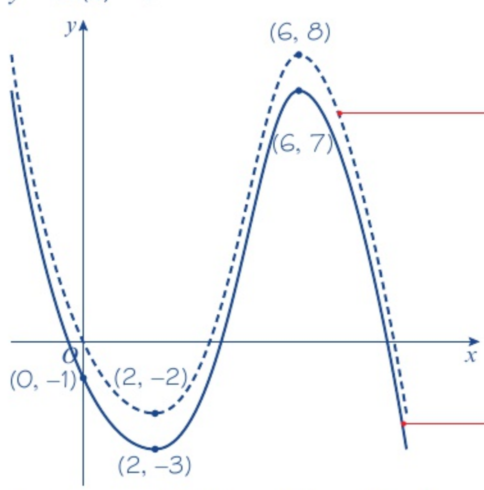
b $y = f(x + 2) + 2$

c $y = \frac{1}{4}f(2x)$

d $y = -f(x - 1)$

In each case, find the coordinates of the images of the points O, A and B.

a $y = 2f(x) - 1$



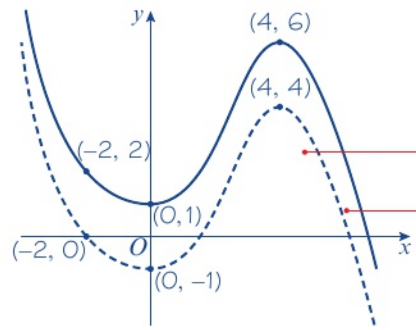
The images of O, A and B are $(0, -1)$, $(2, -3)$ and $(6, 7)$ respectively.

Apply the stretch first. The dotted curve is the graph of $y = 2f(x)$, which is a vertical stretch with scale factor 2.

Next apply the translation. The solid curve is the graph of $y = 2f(x) - 1$, as required. This is a translation of $y = 2f(x)$ by vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Watch out The order is important. If you applied the transformations in the opposite order you would have the graph of $y = 2(f(x) - 1)$ or $y = 2f(x) - 2$

b $y = \mathrm{f}(x + 2) + 2$

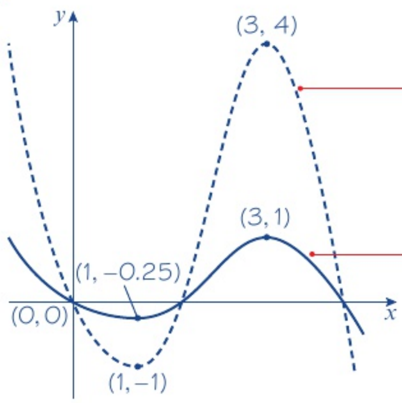


The images of O, A and B are $(-2, 2)$, $(0, 1)$ and $(4, 6)$ respectively.

Apply the translation **inside** the brackets first. The dotted curve is the graph of $y = \mathrm{f}(x + 2)$, which is a translation of $y = \mathrm{f}(x)$ by vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

Next apply the translation **outside** the brackets. The solid curve is the graph of $y = \mathrm{f}(x + 2) + 2$, as required. This is a translation of $y = \mathrm{f}(x + 2)$ by vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

c $y = \frac{1}{4}\mathrm{f}(2x)$

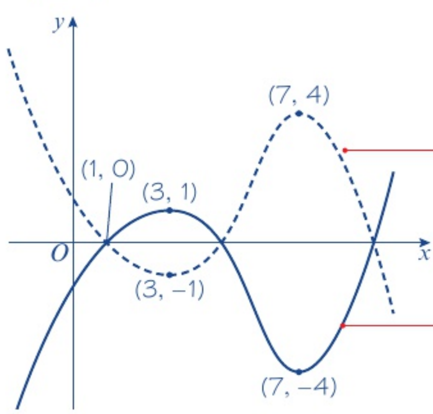


The images of O, A and B are $(0, 0)$, $(1, -0.25)$ and $(3, 1)$ respectively.

Apply the stretch **inside** the brackets first. The dotted curve is the graph of $y = \mathrm{f}(2x)$, which is a horizontal stretch with scale factor $\frac{1}{2}$

Then apply the stretch **outside** the brackets. The solid curve is the graph of $y = \frac{1}{4}\mathrm{f}(2x)$, as required. This is a vertical stretch of $y = \mathrm{f}(2x)$ with scale factor $\frac{1}{4}$

d $y = -\mathrm{f}(x - 1)$



The images of O, A and B are $(1, 0)$, $(3, 1)$ and $(7, -4)$ respectively.

Apply the translation **inside** the brackets first. The dotted curve is the graph of $y = \mathrm{f}(x - 1)$, which is a translation of $y = \mathrm{f}(x)$ by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Then apply the reflection **outside** the brackets. The solid curve is the graph of $y = -\mathrm{f}(x - 1)$, as required. This is a reflection of $y = \mathrm{f}(x - 1)$ in the x-axis.

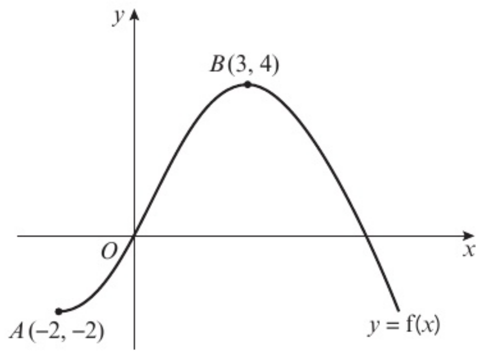
Exercise 2F SKILLS INTERPRETATION

1 The diagram shows a sketch of the graph $y = f(x)$. The curve passes through the origin O, the point $A(-2, -2)$ and the point $B(3, 4)$.

On separate axes, sketch the graphs of:

a $y = 3f(x) + 2$
b $y = f(x - 2) - 5$
c $y = \frac{1}{2}f(x + 1)$
d $y = -f(2x)$
e $y = |f(x)|$
f $y = |f(-x)|$

In each case, find the coordinates of the images of the points O, A and B.

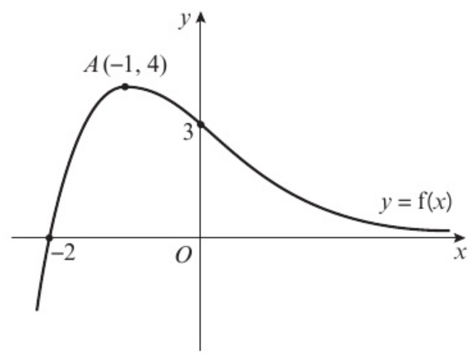


2 The diagram shows a sketch of the graph $y = f(x)$. The curve has a maximum at the point $A(-1, 4)$ and crosses the axes at the points $(0, 3)$ and $(-2, 0)$.

On separate axes, sketch the graphs of:

a $y = 3f(x - 2)$
b $y = \frac{1}{2}f\left(\frac{1}{2}x\right)$
c $y = -f(x) + 4$
d $y = -2f(x + 1)$
e $y = 2f(|x|)$

For each graph, find, where possible, the coordinates of the maximum or minimum and the coordinates of the intersection points with the axes.

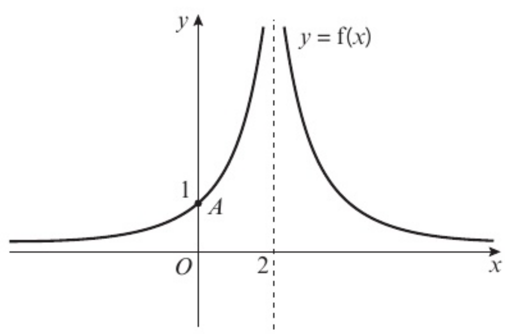


3 The diagram shows a sketch of the graph $y = f(x)$. The lines $x = 2$ and $y = 0$ (the x-axis) are asymptotes to the curve.

On separate axes, sketch the graphs of:

a $y = 3f(x) - 1$
b $y = f(x + 2) + 4$
c $y = -f(2x)$
d $y = f(|x|)$

For each part, state the equations of the asymptotes and the new coordinates of the point A.



(E) **4** The function g is defined by $g: x \mapsto (x - 2)^2 - 9, x \in \mathbb{R}$

a Draw a sketch of the graph of $y = g(x)$, labelling the **turning points** and the x- and y-intercepts. **(3 marks)**

b Write down the coordinates of the turning point when the curve is transformed as follows:

i $2g(x - 4)$ **(2 marks)**
ii $g(2x)$ **(2 marks)**
iii $|g(x)|$ **(2 marks)**

c Sketch the curve with equation $y = g(|x|)$. On your sketch, show the coordinates of all turning points and all x- and y-intercepts. **(4 marks)**

5 $h(x) = 2\sin x$, $-180° \leqslant x \leqslant 180°$

a Sketch the graph of $y = h(x)$

b Write down the coordinates of the minimum, A, and the maximum, B.

c Sketch the graphs of:

i $h(x - 90°) + 1$ **ii** $\frac{1}{4}h\left(\frac{1}{2}x\right)$ **iii** $\frac{1}{2}|h(-x)|$

In each case, find the coordinates of the images of the points O, A and B, with O being the origin.

2.7 Solving modulus problems

You can use combinations of transformations together with $|f(x)|$ and $f(|x|)$ and an understanding of domain and range to solve problems.

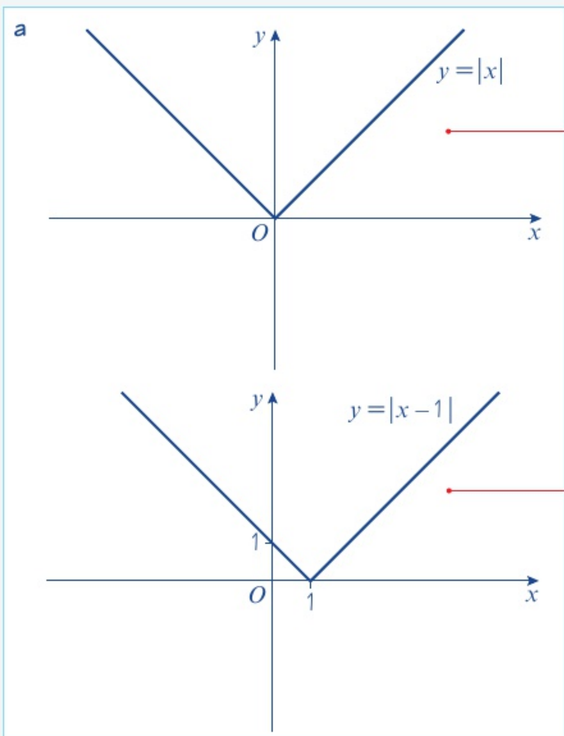
Example 21

Given the function $t(x) = 3|x - 1| - 2$, $x \in \mathbb{R}$:

a sketch the graph of the function

b state the range of the function

c solve the equation $t(x) = \frac{1}{2}x + 3$

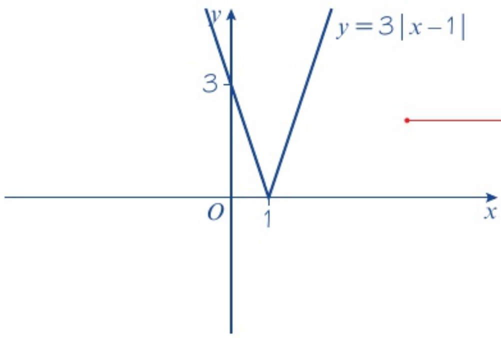


Problem-solving

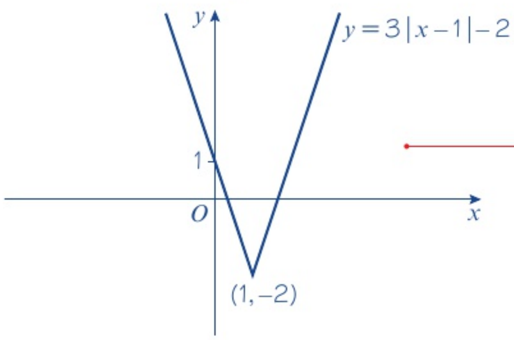
Use transformations to sketch the graph of $y = 3|x - 1| - 2$

Sketch the graph of $y = |x|$

Step 1
Horizontal translation by vector $\begin{pmatrix}1\\0\end{pmatrix}$



Step 2
Vertical stretch, scale factor 3

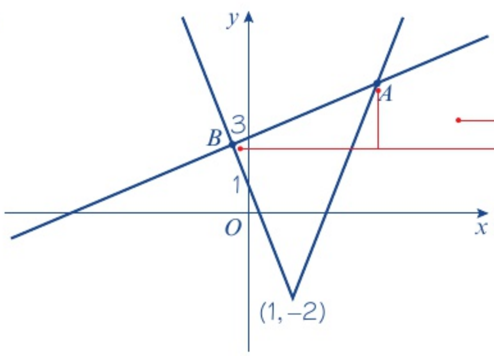


Step 3
Vertical translation by vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

b The range of the function t(x) is $y \in \mathbb{R}$, $y \geqslant -2$

The graph has a minimum at (1, −2).

c



First draw a sketch of $y = 3|x - 1| - 2$ and the line $y = \frac{1}{2}x + 3$

The sketch shows there are two solutions, at A and B, the points of intersection.

At A: $3(x - 1) - 2 = \frac{1}{2}x + 3$

$$3x - 5 = \frac{1}{2}x + 3$$

$$\frac{5}{2}x = 8$$

$$x = \frac{16}{5}$$

This is the solution on the original part of the graph.

At B: $-3(x - 1) - 2 = \frac{1}{2}x + 3$

When f(x) < 0, |f(x)| = −f(x), so use −(3x − 1) − 2 to find the solution on the reflected part of the graph.

$$-3x + 3 - 2 = \frac{1}{2}x + 3$$

$$-\frac{7}{2}x = 2$$

$$x = -\frac{4}{7}$$

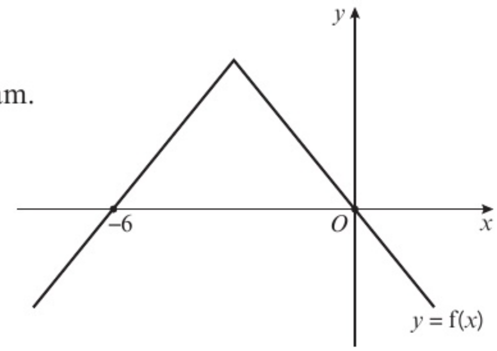
This is the solution corresponding to point B on the sketch.

The solutions are $x = \frac{16}{5}$ and $x = -\frac{4}{7}$

Example 22

The function f is defined by f: $x \mapsto 6 - 2|x + 3|$

A sketch of the graph of the function is shown in the diagram.



a State the range of f.

b Give a reason why f^{-1} does not exist.

c Solve the inequality $f(x) > 5$

a The range of f(x) is $f(x) \leqslant 6$

The greatest value f(x) can take is 6 (when $x = -3$).

b f(x) is a many-to-one function. Therefore, f^{-1} does not exist.

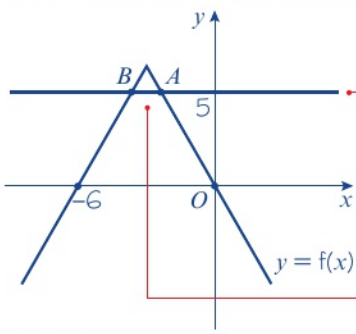
For example, $f(0) = f(-6) = 0$

Problem-solving

Only one-to-one functions have inverses.

c $f(x) = 5$ at the points A and B.

$f(x) > 5$ between the points A and B.



Add the line $y = 5$ to the graph of $y = f(x)$

Between the points A and B, the graph of $y = f(x)$ is above the line $y = 5$

At A: $6 - 2(x + 3) = 5$

$-2(x + 3) = -1$

$x + 3 = \frac{1}{2}$

$x = -\frac{5}{2}$

This is the solution on the original part of the graph.

At B: $6 - (-2(x + 3)) = 5$

When $f(x) < 0$, $|f(x)| = -f(x)$, so use the negative argument, $-2(x + 3)$

$2(x + 3) = -1$

$x + 3 = -\frac{1}{2}$

$x = -\frac{7}{2}$

This is the solution on the reflected part of the graph.

The solution to the inequality $f(x) > 5$ is

$-\frac{7}{2} < x < -\frac{5}{2}$

Online Explore the solution using technology.

Exercise 2G SKILLS INTERPRETATION

(P) **1** For each function:

i sketch the graph of $y = f(x)$

ii state the range of the function.

a $f: x \mapsto 4|x| - 3, x \in \mathbb{R}$

b $f(x) = \frac{1}{3}|x + 2| - 1, x \in \mathbb{R}$

c $f(x) = -2|x - 1| + 6, x \in \mathbb{R}$

d $f: x \mapsto -\frac{5}{2}|x| + 4, x \in \mathbb{R}$

Hint For part **b**, transform the graph of $y = |x|$ by:

- a translation by vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
- a vertical stretch with scale factor $\frac{1}{3}$
- a translation by vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

2 Given that $p(x) = 2|x + 4| - 5, x \in \mathbb{R}$:

a sketch the graph of $y = p(x)$

b shade the region of the graph that satisfies $y \geqslant p(x)$

3 Given that $q(x) = -3|x| + 6, x \in \mathbb{R}$:

a sketch the graph of $y = q(x)$

b shade the region of the graph that satisfies $y < q(x)$

4 The function f is defined as:

$f: x \mapsto 4|x + 6| + 1, x \in \mathbb{R}$

a Sketch the graph of $y = f(x)$

b State the range of the function.

c Solve the equation $f(x) = -\frac{1}{2}x + 1$

5 Given that $g(x) = -\frac{5}{2}|x - 2| + 7, x \in \mathbb{R}$:

a sketch the graph of $y = g(x)$

b state the range of the function

c solve the equation $g(x) = x + 1$

(E/P) **6** The functions m and n are defined as:

$m(x) = -2x + k, x \in \mathbb{R}$

$n(x) = 3|x - 4| + 6, x \in \mathbb{R}$

where k is a constant.

The equation $m(x) = n(x)$ has no real roots.

Find the range of possible values for the constant k. **(4 marks)**

Problem-solving

'$m(x) = n(x)$ has no real **roots**' means that $y = m(x)$ and $y = n(x)$ do not intersect.

E/P **7** The functions s and t are defined as:

$s(x) = -10 - x, x \in \mathbb{R}$

$t(x) = 2|x + b| - 8, x \in \mathbb{R}$

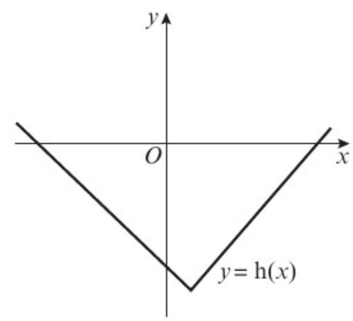
where b is a constant.

The equation $s(x) = t(x)$ has exactly one real root. Find the value of b. **(4 marks)**

E/P **8** The function h is defined by:

$h(x) = \frac{2}{3}|x - 1| - 7, x \in \mathbb{R}$

The diagram shows a sketch of the graph $y = h(x)$



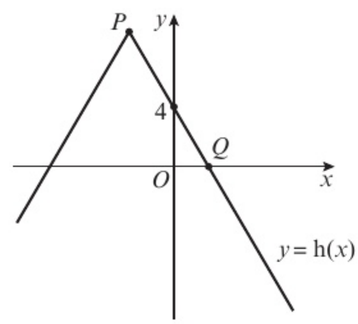
a State the range of h. **(1 mark)**

b Give a reason why h^{-1} does not exist. **(1 mark)**

c Solve the inequality $h(x) < -6$ **(4 marks)**

d State the range of values of k for which the equation $h(x) = \frac{2}{3}x + k$ has no solutions. **(4 marks)**

E/P **9** The diagram shows a sketch of part of the graph $y = h(x)$, where $h(x) = a - 2|x + 3|, x \in \mathbb{R}$



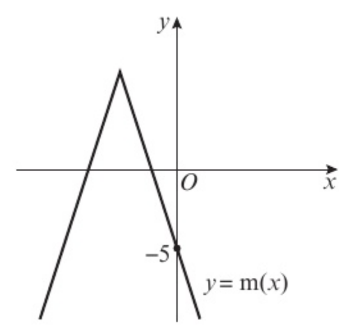
The graph crosses the y-axis at (0, 4).

a Find the value of a. **(2 marks)**

b Find the coordinates of P and Q. **(3 marks)**

c Solve $h(x) = \frac{1}{3}x + 6$ **(5 marks)**

E/P **10** The diagram shows a sketch of part of the graph $y = m(x)$, where $m(x) = -4|x + 3| + 7, x \in \mathbb{R}$



a State the range of m. **(1 mark)**

b Solve the equation $m(x) = \frac{3}{5}x + 2$ **(4 marks)**

c Given that $m(x) = k$, where k is a constant, has two distinct roots, state the set of possible values for k. **(4 marks)**

Challenge

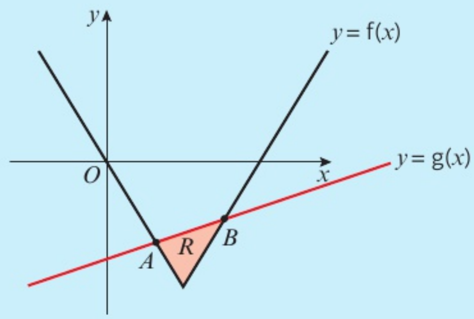
SKILLS CREATIVITY

1 The functions f and g are defined by:

$f(x) = 2|x - 4| - 8, x \in \mathbb{R}$

$g(x) = x - 9, x \in \mathbb{R}$

The diagram shows a sketch of the graphs of $y = f(x)$ and $y = g(x)$



a Find the coordinates of the points A and B.

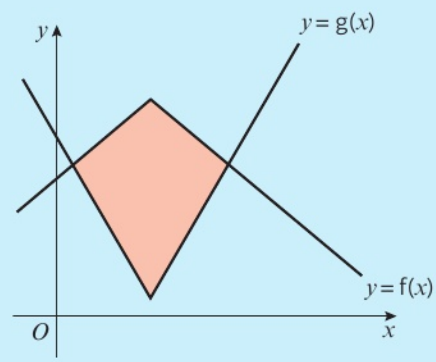
b Find the area of the region R.

2 The functions f and g are defined as:

$f(x) = -|x - 3| + 10, x \in \mathbb{R}$

$g(x) = 2|x - 3| + 2, x \in \mathbb{R}$

Show that the area of the shaded region is $\frac{64}{3}$



Chapter review 2

1 **a** On the same axes, sketch the graphs of $y = 2 - x$ and $y = 2|x + 1|$

b Hence, or otherwise, find the values of x for which $2 - x = 2|x + 1|$

E/P **2** The equation $|2x - 11| = \frac{1}{2}x + k$ has exactly two distinct solutions.

Find the range of possible values of k. **(4 marks)**

E/P **3** Solve $|5x - 2| = -\frac{1}{4}x + 8$ **(4 marks)**

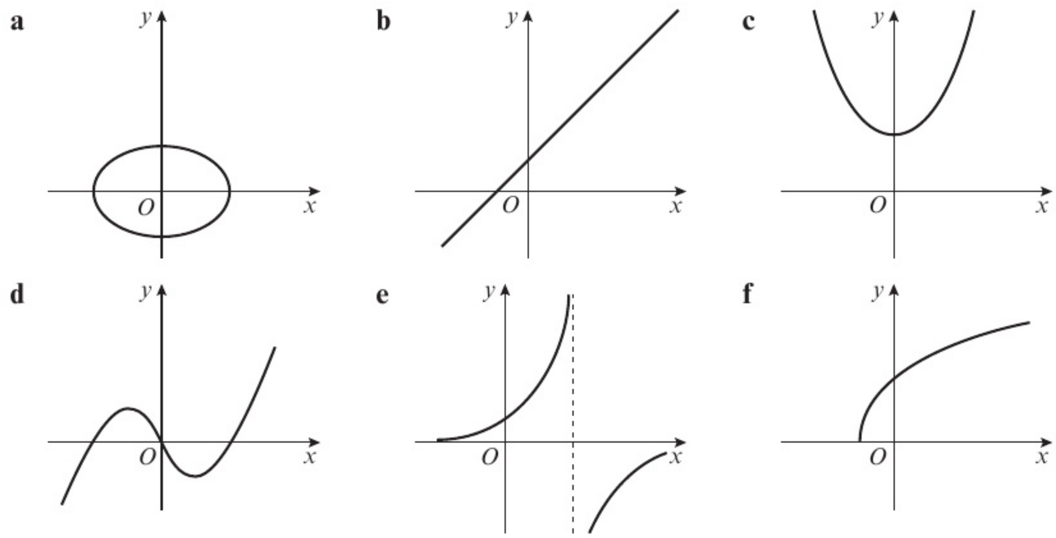
E/P **4** **a** On the same set of axes, sketch $y = |12 - 5x|$ and $y = -2x + 3$ **(3 marks)**

b State, with a reason, whether there are any solutions to the equation $|12 - 5x| = -2x + 3$ **(2 marks)**

5 For each of the following mappings:

i state whether the mapping is one-to-one, many-to-one or one-to-many

ii state whether or not the mapping could represent a function.



E **6** The function f(x) is defined by: $f(x) = \begin{cases} -x, & x \leqslant 1 \\ x - 2, & x > 1 \end{cases}$

a Sketch the graph of f(x) for $-2 \leqslant x \leqslant 6$ **(4 marks)**

b Find the values of x for which $f(x) = -\frac{1}{2}$ **(3 marks)**

E **7** The functions p and q are defined by:

$p: x \mapsto x^2 + 3x - 4, x \in \mathbb{R}$

$q: x \mapsto 2x + 1, x \in \mathbb{R}$

a Find an expression for pq(x). **(2 marks)**

b Solve pq(x) = qq(x) **(3 marks)**

E **8** The function g(x) is defined as $g(x) = 2x + 7, x \in \mathbb{R}, x \geqslant 0$

a Sketch $y = g(x)$, and find the range. **(3 marks)**

b Determine $y = g^{-1}(x)$, stating its range. **(3 marks)**

c Sketch $y = g^{-1}(x)$ on the same axes as $y = g(x)$, stating the relationship between the two graphs. **(2 marks)**

E **9** The function f is defined by:

$$f: x \mapsto \frac{2x + 3}{x - 1}, \quad x \in \mathbb{R}, x > 1$$

a Find $f^{-1}(x)$. **(4 marks)**

b Find: **i** the range of $f^{-1}(x)$

ii the domain of $f^{-1}(x)$ **(2 marks)**

E/P **10** The functions f and g are given by:

$$f: x \mapsto \frac{x}{x^2 - 1} - \frac{1}{x + 1}, \quad x \in \mathbb{R}, x > 1$$

$$g: x \mapsto \frac{2}{x}, \ x \in \mathbb{R}, x > 0$$

a Show that $f(x) = \frac{1}{(x - 1)(x + 1)}$ **(3 marks)**

b Find the range of $f(x)$. **(1 mark)**

c Solve $gf(x) = 70$ **(4 marks)**

P **11** The following functions $f(x)$, $g(x)$ and $h(x)$ are defined by:

$f(x) = 4(x - 2), x \in \mathbb{R}, x \geqslant 0$

$g(x) = x^3 + 1, x \in \mathbb{R}$

$h(x) = 3^x, x \in \mathbb{R}$

a Find $f(7)$, $g(3)$ and $h(-2)$.

b Find the range of $f(x)$ and the range of $g(x)$.

c Find $g^{-1}(x)$.

d Find the composite function $fg(x)$.

e Solve $gh(a) = 244$

E/P **12** The function $f(x)$ is defined by $f: x \mapsto x^2 + 6x - 4, x \in \mathbb{R}, x > a$, for some constant a.

a State the least value of a for which f^{-1} exists. **(4 marks)**

b Given that $a = 0$, find f^{-1}, stating its domain. **(4 marks)**

E/P **13** The functions f and g are given by: $f: x \mapsto 4x - 1, x \in \mathbb{R}$

$$g: x \mapsto \frac{3}{2x - 1}, x \in \mathbb{R}, x \neq \frac{1}{2}$$

Find in its simplest form:

a the inverse function f^{-1} **(2 marks)**

b the composite function gf, stating its domain **(3 marks)**

c the values of x for which $2f(x) = g(x)$, giving your answers to 3 decimal places. **(4 marks)**

E **14** The functions f and g are given by

$$f: x \mapsto \frac{x}{x - 2}, x \in \mathbb{R}, x \neq 2$$

$$g: x \mapsto \frac{3}{x}, x \in \mathbb{R}, x \neq 0$$

a Find an expression for $f^{-1}(x)$ **(2 marks)**

b Write down the range of $f^{-1}(x)$ **(1 mark)**

c Calculate $gf(1.5)$ **(2 marks)**

d Use algebra to find the values of x for which $g(x) = f(x) + 4$ **(4 marks)**

15 The function $n(x)$ is defined by:

$$n(x) = \begin{cases} 5 - x, & x \leqslant 0 \\ x^2, & x > 0 \end{cases}$$

a Find $n(-3)$ and $n(3)$.

b Solve the equation $n(x) = 50$

16 $g(x) = \tan x, -180° \leqslant x \leqslant 180°$

a Sketch the graph of $y = g(x)$

b Sketch the graph of $y = |g(x)|$

c Sketch the graph of $y = g(|x|)$

(E) **17** The diagram shows the graph of $f(x)$.

The points $A(4, -3)$ and $B(9, 3)$ are turning points on the graph.

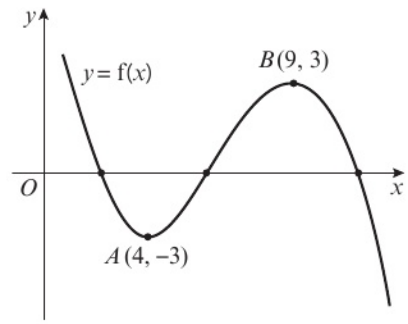
Sketch, on separate diagrams, the graphs of:

a $y = f(2x) + 1$ **(3 marks)**

b $y = |f(x)|$ **(3 marks)**

c $y = -f(x - 2)$ **(3 marks)**

Indicate on each diagram the coordinates of any turning points on your sketch.



(E) **18** Functions f and g are defined by:

$f: x \mapsto 4 - x, x \in \mathbb{R}$

$g: x \mapsto 3x^2, x \in \mathbb{R}$

a Write down the range of g. **(1 mark)**

b Solve $gf(x) = 48$ **(4 marks)**

c Sketch the graph of $y = |f(x)|$ and hence find the values of x for which $|f(x)| = 2$ **(4 marks)**

(E/P) **19** The function f is defined by $f: x \mapsto |2x - a|, x \in \mathbb{R}$, where a is a positive constant.

a Sketch the graph of $y = f(x)$, showing the coordinates of the points where the graph cuts the axes. **(3 marks)**

b On a separate diagram, sketch the graph of $y = f(2x)$, showing the coordinates of the points where the graph cuts the axes. **(2 marks)**

c Given that a solution of the equation $f(x) = \frac{1}{2}x$ is $x = 4$, find the two possible values of a. **(4 marks)**

(E/P) **20** **a** Sketch the graph of $y = |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph meets the axes. **(3 marks)**

b Using algebra, solve, for x in terms of a, $|x - 2a| = \frac{1}{3}x$ **(4 marks)**

c On a separate diagram, sketch the graph of $y = a - |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph cuts the axes. **(4 marks)**

(E/P) **21** **a** Sketch the graph of $y = |2x + a|$, $a > 0$, showing the coordinates of the points where the graph meets the coordinate axes. **(3 marks)**

b On the same axes, sketch the graph of $y = \frac{1}{x}$ **(2 marks)**

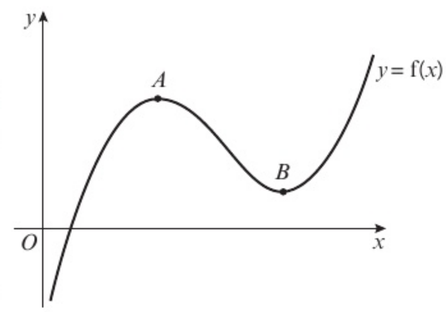
c Explain how your graphs show that there is only one solution of the equation $x|2x + a| - 1 = 0$ **(2 marks)**

d Find, using algebra, the value of x for which $x|2x + a| - 1 = 0$. **(3 marks)**

E/P **22** The diagram shows part of the curve with equation $y = f(x)$, where

$$f(x) = x^2 - 7x + 5\ln x + 8, \quad x > 0$$

The points A and B are the **stationary points** of the curve.

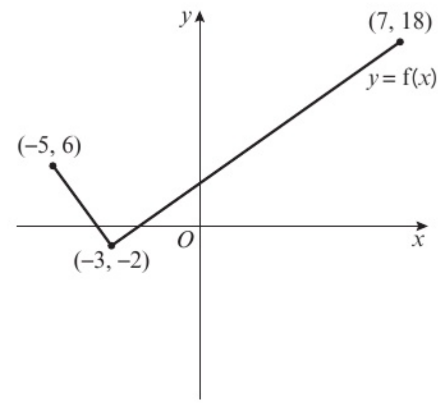


a Using calculus and showing your working, find the coordinates of the points A and B. **(4 marks)**

b Sketch the curve with equation $y = -3f(x - 2)$ **(3 marks)**

c Find the coordinates of the stationary points of the curve with equation $y = -3f(x - 2)$. State, without proof, which point is a maximum and which point is a minimum. **(3 marks)**

E/P **23** The function f has domain $-5 \leqslant x \leqslant 7$ and is linear from $(-5, 6)$ to $(-3, -2)$ and from $(-3, -2)$ to $(7, 18)$.
The diagram shows a sketch of the function.



a Write down the range of f. **(1 mark)**

b Find $ff(-3)$. **(2 marks)**

c Sketch the graph of $y = |f(x)|$, marking the points at which the graph meets or cuts the axes. **(3 marks)**

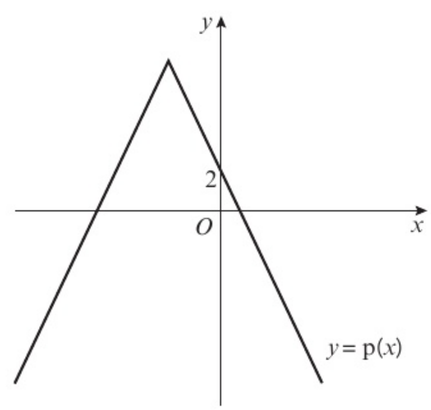
The function g is defined by $g: x \mapsto x^2 - 7x + 10$

d Solve the equation $fg(x) = 2$ **(3 marks)**

P **24** The function p is defined by:

$$p: x \mapsto -2|x + 4| + 10$$

The diagram shows a sketch of the graph.



a State the range of p. **(1 mark)**

b Give a reason why p^{-1} does not exist. **(1 mark)**

c Solve the inequality $p(x) > -4$ **(4 marks)**

d State the range of values of k for which the equation $p(x) = -\frac{1}{2}x + k$ has no solutions. **(4 marks)**

Challenge

SKILLS CREATIVITY

a Sketch, on a single diagram, the graphs of $y = a^2 - x^2$ and $y = |x + a|$, where a is a constant and $a > 1$.

b Write down the coordinates of the points where the graph of $y = a^2 - x^2$ cuts the coordinate axes.

c Given that the two graphs intersect at $x = 4$, calculate the value of a.

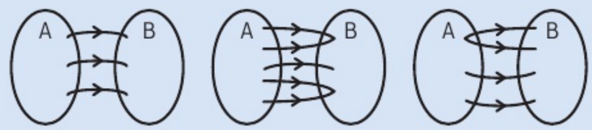
Summary of key points

1 A modulus function is, in general, a function of the type $y = |f(x)|$

- When $f(x) \geqslant 0$, $|f(x)| = f(x)$
- When $f(x) < 0$, $|f(x)| = -f(x)$

2 To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ and then reflect the section of the graph below the x-axis in the x-axis.

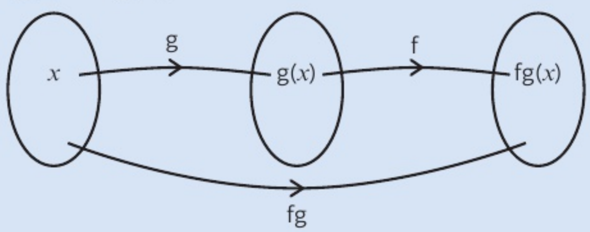
3 A mapping is a **function** if every input has a distinct output. Functions can either be **one-to-one** or **many-to-one**.



one-to-one function | many-to-one function | not a function

4 fg(x) means apply g first, then apply f.

$fg(x) = f(g(x))$



5 Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = x$ and $f^{-1}f(x) = x$

6 The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$

7 The domain of $f(x)$ is the range of $f^{-1}(x)$.

8 The range of $f(x)$ is the domain of $f^{-1}(x)$.

9 To sketch the graph of $y = |f(x)|$:

- sketch the graph of $y = f(x)$
- reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x-axis
- delete the parts below the x-axis.

10 To sketch the graph of $y = f(|x|)$:

- sketch the graph of $y = f(x)$ for $x \geqslant 0$
- reflect this in the y-axis.

11 $f(x + a)$ is a horizontal translation by $-a$.

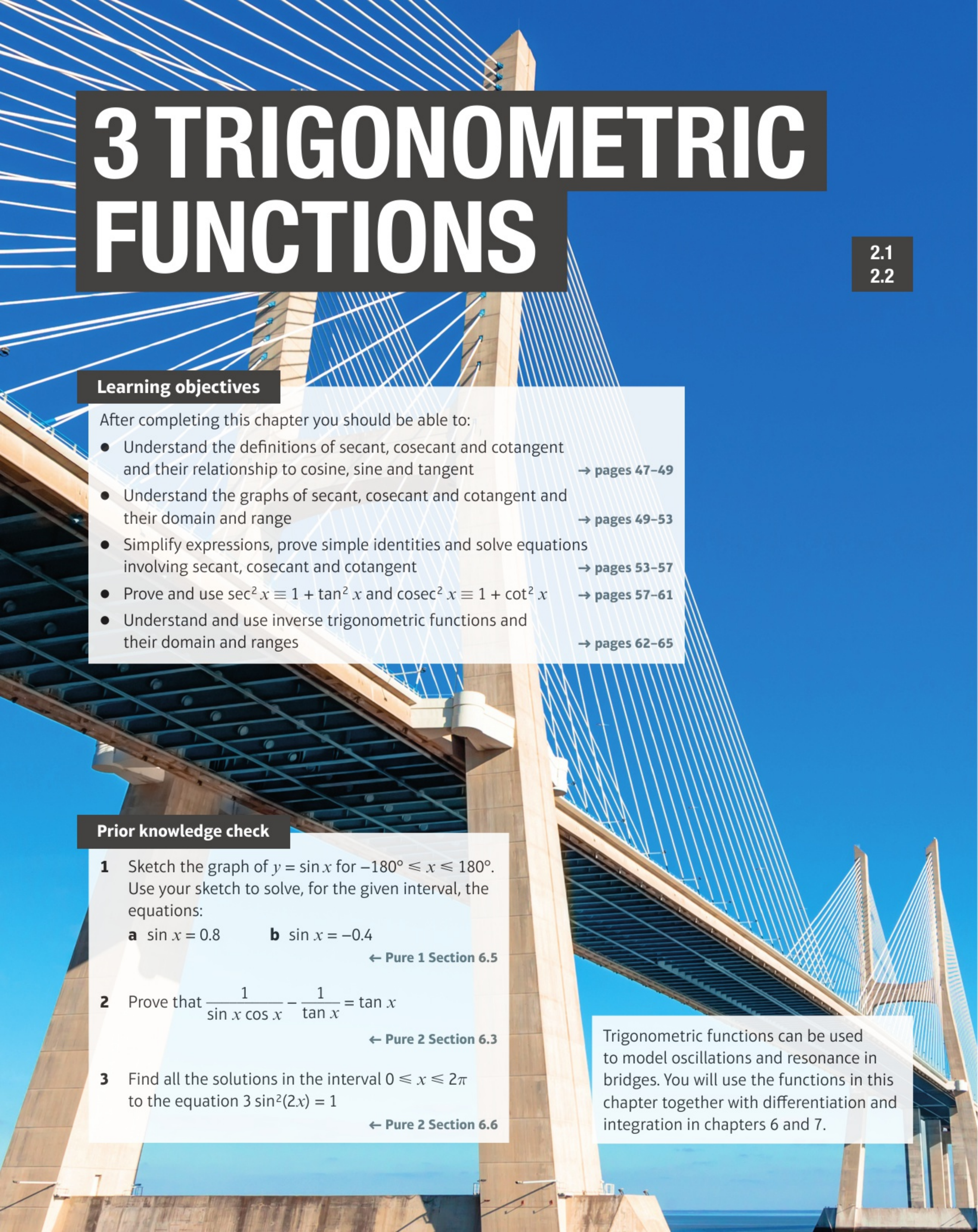
12 $f(x) + a$ is a vertical translation by $+a$.

13 $f(ax)$ is a horizontal stretch of scale factor $\frac{1}{a}$

14 $af(x)$ is a vertical stretch of scale factor a.

15 $f(-x)$ reflects $f(x)$ in the y-axis.

16 $-f(x)$ reflects $f(x)$ in the x-axis.



3 TRIGONOMETRIC FUNCTIONS

2.1
2.2

Learning objectives

After completing this chapter you should be able to:

- Understand the definitions of secant, cosecant and cotangent and their relationship to cosine, sine and tangent → pages 47–49
- Understand the graphs of secant, cosecant and cotangent and their domain and range → pages 49–53
- Simplify expressions, prove simple identities and solve equations involving secant, cosecant and cotangent → pages 53–57
- Prove and use $\sec^2 x \equiv 1 + \tan^2 x$ and $\text{cosec}^2 x \equiv 1 + \cot^2 x$ → pages 57–61
- Understand and use inverse trigonometric functions and their domain and ranges → pages 62–65

Prior knowledge check

1 Sketch the graph of $y = \sin x$ for $-180° \leqslant x \leqslant 180°$. Use your sketch to solve, for the given interval, the equations:

a $\sin x = 0.8$ **b** $\sin x = -0.4$

← Pure 1 Section 6.5

2 Prove that $\dfrac{1}{\sin x \cos x} - \dfrac{1}{\tan x} = \tan x$

← Pure 2 Section 6.3

3 Find all the solutions in the interval $0 \leqslant x \leqslant 2\pi$ to the equation $3\sin^2(2x) = 1$

← Pure 2 Section 6.6

Trigonometric functions can be used to model oscillations and resonance in bridges. You will use the functions in this chapter together with differentiation and integration in chapters 6 and 7.

3.1 Secant, cosecant and cotangent

Secant (sec), cosecant (cosec) and cotangent (cot) are known as the **reciprocal** trigonometric functions.

- $\sec x = \frac{1}{\cos x}$ (**undefined** for values of x for which $\cos x = 0$)
- $\text{cosec } x = \frac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)
- $\cot x = \frac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)

You can also write $\cot x$ in terms of $\sin x$ and $\cos x$.

- $\cot x = \frac{\cos x}{\sin x}$

Example 1

Use your calculator to write down the values of:

a sec 280° **b** cot 115°

a $\sec 280° = \frac{1}{\cos 280°} = 5.76$ (3 s.f.)

b $\cot 115° = \frac{1}{\tan 115°} = -0.466$ (3 s.f.)

Make sure your calculator is in degrees mode.

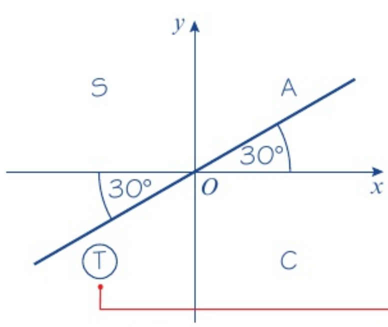
Example 2

Work out the exact values of:

a sec 210° **b** $\text{cosec } \frac{3\pi}{4}$

Exact here means give in surd form.

a $\sec 210° = \frac{1}{\cos 210°}$



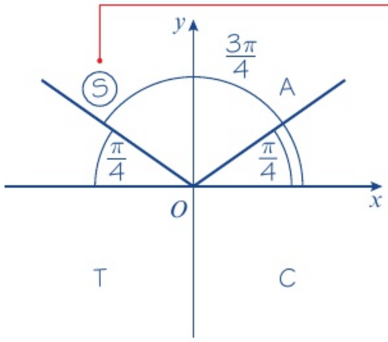
210° is in the 3rd quadrant, so cos 210° = −cos 30°

$\cos 30° = \frac{\sqrt{3}}{2}$ so $-\cos 30° = -\frac{\sqrt{3}}{2}$

So $\sec 210° = -\frac{2}{\sqrt{3}}$

Or $\sec 210° = -\frac{2\sqrt{3}}{3}$ if you rationalise the denominator.

b $\text{cosec}\,\frac{3\pi}{4} = \frac{1}{\sin\left(\frac{3\pi}{4}\right)}$



$\frac{3\pi}{4}$ is in the 2nd quadrant, so $\sin\frac{3\pi}{4} = +\sin\frac{\pi}{4}$

So $\text{cosec}\,\frac{3\pi}{4} = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$

$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

So $\text{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}$

Exercise 3A

SKILLS ANALYSIS

1 Without using your calculator, write down the sign of:

a sec 300° **b** cosec 190° **c** cot 110°
d cot 200° **e** sec 95°

2 Use your calculator to find, to 3 significant figures, the values of:

a sec 100° **b** cosec 260° **c** cosec 280°
d cot 550° **e** $\cot\frac{4\pi}{3}$ **f** sec 2.4 rad
g $\text{cosec}\,\frac{11\pi}{10}$ **h** sec 6 rad

3 Find the exact value (as an integer, fraction or surd) of each of the following:

a cosec 90° **b** cot 135° **c** sec 180°
d sec 240° **e** cosec 300° **f** cot (−45°)
g sec 60° **h** cosec (−210°) **i** sec 225°
j $\cot\frac{4\pi}{3}$ **k** $\sec\frac{11\pi}{6}$ **l** $\text{cosec}\left(-\frac{3\pi}{4}\right)$

(P) **4** Prove that $\text{cosec}(\pi - x) \equiv \text{cosec}\,x$

(P) **5** Show that cot 30° sec 30° = 2

(P) **6** Show that $\text{cosec}\,\frac{2\pi}{3} + \sec\frac{2\pi}{3} = a + b\sqrt{3}$, where a and b are real numbers to be found.

Challenge

SKILLS CREATIVITY

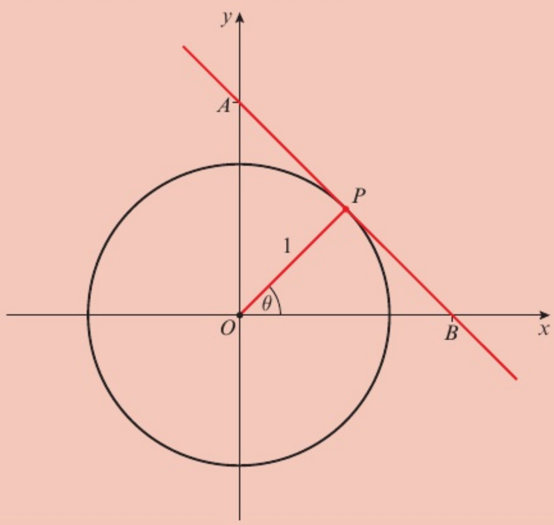
The point P lies on the unit circle, centre O. The radius OP makes an **acute angle** of θ with the positive x-axis. The tangent to the circle at P intersects the coordinate axes at points A and B.

Prove that:

a $OB = \sec\theta$

b $OA = \operatorname{cosec}\theta$

c $AP = \cot\theta$

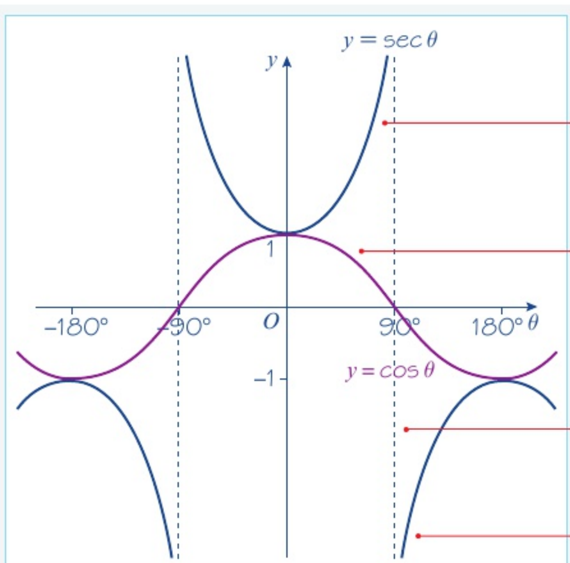


3.2 Graphs of sec x, cosec x and cot x

You can use the graphs of $y = \cos x$, $y = \sin x$ and $y = \tan x$ to sketch the graphs of their reciprocal functions.

Example 3 SKILLS INTERPRETATION

Sketch, in the **interval** $-180° \leqslant \theta \leqslant 180°$, the graph of $y = \sec\theta$



First draw the graph $y = \cos\theta$

For each value of θ, the value of $\sec\theta$ is the reciprocal of the corresponding value of $\cos\theta$.

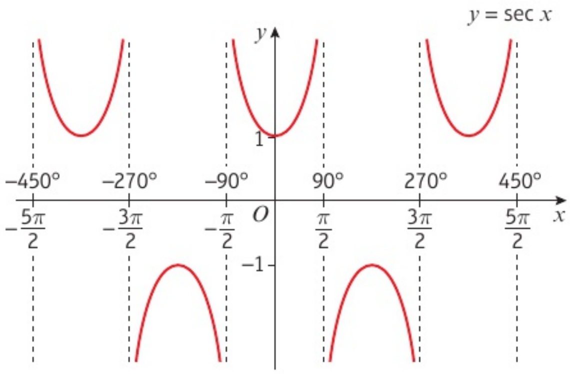
In particular: $\cos 0° = 1$, so $\sec 0° = 1$;
and $\cos 180° = -1$, so $\sec 180° = -1$

As θ approaches 90° from the left, $\cos\theta$ is +ve but approaches zero, and so $\sec\theta$ is +ve but becomes increasingly large.

At $\theta = 90°$, $\sec\theta$ is undefined and there is a vertical asymptote. This is also true for $\theta = -90°$

As θ approaches 90° from the right, $\cos\theta$ is –ve but approaches zero, and so $\sec\theta$ is –ve but becomes increasingly large negative.

- The graph of $y = \sec x$, $x \in \mathbb{R}$, has **symmetry** in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$

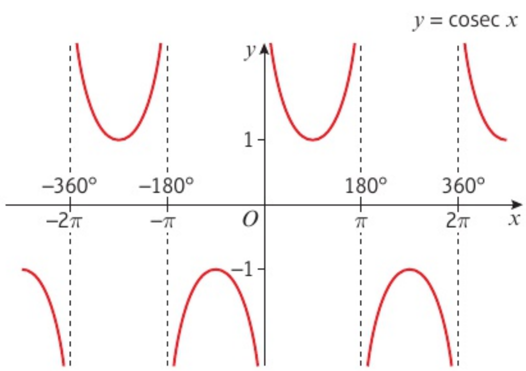


Notation The domain can also be given as $x \in \mathbb{R}, x \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$

$\mathbb{Z}$ is the symbol used for **integers**, which are the positive and negative whole numbers including 0.

- The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$ or any odd multiple of 90°
- In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or any odd multiple of $\frac{\pi}{2}$
- The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$

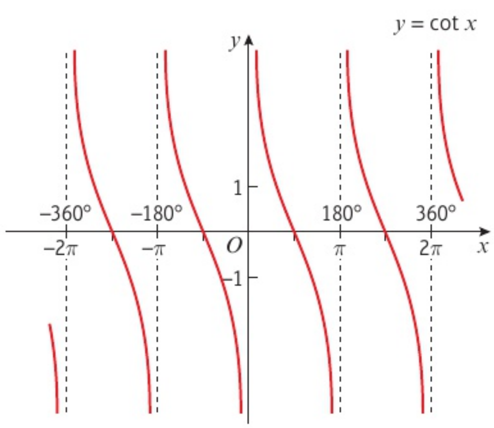
- The graph of $y = \operatorname{cosec} x$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$



Notation The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$

- The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π
- The range of $y = \operatorname{cosec} x$ is $y \leqslant -1$ or $y \geqslant 1$

- The graph of $y = \cot x$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$



- The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π
- The range of $y = \cot x$ is $y \in \mathbb{R}$

Notation The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$

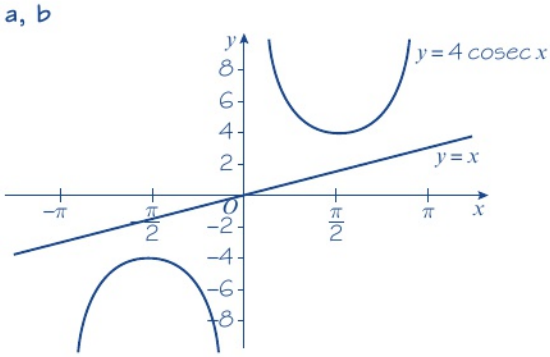
Example 4

a Sketch the graph of $y = 4\operatorname{cosec} x$, $-\pi \leqslant x \leqslant \pi$

b On the same axes, sketch the line $y = x$

c State the number of solutions to the equation $4\operatorname{cosec} x - x = 0$, $-\pi \leqslant x \leqslant \pi$

a, b



c $4\operatorname{cosec} x - x = 0$

$4\operatorname{cosec} x = x$

$y = 4\operatorname{cosec} x$ and $y = x$ do not intersect for $-\pi \leqslant x \leqslant \pi$ so the equation has no solutions in the given range.

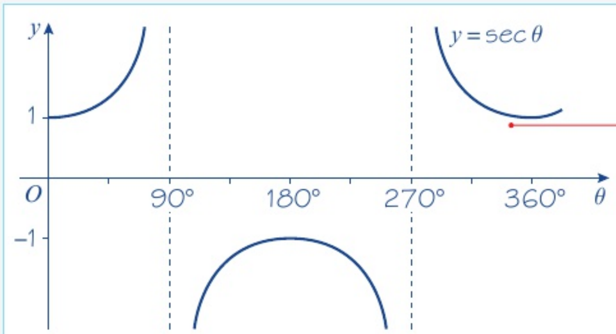
$y = 4\operatorname{cosec} x$ is a stretch of the graph of $y = \operatorname{cosec} x$, scale factor 4 in the y-direction. You only need to draw the graph for $-\pi \leqslant x \leqslant \pi$

Problem-solving

The solutions to the equation $f(x) = g(x)$ correspond to the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$

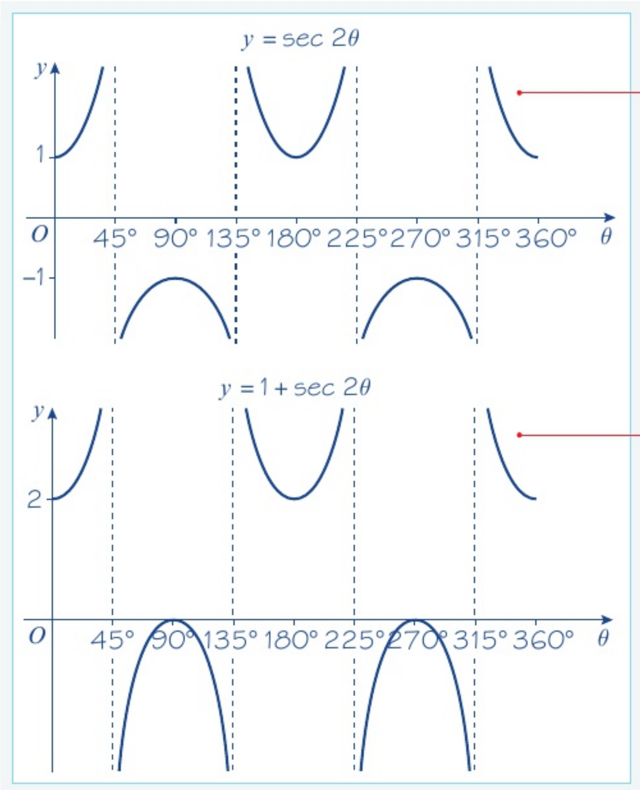
Example 5

Sketch, in the interval $0° \leqslant \theta \leqslant 360°$, the graph of $y = 1 + \sec 2\theta$



Online Explore transformations of the graphs of reciprocal trigonometric functions using technology.

Step 1
Draw the graph of $y = \sec \theta$



Step 2
Stretch in the θ-direction with scale factor $\frac{1}{2}$

Step 3
Translate by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Exercise 3B

SKILLS INTERPRETATION

1 Sketch, in the interval $-540° \leqslant \theta \leqslant 540°$, the graphs of:

a $y = \sec\theta$ **b** $y = \text{cosec}\,\theta$ **c** $y = \cot\theta$

2 **a** Sketch, on the same set of axes, in the interval $-\pi \leqslant x \leqslant \pi$, the graphs of $y = \cot x$ and $y = -x$

b **Deduce** the number of solutions of the equation $\cot x + x = 0$ in the interval $-\pi \leqslant x \leqslant \pi$

3 **a** Sketch, on the same set of axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of $y = \sec\theta$ and $y = -\cos\theta$

b Explain how your graphs show that $\sec\theta = -\cos\theta$ has no solutions.

4 **a** Sketch, on the same set of axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of $y = \cot\theta$ and $y = \sin 2\theta$

b Deduce the number of solutions of the equation $\cot\theta = \sin 2\theta$ in the interval $0° \leqslant \theta \leqslant 360°$

5 **a** Sketch on separate axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of $y = \tan\theta$ and $y = \cot(\theta + 90°)$

b Hence, state a relationship between $\tan\theta$ and $\cot(\theta + 90°)$

P **6 a** Describe the relationships between the graphs of:

i $y = \tan\left(\theta + \frac{\pi}{2}\right)$ and $y = \tan\theta$ **ii** $y = \cot(-\theta)$ and $y = \cot\theta$

iii $y = \operatorname{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $y = \operatorname{cosec}\theta$ **iv** $y = \sec\left(\theta - \frac{\pi}{4}\right)$ and $y = \sec\theta$

b By considering the graphs of $y = \tan\left(\theta + \frac{\pi}{2}\right)$, $y = \cot(-\theta)$, $y = \operatorname{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $y = \sec\left(\theta - \frac{\pi}{4}\right)$, state which pairs of functions are equal.

P **7** Sketch on separate axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of:

a $y = \sec 2\theta$ **b** $y = -\operatorname{cosec}\theta$ **c** $y = 1 + \sec\theta$

d $y = \operatorname{cosec}(\theta - 30°)$ **e** $y = 2\sec(\theta - 60°)$ **f** $y = \operatorname{cosec}(2\theta + 60°)$

g $y = -\cot(2\theta)$ **h** $y = 1 - 2\sec\theta$

In each case, show the coordinates of any maximum and minimum points, and of any points at which the curve meets the axes.

8 Write down the periods of the following functions. Give your answers in terms of π.

a $\sec 3\theta$ **b** $\operatorname{cosec}\frac{1}{2}\theta$ **c** $2\cot\theta$ **d** $\sec(-\theta)$

E/P **9 a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 3 + 5\operatorname{cosec}x$ **(3 marks)**

b Hence deduce the range of values of k for which the equation $3 + 5\operatorname{cosec}x = k$ has no solutions. **(2 marks)**

E/P **10 a** Sketch the graph of $y = 1 + 2\sec\theta$ in the interval $-\pi \leqslant \theta \leqslant 2\pi$ **(3 marks)**

b Write down the θ-coordinates of points at which the gradient is zero. **(2 marks)**

c Deduce the maximum and minimum values of $\frac{1}{1 + 2\sec\theta}$ and give the smallest positive values of θ at which they occur. **(4 marks)**

3.3 Using sec x, cosec x and cot x

You need to be able to simplify expressions, prove identities and solve equations involving sec x, cosec x and cot x.

- **sec x = k and cosec x = k have no solutions for $-1 < k < 1$**

Example 6

Simplify:

a $\sin\theta\cot\theta\sec\theta$

b $\sin\theta\cos\theta(\sec\theta + \operatorname{cosec}\theta)$

a $\sin\theta \cot\theta \sec\theta$

$\equiv \sin\theta \times \frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos\theta}$

$\equiv 1$

Write the expression in terms of sin and cos, using $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$ and $\sec\theta \equiv \frac{1}{\cos\theta}$

b $\sec\theta + \operatorname{cosec}\theta \equiv \frac{1}{\cos\theta} + \frac{1}{\sin\theta}$

Write the expression in terms of sin and cos, using $\sec\theta \equiv \frac{1}{\cos\theta}$ and $\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$

$\equiv \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}$

Put over a common denominator.

So $\sin\theta\cos\theta(\sec\theta + \operatorname{cosec}\theta) = \sin\theta + \cos\theta$

Multiply both sides by $\sin\theta\cos\theta$.

Example 7

a Prove that $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta} \equiv \cos^3\theta$

b Hence explain why the equation $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta} = 8$ has no solutions.

a Consider the LHS:

The numerator $\cot\theta \operatorname{cosec}\theta$

$\equiv \frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} \equiv \frac{\cos\theta}{\sin^2\theta}$

Write the expression in terms of sin and cos, using $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$ and $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

The denominator $\sec^2\theta + \operatorname{cosec}^2\theta$

$\equiv \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$

Write the expression in terms of sin and cos, using $\sec^2\theta \equiv \left(\frac{1}{\cos\theta}\right)^2 \equiv \frac{1}{\cos^2\theta}$ and $\operatorname{cosec}^2\theta \equiv \frac{1}{\sin^2\theta}$

$\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$

$\equiv \frac{1}{\cos^2\theta\sin^2\theta}$

Remember that $\sin^2\theta + \cos^2\theta \equiv 1$

So $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta}$

$\equiv \left(\frac{\cos\theta}{\sin^2\theta}\right) \div \left(\frac{1}{\cos^2\theta\sin^2\theta}\right)$

$\equiv \frac{\cos\theta}{\sin^2\theta} \times \frac{\cos^2\theta\sin^2\theta}{1}$

Remember to invert the fraction when changing from ÷ sign to ×.

$\equiv \cos^3\theta$

b Since $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta} \equiv \cos^3\theta$ we are required to solve the equation $\cos^3\theta = 8$

$\cos^3\theta = 8 \Rightarrow \cos\theta = 2$ which has no solutions since $-1 \leqslant \cos\theta \leqslant 1$

Problem-solving

Write down the equivalent equation, and state the range of possible values for $\cos\theta$.

Example 8

Solve the equations:

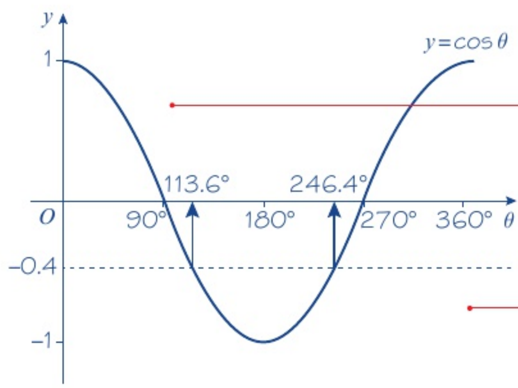
a $\sec\theta = -2.5$ **b** $\cot 2\theta = 0.6$

in the interval $0° \leqslant \theta \leqslant 360°$

a $\frac{1}{\cos\theta} = -2.5$

Substitute $\frac{1}{\cos\theta}$ for $\sec\theta$ and then simplify to get an equation in the form $\cos\theta = k$

$\cos\theta = \frac{1}{-2.5} = -0.4$



Sketch the graph of $y = \cos\theta$ for the given interval. The graph is **symmetrical** about $\theta = 180°$
Find the principal value using your calculator then subtract this from 360° to find the second solution.

You could also find all the solutions using a CAST diagram. This method is shown for part **b** below.

$\theta = 113.6°, 246.4° = 114°, 246°$ (3 s.f.)

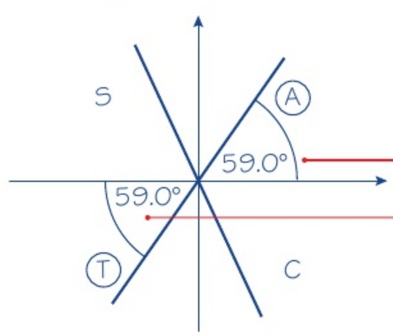
b $\frac{1}{\tan 2\theta} = 0.6$

Substitute $\frac{1}{\tan 2\theta}$ for $\cot 2\theta$ and then simplify to get an equation in the form $\tan 2\theta = k$

$\tan 2\theta = \frac{1}{0.6} = \frac{5}{3}$

Let $X = 2\theta$, so that you are solving

$\tan X = \frac{5}{3}$, in the interval $0° \leqslant X \leqslant 720°$



Draw the CAST diagram, with the acute angle $X = \tan^{-1}\left(\frac{5}{3}\right)$ drawn to the horizontal in the 1st and 3rd quadrants.

$X = 59.0°, 239.0°, 419.0°, 599.0°$

Remember that $X = 2\theta$

so $\theta = 29.5°, 120°, 210°, 300°$ (3 s.f.)

Exercise 3C

SKILLS ANALYSIS

1 Rewrite the following as powers of $\sec\theta$, $\csc\theta$ or $\cot\theta$.

a $\dfrac{1}{\sin^3\theta}$ **b** $\dfrac{4}{\tan^6\theta}$ **c** $\dfrac{1}{2\cos^2\theta}$ **d** $\dfrac{1-\sin^2\theta}{\sin^2\theta}$

e $\dfrac{\sec\theta}{\cos^4\theta}$ **f** $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$ **g** $\dfrac{2}{\sqrt{\tan\theta}}$ **h** $\dfrac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}$

2 Write down the value(s) of $\cot x$ in each of the following equations:

a $5\sin x = 4\cos x$ **b** $\tan x = -2$ **c** $\dfrac{3\sin x}{\cos x} = \dfrac{\cos x}{\sin x}$

3 Using the definitions of sec, cosec, cot and tan, simplify the following expressions.

a $\sin\theta\cot\theta$ **b** $\tan\theta\cot\theta$

c $\tan 2\theta \operatorname{cosec} 2\theta$ **d** $\cos\theta\sin\theta(\cot\theta+\tan\theta)$

e $\sin^3 x\operatorname{cosec} x + \cos^3 x\sec x$ **f** $\sec A - \sec A\sin^2 A$

g $\sec^2 x\cos^5 x + \cot x\operatorname{cosec} x\sin^4 x$

(P) **4** Prove that:

a $\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$ **b** $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta\sec\theta$

c $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$ **d** $(1-\cos x)(1+\sec x) \equiv \sin x\tan x$

e $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv 2\sec x$ **f** $\dfrac{\cos\theta}{1+\cot\theta} \equiv \dfrac{\sin\theta}{1+\tan\theta}$

(P) **5** Solve the following equations for values of θ in the interval $0° \leqslant \theta \leqslant 360°$
Give your answers to 3 significant figures where necessary.

a $\sec\theta = \sqrt{2}$ **b** $\operatorname{cosec}\theta = -3$ **c** $5\cot\theta = -2$ **d** $\operatorname{cosec}\theta = 2$

e $3\sec^2\theta - 4 = 0$ **f** $5\cos\theta = 3\cot\theta$ **g** $\cot^2\theta - 8\tan\theta = 0$ **h** $2\sin\theta = \operatorname{cosec}\theta$

(P) **6** Solve the following equations for values of θ in the interval $-180° \leqslant \theta \leqslant 180°$

a $\operatorname{cosec}\theta = 1$ **b** $\sec\theta = -3$ **c** $\cot\theta = 3.45$

d $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$ **e** $\sec\theta = 2\cos\theta$ **f** $3\cot\theta = 2\sin\theta$

g $\operatorname{cosec} 2\theta = 4$ **h** $2\cot^2\theta - \cot\theta - 5 = 0$

(P) **7** Solve the following equations for values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$
Give your answers in terms of π.

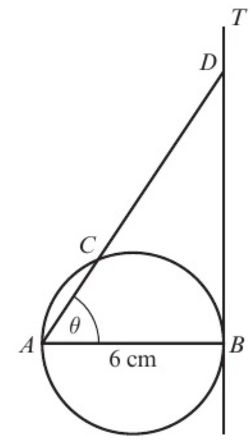
a $\sec\theta = -1$ **b** $\cot\theta = -\sqrt{3}$

c $\operatorname{cosec}\dfrac{\theta}{2} = \dfrac{2\sqrt{3}}{3}$ **d** $\sec\theta = \sqrt{2}\tan\theta,\ \theta \neq \dfrac{\pi}{2},\ \theta \neq \dfrac{3\pi}{2}$

E/P **8** In the diagram, $AB = 6$ cm is the diameter of the circle and BT is the tangent to the circle at B. The **chord** AC is extended to meet this tangent at D and $\angle DAB = \theta$

a Show that $CD = 6(\sec\theta - \cos\theta)$ cm. **(4 marks)**

b Given that $CD = 16$ cm, calculate the length of the chord AC. **(3 marks)**



Problem-solving

AB is the diameter of the circle, so $\angle ACB = 90°$

E/P **9** **a** Prove that $\frac{\operatorname{cosec} x - \cot x}{1 - \cos x} \equiv \operatorname{cosec} x$ **(4 marks)**

b Hence solve, in the interval $-\pi \leqslant x \leqslant \pi$, the equation $\frac{\operatorname{cosec} x - \cot x}{1 - \cos x} = 2$ **(3 marks)**

E/P **10** **a** Prove that $\frac{\sin x \tan x}{1 - \cos x} - 1 \equiv \sec x$ **(4 marks)**

b Hence explain why the equation $\frac{\sin x \tan x}{1 - \cos x} - 1 = -\frac{1}{2}$ has no solutions. **(1 mark)**

E/P **11** Solve, in the interval $0° \leqslant x \leqslant 360°$, the equation $\frac{1 + \cot x}{1 + \tan x} = 5$ **(8 marks)**

Problem-solving

Use the relationship $\cot x = \frac{1}{\tan x}$ to form a quadratic equation in $\tan x$. ← **Pure 1 Section 2.1**

3.4 Trigonometric identities

You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities.

- **$1 + \tan^2 x \equiv \sec^2 x$**
- **$1 + \cot^2 x \equiv \operatorname{cosec}^2 x$**

Example 9 SKILLS ANALYSIS

a Prove that $1 + \tan^2 x \equiv \sec^2 x$

b Prove that $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

a $\sin^2 x + \cos^2 x \equiv 1$

$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x}$

$\left(\frac{\sin x}{\cos x}\right)^2 + 1 \equiv \left(\frac{1}{\cos x}\right)^2$

so $1 + \tan^2 x \equiv \sec^2 x$

b $\sin^2 x + \cos^2 x \equiv 1$

$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \equiv \frac{1}{\sin^2 x}$

$1 + \left(\frac{\cos x}{\sin x}\right)^2 \equiv \left(\frac{1}{\sin x}\right)^2$

so $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

Unless otherwise stated, you can assume the identity $\sin^2 x + \cos^2 x \equiv 1$ in proofs involving cosec, sec and cot in your exam.

Divide both sides of the identity by $\cos^2 x$.

Use $\tan x \equiv \frac{\sin x}{\cos x}$ and $\sec x \equiv \frac{1}{\cos x}$

Divide both sides of the identity by $\sin^2 x$.

Use $\cot x \equiv \frac{\cos x}{\sin x}$ and $\operatorname{cosec} x \equiv \frac{1}{\sin x}$

Example 10

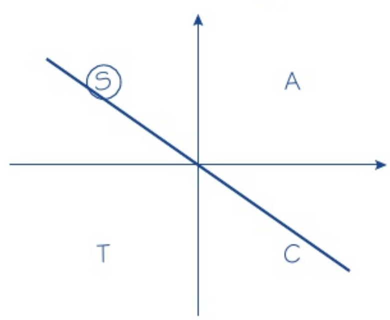
Given that $\tan A = -\frac{5}{12}$, and that angle A is **obtuse**, find the exact values of:

a $\sec A$ **b** $\sin A$

a Using $1 + \tan^2 A \equiv \sec^2 A$

$\sec^2 A = 1 + \frac{25}{144} = \frac{169}{144}$

$\sec A = \pm\frac{13}{12}$



$\sec A = -\frac{13}{12}$

b Using $\tan A \equiv \frac{\sin A}{\cos A}$

$\sin A \equiv \tan A \cos A$

So $\sin A = \left(-\frac{5}{12}\right) \times \left(-\frac{12}{13}\right)$

$= \frac{5}{13}$

$\tan^2 A = \frac{25}{144}$

Problem-solving

You are told that A is obtuse. This means it lies in the second quadrant, so $\cos A$ is negative, and $\sec A$ is also negative.

$\cos A = -\frac{12}{13}$, since $\cos A = \frac{1}{\sec A}$

Example 11

Prove the identities:

a $\operatorname{cosec}^4\theta - \cot^4\theta \equiv \dfrac{1+\cos^2\theta}{1-\cos^2\theta}$

b $\sec^2\theta - \cos^2\theta \equiv \sin^2\theta(1+\sec^2\theta)$

a LHS $= \operatorname{cosec}^4\theta - \cot^4\theta$

$\equiv (\operatorname{cosec}^2\theta + \cot^2\theta)(\operatorname{cosec}^2\theta - \cot^2\theta)$

$\equiv \operatorname{cosec}^2\theta + \cot^2\theta$

$\equiv \dfrac{1}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta}$

$\equiv \dfrac{1+\cos^2\theta}{\sin^2\theta}$

$\equiv \dfrac{1+\cos^2\theta}{1-\cos^2\theta} =$ RHS

This is the difference of two squares, so factorise.

As $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$, so $\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$

Using $\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$, $\cot\theta \equiv \dfrac{\cos\theta}{\sin\theta}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$

b RHS $= \sin^2\theta + \sin^2\theta\sec^2\theta$

$\equiv \sin^2\theta + \dfrac{\sin^2\theta}{\cos^2\theta}$

$\equiv \sin^2\theta + \tan^2\theta$

$\equiv (1-\cos^2\theta) + (\sec^2\theta - 1)$

$\equiv \sec^2\theta - \cos^2\theta$

$\equiv$ LHS

Write in terms of $\sin\theta$ and $\cos\theta$.

Use $\sec\theta \equiv \dfrac{1}{\cos\theta}$

$\dfrac{\sin^2\theta}{\cos^2\theta} \equiv \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 \equiv \tan^2\theta$

Look at LHS. It is in terms of $\cos^2\theta$ and $\sec^2\theta$, so use $\sin^2\theta + \cos^2\theta \equiv 1$ and $1 + \tan^2\theta \equiv \sec^2\theta$

Problem-solving

You can start from either the LHS or the RHS when proving an identity. Try starting with the LHS using $\cos^2\theta \equiv 1 - \sin^2\theta$ and $\sec^2\theta \equiv 1 + \tan^2\theta$

Example 12

Solve the equation $4\operatorname{cosec}^2\theta - 9 = \cot\theta$ in the interval $0° \leqslant \theta \leqslant 360°$

The equation can be rewritten as

$4(1+\cot^2\theta) - 9 = \cot\theta$

So $4\cot^2\theta - \cot\theta - 5 = 0$

$(4\cot\theta - 5)(\cot\theta + 1) = 0$

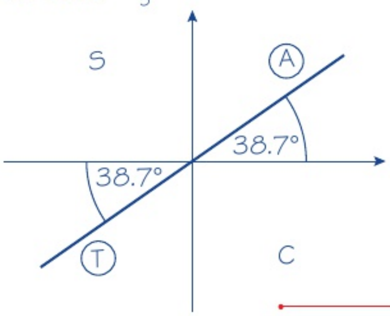
So $\cot\theta = \frac{5}{4}$ or $\cot\theta = -1$

$\therefore$ $\tan\theta = \frac{4}{5}$ or $\tan\theta = -1$

This is a quadratic equation. You need to write it in terms of one trigonometric function only, so use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Factorise, or solve using the quadratic formula.

For $\tan\theta = \frac{4}{5}$

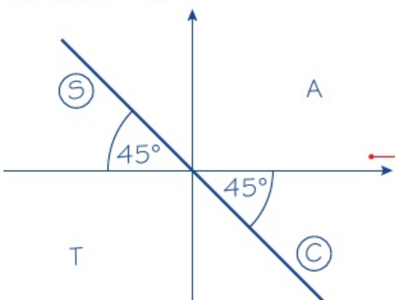


$\theta = 38.7°, 219°$ (3 s.f.)

As $\tan\theta$ is +ve, θ is in the 1st and 3rd quadrants. The acute angle to the horizontal is $\tan^{-1}\left(\frac{4}{5}\right) = 38.7°$

If α is the value the calculator gives for $\tan^{-1}\left(\frac{4}{5}\right)$, then the solutions are α and $(180° + \alpha)$

For $\tan\theta = -1$



$\theta = 135°, 315°$

As $\tan\theta$ is –ve, θ is in the 2nd and 4th quadrants. The acute angle to the horizontal is $\tan^{-1} 1 = 45°$

If α is the value the calculator gives for $\tan^{-1}(-1)$, then the solutions are $(180° + \alpha)$ and $(360° + \alpha)$, as α is not in the given interval.

Online Solve this equation numerically using your calculator.

Exercise 3D

SKILLS ANALYSIS

Give answers to 3 significant figures where necessary.

1 Simplify each of the following expressions.

a $1 + \tan^2\left(\frac{\theta}{2}\right)$ b $(\sec\theta - 1)(\sec\theta + 1)$ c $\tan^2\theta(\operatorname{cosec}^2\theta - 1)$

d $(\sec^2\theta - 1)\cot\theta$ e $(\operatorname{cosec}^2\theta - \cot^2\theta)^2$ f $2 - \tan^2\theta + \sec^2\theta$

g $\dfrac{\tan\theta\sec\theta}{1 + \tan^2\theta}$ h $(1 - \sin^2\theta)(1 + \tan^2\theta)$ i $\dfrac{\operatorname{cosec}\theta\cot\theta}{1 + \cot^2\theta}$

j $(\sec^4\theta - 2\sec^2\theta\tan^2\theta + \tan^4\theta)$ k $4\operatorname{cosec}^2 2\theta + 4\operatorname{cosec}^2 2\theta\cot^2 2\theta$

(P) 2 Given that $\operatorname{cosec} x = \dfrac{k}{\operatorname{cosec} x}$, where $k > 1$, find, in terms of k, possible values of $\cot x$.

3 Given that $\cot\theta = -\sqrt{3}$, and that $90° < \theta < 180°$, find the exact values of:

a $\sin\theta$ b $\cos\theta$

4 Given that $\tan\theta = \frac{3}{4}$, and that $180° < \theta < 270°$, find the exact values of:

a $\sec\theta$ b $\cos\theta$ c $\sin\theta$

5 Given that $\cos\theta = \frac{24}{25}$, and that θ is a reflex angle, find the exact values of:

a $\tan\theta$ b $\operatorname{cosec}\theta$

(P) **6** Prove the following identities:

a $\sec^4\theta - \tan^4\theta \equiv \sec^2\theta + \tan^2\theta$

b $\operatorname{cosec}^2 x - \sin^2 x \equiv \cot^2 x + \cos^2 x$

c $\sec^2 A(\cot^2 A - \cos^2 A) \equiv \cot^2 A$

d $1 - \cos^2\theta \equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$

e $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} \equiv 1 - 2\sin^2 A$

f $\sec^2\theta + \operatorname{cosec}^2\theta \equiv \sec^2\theta \operatorname{cosec}^2\theta$

g $\operatorname{cosec} A \sec^2 A \equiv \operatorname{cosec} A + \tan A \sec A$

h $(\sec\theta - \sin\theta)(\sec\theta + \sin\theta) \equiv \tan^2\theta + \cos^2\theta$

(P) **7** Given that $3\tan^2\theta + 4\sec^2\theta = 5$, and that θ is obtuse, find the exact value of $\sin\theta$.

(P) **8** Solve the following equations in the given intervals:

a $\sec^2\theta = 3\tan\theta,\ 0° \leqslant \theta \leqslant 360°$

b $\tan^2\theta - 2\sec\theta + 1 = 0,\ -\pi \leqslant \theta \leqslant \pi$

c $\operatorname{cosec}^2\theta + 1 = 3\cot\theta,\ -180° \leqslant \theta \leqslant 180°$

d $\cot\theta = 1 - \operatorname{cosec}^2\theta,\ 0 \leqslant \theta \leqslant 2\pi$

e $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta,\ 0° \leqslant \theta \leqslant 360°$

f $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta,\ 0 \leqslant \theta \leqslant \pi$

g $\tan^2 2\theta = \sec 2\theta - 1,\ 0° \leqslant \theta \leqslant 180°$

h $\sec^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 1,\ 0 \leqslant \theta \leqslant 2\pi$

(E/P) **9** Given that $\tan^2 k = 2\sec k$,

a find the value of $\sec k$ **(4 marks)**

b deduce that $\cos k = \sqrt{2} - 1$. **(2 marks)**

c Hence solve, in the interval $0° \leqslant k \leqslant 360°$, $\tan^2 k = 2\sec k$, giving your answers to 1 decimal place. **(3 marks)**

(E/P) **10** Given that $a = 4\sec x$, $b = \cos x$ and $c = \cot x$,

a express b in terms of a **(2 marks)**

b show that $c^2 = \dfrac{16}{a^2 - 16}$ **(3 marks)**

(E/P) **11** Given that $x = \sec\theta + \tan\theta$,

a show that $\dfrac{1}{x} = \sec\theta - \tan\theta$ **(3 marks)**

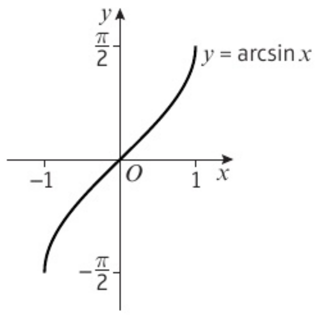
b Hence express $x^2 + \dfrac{1}{x^2} + 2$ in terms of θ, in its simplest form. **(5 marks)**

(E/P) **12** Given that $2\sec^2\theta - \tan^2\theta = p$, show that $\operatorname{cosec}^2\theta = \dfrac{p - 1}{p - 2}$, $p \neq 2$ **(5 marks)**

3.5 Inverse trigonometric functions

You need to understand and use the inverse trigonometric functions arcsin x, arccos x and arctan x and their graphs.

- **The inverse function of sin x is called arcsin x.**



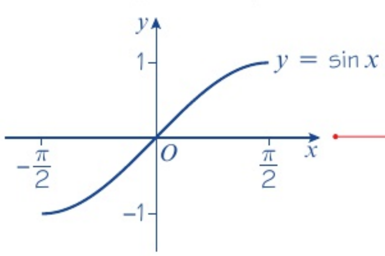
Hint The $\sin^{-1}$ function on your calculator will give principal values in the same range as arcsin.

- **The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$**
- **The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$**

Example 13

Sketch the graph of $y = \arcsin x$

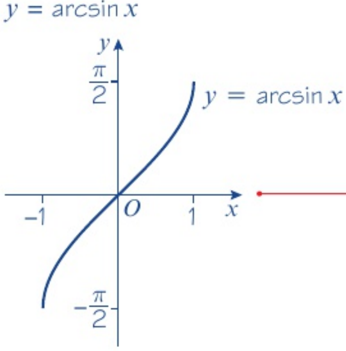
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$



Step 1
Draw the graph of $y = \sin x$, with the restricted domain of $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Restricting the domain ensures that the inverse function exists since $y = \sin x$ is a **one-to-one** function for the restricted domain. Only one-to-one functions have inverses. ← Pure 1 Section 2.3

$y = \arcsin x$



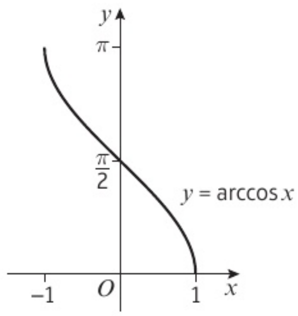
Step 2
Reflect in the line $y = x$

The domain of arcsin x is $-1 \leqslant x \leqslant 1$; the range is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$

Remember that the x and y coordinates of points interchange (swap) when reflecting in $y = x$
For example:

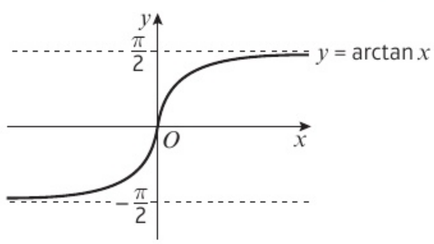
$$\left(\frac{\pi}{2}, 1\right) \rightarrow \left(1, \frac{\pi}{2}\right)$$

■ The inverse function of $\cos x$ is called $\arccos x$.



- The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$

■ The inverse function of $\tan x$ is called $\arctan x$.



Watch out Unlike $\arcsin x$ and $\arccos x$, the function $\arctan x$ is defined for all real values of x.

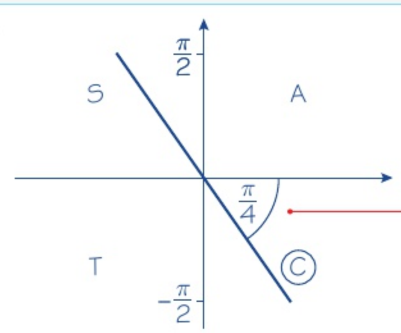
- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} \leqslant \arctan x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arctan x \leqslant 90°$

Example 14

Work out, in radians, the values of:

a $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ **b** $\arccos(-1)$ **c** $\arctan(\sqrt{3})$

a



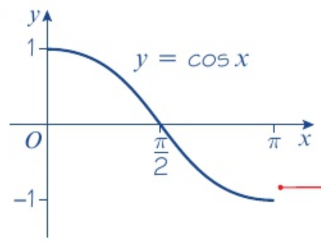
$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

You need to solve, in the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, the equation $\sin x = -\frac{\sqrt{2}}{2}$

The angle to the horizontal is $\frac{\pi}{4}$ and, as sin is −ve, it is in the 4th quadrant.

Online Use your calculator to evaluate inverse trigonometric functions in radians.

b

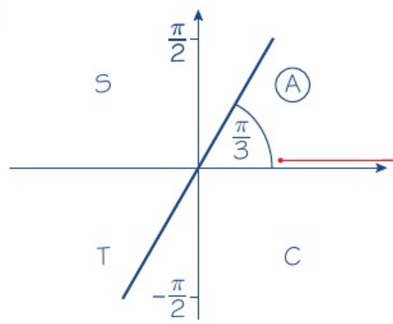


You need to solve, in the interval $0 \leqslant x \leqslant \pi$, the equation $\cos x = -1$

Draw the graph of $y = \cos x$

$\arccos(-1) = \pi$

c



You need to solve, in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the equation $\tan x = \sqrt{3}$

The angle to the horizontal is $\frac{\pi}{3}$ and, as tan is +ve, it is in the 1st quadrant.

$\arctan(\sqrt{3}) = \frac{\pi}{3}$

You can verify these results using the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ functions on your calculator.

Exercise 3E SKILLS INTERPRETATION

In this exercise, all angles are given in radians.

1 Without using a calculator, work out, giving your answer in terms of π:

a $\arccos(0)$ b $\arcsin(1)$ c $\arctan(-1)$ d $\arcsin\left(-\frac{1}{2}\right)$

e $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ f $\arctan-\frac{1}{\sqrt{3}}$ g $\arcsin\left(\sin\frac{\pi}{3}\right)$ h $\arcsin\left(\sin\frac{2\pi}{3}\right)$

2 Find:

a $\arcsin\left(\frac{1}{2}\right) + \arcsin\left(-\frac{1}{2}\right)$ b $\arccos\left(\frac{1}{2}\right) - \arccos\left(-\frac{1}{2}\right)$ c $\arctan(1) - \arctan(-1)$

P 3 Without using a calculator, work out the values of:

a $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$ b $\sin\left(\arcsin\left(-\frac{1}{2}\right)\right)$

c $\tan(\arctan(-1))$ d $\cos(\arccos 0)$

P 4 Without using a calculator, work out the exact values of:

a $\sin\left(\arccos\left(\frac{1}{2}\right)\right)$ b $\cos\left(\arcsin\left(-\frac{1}{2}\right)\right)$ c $\tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

d $\sec(\arctan(\sqrt{3}))$ e $\operatorname{cosec}(\arcsin(-1))$ f $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$

P **5** Given that $\arcsin k = \alpha$, where $0 < k < 1$, write down the first two positive values of x satisfying the equation $\sin x = k$

E/P **6** Given that x satisfies $\arcsin x = k$, where $0 < k < \frac{\pi}{2}$,

a state the range of possible values of x **(1 mark)**

b express, in terms of x,

i $\cos k$ **ii** $\tan k$ **(4 marks)**

Given, instead, that $-\frac{\pi}{2} < k < 0$,

c how, if at all, are your answers to part **b** affected? **(2 marks)**

P **7** Sketch the graphs of:

a $y = \frac{\pi}{2} + 2\arcsin x$ **b** $y = \pi - \arctan x$

c $y = \arccos(2x + 1)$ **d** $y = -2\arcsin(-x)$

E/P **8** The function f is defined as $f: x \mapsto \arcsin x, -1 \leqslant x \leqslant 1$, and the function g is such that $g(x) = f(2x)$

a Sketch the graph of $y = f(x)$ and state the range of f. **(3 marks)**

b Sketch the graph of $y = g(x)$ **(2 marks)**

c Define g in the form $g: x \mapsto \ldots$ and give the domain of g. **(3 marks)**

d Define g^{-1} in the form $g^{-1}: x \mapsto \ldots$ **(2 marks)**

E/P **9** **a** Prove that for $0 \leqslant x \leqslant 1$, $\arccos x = \arcsin\sqrt{1 - x^2}$ **(4 marks)**

b Give a reason why this result is not true for $-1 \leqslant x \leqslant 0$ **(2 marks)**

Challenge

SKILLS INTERPRETATION

a Sketch the graph of $y = \sec x$, with the restricted domain $0 \leqslant x \leqslant \pi, x \neq \frac{\pi}{2}$

b Given that arcsec x is the inverse function of $\sec x, 0 \leqslant x \leqslant \pi, x \neq \frac{\pi}{2}$, sketch the graph of $y = \text{arcsec}\, x$ and state the range of arcsec x.

Chapter review 3

Give any non-exact answers to equations to 1 decimal place.

E/P **1** Solve $\tan x = 2\cot x$, in the interval $-180° \leqslant x \leqslant 90°$ **(4 marks)**

E/P **2** Given that $p = 2\sec\theta$ and $q = 4\cos\theta$, express p in terms of q. **(4 marks)**

E/P **3** Given that $p = \sin\theta$ and $q = 4\cot\theta$, show that $p^2q^2 = 16(1 - p^2)$ **(4 marks)**

P **4** **a** Solve, in the interval $0° < \theta < 180°$,

i $\operatorname{cosec}\theta = 2\cot\theta$ **ii** $2\cot^2\theta = 7\operatorname{cosec}\theta - 8$

b Solve, in the interval $0° \leqslant \theta \leqslant 360°$,

i $\sec(2\theta - 15°) = \operatorname{cosec} 135°$ **ii** $\sec^2\theta + \tan\theta = 3$

c Solve, in the interval $0 \leqslant x \leqslant 2\pi$,

i $\operatorname{cosec}\left(x + \frac{\pi}{15}\right) = -\sqrt{2}$ **ii** $\sec^2 x = \frac{4}{3}$

E/P **5** Given that $5\sin x\cos y + 4\cos x\sin y = 0$, and that $\cot x = 2$, find the value of $\cot y$. **(5 marks)**

P **6** Prove that:

a $(\tan\theta + \cot\theta)(\sin\theta + \cos\theta) \equiv \sec\theta + \operatorname{cosec}\theta$

b $\dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x} \equiv \sec^2 x$

c $(1 - \sin x)(1 + \operatorname{cosec} x) \equiv \cos x\cot x$

d $\dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\cos x}{1 + \sin x} \equiv 2\tan x$

e $\dfrac{1}{\operatorname{cosec}\theta - 1} + \dfrac{1}{\operatorname{cosec}\theta + 1} \equiv 2\sec\theta\tan\theta$

f $\dfrac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{1 + \tan^2\theta} \equiv \cos^2\theta$

E/P **7** **a** Prove that $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} \equiv 2\operatorname{cosec} x$ **(4 marks)**

b Hence solve, in the interval $-2\pi \leqslant x \leqslant 2\pi$, $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = -\dfrac{4}{\sqrt{3}}$ **(4 marks)**

E/P **8** Prove that $\dfrac{1 + \cos\theta}{1 - \cos\theta} \equiv (\operatorname{cosec}\theta + \cot\theta)^2$ **(4 marks)**

E **9** Given that $\sec A = -3$, where $\frac{\pi}{2} < A < \pi$,

a calculate the exact value of $\tan A$ **(3 marks)**

b show that $\operatorname{cosec} A = \dfrac{3\sqrt{2}}{4}$ **(3 marks)**

10 Given that $\sec\theta = k$, $|k| \geqslant 1$, and that θ is obtuse, express in terms of k:

a $\cos\theta$ **b** $\tan^2\theta$ **c** $\cot\theta$ **d** $\operatorname{cosec}\theta$

E **11** Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec\left(x + \frac{\pi}{4}\right) = 2$, giving your answers in terms of π. **(5 marks)**

E/P **12** Find, in terms of π, the value of $\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$ **(4 marks)**

E/P **13** Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec^2 x - \frac{2\sqrt{3}}{3}\tan x - 2 = 0$, giving your answers in terms of π. **(5 marks)**

E/P **14** **a** Factorise $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2$ **(2 marks)**

b Hence solve $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2 = 0$ in the interval $0° \leqslant x \leqslant 360°$ **(4 marks)**

E/P **15** Given that $\arctan(x - 2) = -\frac{\pi}{3}$, find the value of x. **(3 marks)**

E **16** On the same set of axes, sketch the graphs of $y = \cos x$, $0 \leqslant x \leqslant \pi$, and $y = \arccos x$, $-1 \leqslant x \leqslant 1$, showing the coordinates of points at which the curves meet the axes. **(4 marks)**

E/P **17** **a** Given that $\sec x + \tan x = -3$, use the identity $1 + \tan^2 x \equiv \sec^2 x$ to find the value of $\sec x - \tan x$ **(3 marks)**

b Deduce the values of:

i $\sec x$ **ii** $\tan x$ **(3 marks)**

c Hence solve, in the interval $-180° \leqslant x \leqslant 180°$, $\sec x + \tan x = -3$ **(3 marks)**

E/P **18** Given that $p = \sec\theta - \tan\theta$ and $q = \sec\theta + \tan\theta$, show that $p = \frac{1}{q}$ **(4 marks)**

E/P **19** **a** Prove that $\sec^4\theta - \tan^4\theta = \sec^2\theta + \tan^2\theta$ **(3 marks)**

b Hence solve, in the interval $-180° \leqslant \theta \leqslant 180°$, $\sec^4\theta = \tan^4\theta + 3\tan\theta$ **(4 marks)**

P **20** **a** Sketch the graph of $y = \sin x$ and shade in the area representing $\int_0^{\frac{\pi}{2}} \sin x \, dx$.

b Sketch the graph of $y = \arcsin x$ and shade in the area representing $\int_0^1 \arcsin x \, dx$.

c By considering the shaded areas, explain why $\int_0^{\frac{\pi}{2}} \sin x \, dx + \int_0^1 \arcsin x \, dx = \frac{\pi}{2}$

P **21** Show that $\cot 60° \sec 60° = \frac{2\sqrt{3}}{3}$

E/P **22** **a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 2 - 3\sec x$ **(3 marks)**

b Hence deduce the range of values of k for which the equation $2 - 3\sec x = k$ has no solutions. **(2 marks)**

P **23** **a** Sketch the graph of $y = 3\arcsin x - \frac{\pi}{2}$, showing clearly the exact coordinates of the end-points of the curve. **(4 marks)**

b Find the exact coordinates of the point where the curve crosses the x-axis. **(3 marks)**

24 **a** Prove that for $0 < x \leqslant 1$, $\arccos x = \arctan \frac{\sqrt{1-x^2}}{x}$

b Prove that for $-1 \leqslant x < 0$, $\arccos x = k + \arctan \frac{\sqrt{1-x^2}}{x}$, where k is a constant to be found.

Summary of key points

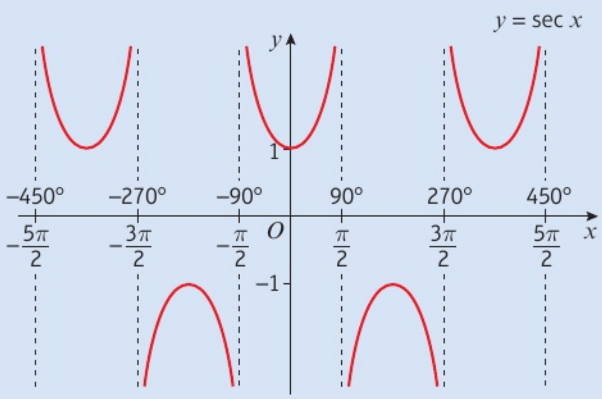
1 • $\sec x = \frac{1}{\cos x}$ (undefined for values of x for which $\cos x = 0$)

• $\text{cosec}\, x = \frac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)

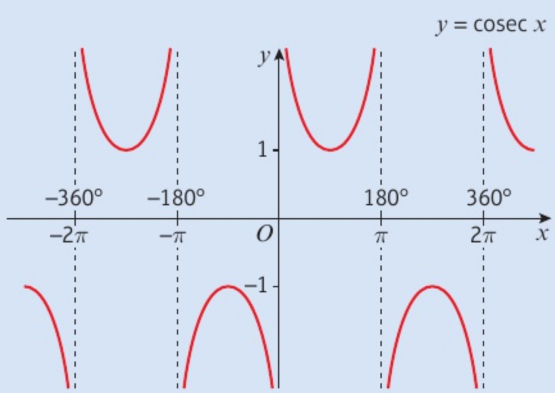
• $\cot x = \frac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)

• $\cot x = \frac{\cos x}{\sin x}$

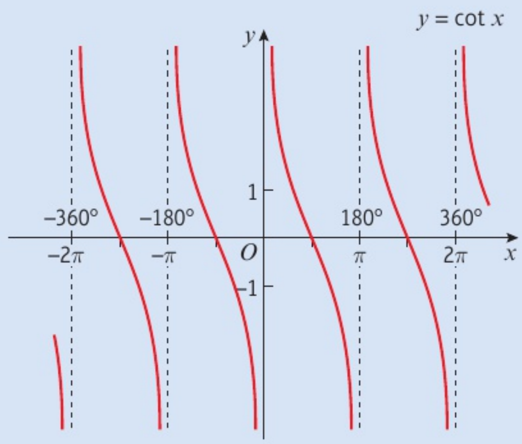
2 The graph of $\boldsymbol{y = \sec x}$, $x \in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$



3 The graph of $\boldsymbol{y = \text{cosec}\, x}$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$



4 The graph of $\boldsymbol{y = \cot x}$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$

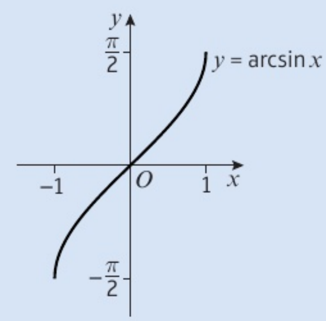


5 You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities:

- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

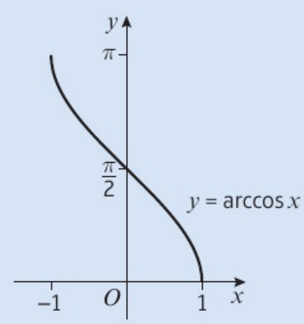
6 The **inverse function** of $\sin x$ is called **arcsin** $\boldsymbol{x}$.

- The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$



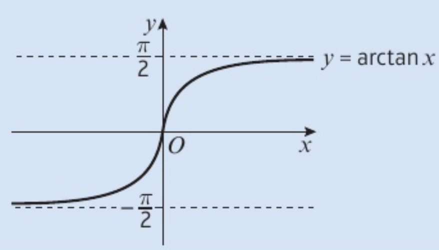
7 The inverse function of $\cos x$ is called **arccos** $\boldsymbol{x}$.

- The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$



8 The inverse function of $\tan x$ is called **arctan** $\boldsymbol{x}$.

- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} \leqslant \arctan x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arctan x \leqslant 90°$



4 TRIGONOMETRIC ADDITION FORMULAE

2.3

Learning objectives

After completing this unit you should be able to:

- Prove and use the addition formulae → pages 71–77
- Understand and use the double-angle formulae → pages 78–81
- Solve trigonometric equations using the double-angle and addition formulae → pages 81–85
- Write expressions of the form $a\cos\theta \pm b\sin\theta$ in the forms $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ → pages 85–90
- Prove trigonometric identities using a variety of identities → pages 90–93

Prior knowledge check

1 Find the exact values of:

a $\sin 45°$ **b** $\cos\frac{\pi}{6}$ **c** $\tan\frac{\pi}{3}$

← Pure 2 Section 6.2

2 Solve the following equations in the interval $0° \leqslant x < 360°$:

a $\sin(x + 50°) = -0.9$ **b** $\cos(2x - 30°) = \frac{1}{2}$

c $2\sin^2 x - \sin x - 3 = 0$

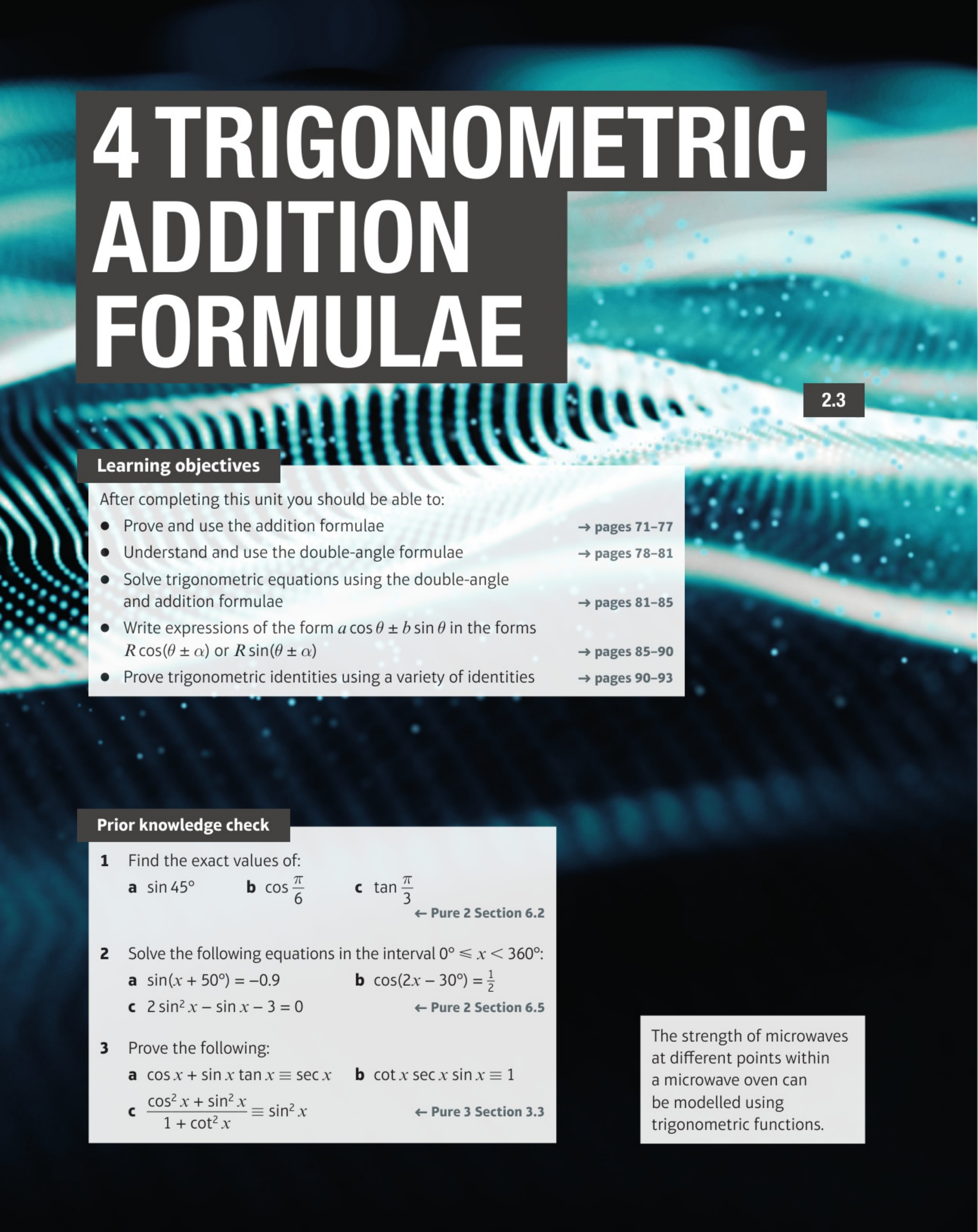
← Pure 2 Section 6.5

3 Prove the following:

a $\cos x + \sin x \tan x \equiv \sec x$ **b** $\cot x \sec x \sin x \equiv 1$

c $\dfrac{\cos^2 x + \sin^2 x}{1 + \cot^2 x} \equiv \sin^2 x$

← Pure 3 Section 3.3



The strength of microwaves at different points within a microwave oven can be modelled using trigonometric functions.

4.1 Addition formulae

The addition formulae for sine, cosine and tangent are defined as follows:

Notation The addition formulae are sometimes called the **compound-angle formulae**.

- $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$
- $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ $\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$
- $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

You can prove these identities using geometric constructions.

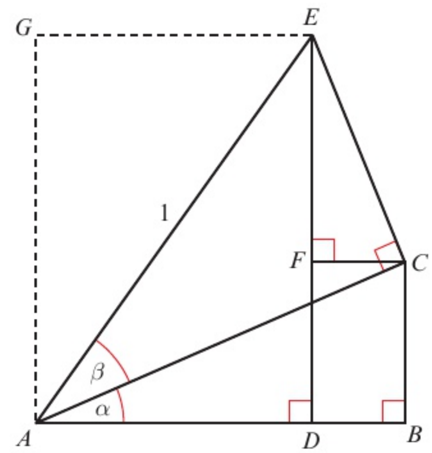
Example 1

In the diagram $\angle BAC = \alpha$, $\angle CAE = \beta$ and $AE = 1$. Additionally, lines AB and BC are perpendicular, lines AB and DE are perpendicular, lines AC and EC are perpendicular and lines EF and FC are perpendicular.

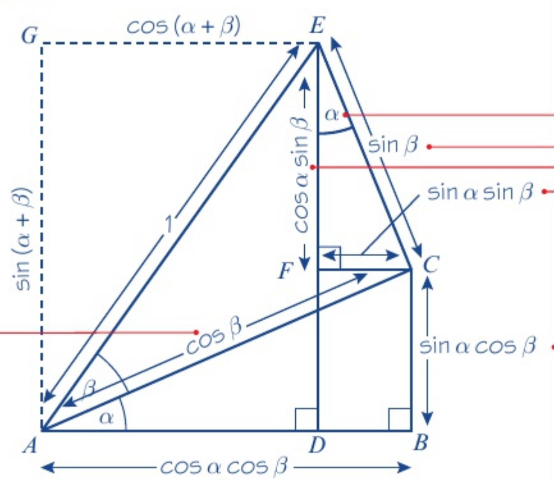
Use the diagram, together with known properties of sine and cosine, to prove the following identities:

a $\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$

b $\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$



The diagram can be labelled with the following lengths using the properties of sine and cosine.



$\angle ACF = \alpha \Rightarrow \angle FCE = 90° - \alpha$. So $\angle FEC = \alpha$

In triangle ACE, $\sin\beta = \dfrac{EC}{AE} \Rightarrow \sin\beta = \dfrac{EC}{1}$
So $EC = \sin\beta$

In triangle FEC, $\cos\alpha = \dfrac{FE}{EC} \Rightarrow \cos\alpha = \dfrac{FE}{\sin\beta}$
So $FE = \cos\alpha\sin\beta$

In triangle FEC, $\sin\alpha = \dfrac{FC}{EC} \Rightarrow \sin\alpha = \dfrac{FC}{\sin\beta}$
So $FC = \sin\alpha\sin\beta$

In triangle ABC, $\sin\alpha = \dfrac{BC}{AC} \Rightarrow \sin\alpha = \dfrac{BC}{\cos\beta}$
So $BC = \sin\alpha\cos\beta$

In triangle ABC, $\cos\alpha = \dfrac{AB}{AC} \Rightarrow \cos\alpha = \dfrac{AB}{\cos\beta}$
So $AB = \cos\alpha\cos\beta$

In triangle ACE, $\cos\beta = \dfrac{AC}{AE} \Rightarrow \cos\beta = \dfrac{AC}{1}$
So $AC = \cos\beta$

a Using triangle ADE

$DE = \sin(\alpha + \beta)$

$AD = \cos(\alpha + \beta)$

$DE = DF + FE$

$\Rightarrow \sin(\alpha + \beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$ as required

b $AD = AB - DB$

$\Rightarrow \cos(\alpha + \beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$ as required

Substitute the lengths from the diagram.

Problem-solving

You are looking for a relationship involving $\sin(\alpha + \beta)$, so consider the right-angled triangle ADE with angle $(\alpha + \beta)$. You can see these relationships more easily on the diagram by looking at $AG = DE$ and $GE = AD$

Online Explore the proof step by step using GeoGebra.

Example 2

Use the results from Example 1 to show that

a $\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$

b $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

a Replace B by $-B$ in

$\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$

$\cos(A + (-B)) \equiv \cos A\cos(-B) - \sin A\sin(-B)$

$\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$

$\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

b $\tan(A + B) \equiv \dfrac{\sin(A + B)}{\cos(A + B)}$

$$\equiv \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

Divide the numerator and denominator by $\cos A\cos B$

$$\equiv \frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

Cancel where possible.

$$\equiv \frac{\tan A + \tan B}{1 - \tan A\tan B} \text{ as required}$$

Example 3

Prove that

$$\frac{\cos A}{\sin B} - \frac{\sin A}{\cos B} \equiv \frac{\cos(A + B)}{\sin B\cos B}$$

$$\text{LHS} \equiv \frac{\cos A}{\sin B} - \frac{\sin A}{\cos B}$$

$$\equiv \frac{\cos A \cos B}{\sin B \cos B} - \frac{\sin A \sin B}{\sin B \cos B}$$

$$\equiv \frac{\cos A \cos B - \sin A \sin B}{\sin B \cos B}$$

$$\equiv \frac{\cos (A + B)}{\sin B \cos B} \equiv \text{RHS}$$

Write both fractions with a common denominator.

Problem-solving

When proving an identity, always keep an eye on the final answer. This can act as a guide as to what to do next.

Use the addition formula in reverse:
$\cos A \cos B - \sin A \sin B \equiv \cos (A + B)$

Example 4

Given that $2 \sin (x + y) = 3 \cos (x - y)$, express $\tan x$ in terms of $\tan y$.

Expanding $\sin (x + y)$ and $\cos (x - y)$ gives

$$2 \sin x \cos y + 2 \cos x \sin y = 3 \cos x \cos y + 3 \sin x \sin y$$

so
$$\frac{2 \sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \frac{2 \cancel{\cos x} \sin y}{\cancel{\cos x} \cos y} = \frac{3 \cancel{\cos x} \cancel{\cos y}}{\cancel{\cos x} \cancel{\cos y}} + \frac{3 \sin x \sin y}{\cos x \cos y}$$

$$2 \tan x + 2 \tan y = 3 + 3 \tan x \tan y$$

$$2 \tan x - 3 \tan x \tan y = 3 - 2 \tan y$$

$$\tan x(2 - 3 \tan y) = 3 - 2 \tan y$$

So
$$\tan x = \frac{3 - 2 \tan y}{2 - 3 \tan y}$$

Remember $\tan x = \frac{\sin x}{\cos x}$

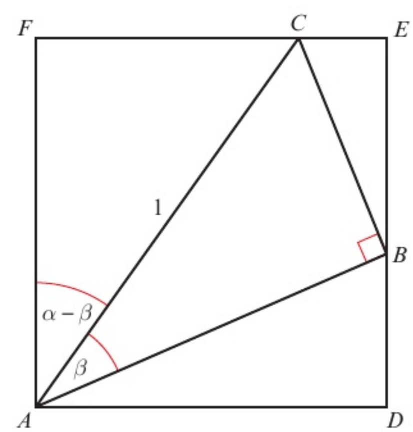
Dividing each term by $\cos x \cos y$ will produce $\tan x$ and $\tan y$ terms.

Collect all $\tan x$ terms on one side of the equation.

Factorise.

Exercise 4A

1 In the diagram $\angle BAC = \beta$, $\angle CAF = \alpha - \beta$ and $AC = 1$. Additionally lines AB and BC are perpendicular.



a Show each of the following:
   i $\angle FAB = \alpha$ ii $\angle ABD = \alpha$ and $\angle ECB = \alpha$
   iii $AB = \cos \beta$ iv $BC = \sin \beta$

b Use $\triangle ABD$ to write an expression for the lengths
   i AD ii BD

c Use $\triangle BEC$ to write an expression for the lengths
   i CE ii BE

d Use $\triangle FAC$ to write an expression for the lengths
   i FC ii FA

e Use your completed diagram to show that:
   i $\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$
   ii $\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

(P) **2** Use the formulae for $\sin(A - B)$ and $\cos(A - B)$ to show that

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(P) **3** By substituting $A = P$ and $B = -Q$ into the addition formula for $\sin(A + B)$, show that $\sin(P - Q) \equiv \sin P \cos Q - \cos P \sin Q$

(P) **4** A student makes the mistake of thinking that $\sin(A + B) \equiv \sin A + \sin B$
Choose non-zero values of A and B to show that this identity is not true.

Watch out This is a common mistake. One counter-example is sufficient to disprove the statement.

(P) **5** Using the expansion of $\cos(A - B)$ with $A = B = \theta$, show that $\sin^2\theta + \cos^2\theta \equiv 1$

(P) **6** **a** Use the expansion of $\sin(A - B)$ to show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

b Use the expansion of $\cos(A - B)$ to show that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

(P) **7** Write $\sin\left(x + \frac{\pi}{6}\right)$ in the form $p\sin x + q\cos x$, where p and q are constants to be found.

(P) **8** Write $\cos\left(x + \frac{\pi}{3}\right)$ in the form $a\cos x + b\sin x$, where a and b are constants to be found.

(P) **9** Express the following as a single sine, cosine or tangent:

a $\sin 15° \cos 20° + \cos 15° \sin 20°$

b $\sin 58° \cos 23° - \cos 58° \sin 23°$

c $\cos 130° \cos 80° - \sin 130° \sin 80°$

d $\dfrac{\tan 76° - \tan 45°}{1 + \tan 76° \tan 45°}$

e $\cos 2\theta \cos\theta + \sin 2\theta \sin\theta$

f $\cos 4\theta \cos 3\theta - \sin 4\theta \sin 3\theta$

g $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{5\theta}{2}\right)$

h $\dfrac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta}$

i $\sin(A + B)\cos B - \cos(A + B)\sin B$

j $\cos\left(\frac{3x + 2y}{2}\right)\cos\left(\frac{3x - 2y}{2}\right) - \sin\left(\frac{3x + 2y}{2}\right)\sin\left(\frac{3x - 2y}{2}\right)$

(P) **10** Use the addition formulae for sine or cosine to write each of the following as a single trigonometric function in the form $\sin(x \pm \theta)$ or $\cos(x \pm \theta)$, where $0 < \theta < \frac{\pi}{2}$

a $\frac{1}{\sqrt{2}}(\sin x + \cos x)$

b $\frac{1}{\sqrt{2}}(\cos x - \sin x)$

c $\frac{1}{2}(\sin x + \sqrt{3}\cos x)$

d $\frac{1}{\sqrt{2}}(\sin x - \cos x)$

(P) **11** Given that $\cos y = \sin(x + y)$, show that $\tan y = \sec x - \tan x$

(P) **12** Given that $\tan(x - y) = 3$, express $\tan y$ in terms of $\tan x$

(P) **13** Given that $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$, find the value of $\tan(x + y)$

Hint First multiply out the brackets.

(P) **14** In each of the following, calculate the exact value of $\tan x$:

a $\tan(x - 45°) = \frac{1}{4}$ **b** $\sin(x - 60°) = 3\cos(x + 30°)$ **c** $\tan(x - 60°) = 2$

(E/P) **15** Given that $\tan\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$, show that $\tan x = 8 - 5\sqrt{3}$ **(3 marks)**

(E/P) **16** Prove that

$$\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right) = 0$$

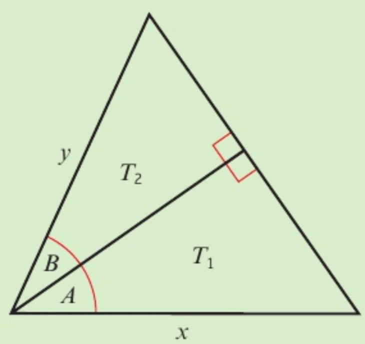
You must show each stage of your working. **(4 marks)**

Challenge

This triangle is constructed from two right-angled triangles T_1 and T_2.

a Find expressions involving x, y, A and B for:
- **i** the area of T_1
- **ii** the area of T_2
- **iii** the area of the large triangle.

b Hence prove that $\sin(A + B) = \sin A\cos B + \cos A\sin B$



Hint For part **a** your expressions should all involve **all four** variables. You will need to use the formula Area $= \frac{1}{2}ab\sin\theta$ in each case.

4.2 Using the angle addition formulae

The addition formulae can be used to find exact values of trigonometric functions of different angles.

Example 5

Show, using the formula for $\sin(A - B)$, that $\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4}$

$$\begin{aligned}\sin 15° &= \sin(45° - 30°) \\ &= \sin 45°\cos 30° - \cos 45°\sin 30° \\ &= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) - \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right) \\ &= \tfrac{1}{4}\left(\sqrt{3}\sqrt{2} - \sqrt{2}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

You know the exact values of sin and cos for many angles, e.g. 30°, 45°, 60°, 90°, 180°, …, so write 15° using two of these angles. You could also use sin (60° – 45°).

Example 6

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$ and B is obtuse, find the value of:

a $\cos(A - B)$ **b** $\tan(A + B)$ **c** $\operatorname{cosec}(A - B)$

a $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

$\cos^2 A \equiv 1 - \sin^2 A = 1 - \left(-\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$\sin^2 B \equiv 1 - \cos^2 = 1 - \left(-\frac{12}{13}\right)^2 = 1 - \frac{144}{169} = \frac{25}{169}$

$\cos A = \pm\frac{4}{5}$ $\sin B = \pm\frac{5}{13}$

$180° < A < 270°$ so $\cos A = -\frac{4}{5}$ B is obtuse so $\sin B = \frac{5}{13}$

$\cos(A - B) = \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(+\frac{5}{13}\right)$

$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$

b $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

So $\tan(A + B) = \dfrac{\frac{3}{4} + \left(-\frac{5}{12}\right)}{1 - \left(\frac{3}{4}\right)\left(-\frac{5}{12}\right)}$

$= \dfrac{\frac{1}{3}}{\frac{21}{16}} = \frac{1}{3} \times \frac{16}{21} = \frac{16}{63}$

c $\operatorname{cosec}(A - B) \equiv \dfrac{1}{\sin(A - B)}$

$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

$\sin(A - B) = \left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right) - \left(\frac{-4}{5}\right)\left(\frac{5}{13}\right) = \frac{56}{65}$

$\operatorname{cosec}(A - B) = \dfrac{1}{\left(\frac{56}{65}\right)} = \frac{65}{56}$

You know $\sin A$ and $\cos B$, but need to find $\sin B$ and $\cos A$.

Use $\sin^2 x + \cos^2 x \equiv 1$ to determine $\cos A$ and $\sin B$.

Problem-solving

Remember there are two possible solutions to $\cos^2 A = \frac{16}{25}$. Use a CAST diagram to determine which one to use.

$\cos x$ is negative in the third quadrant, so choose the negative square root $-\frac{4}{5}$. $\sin x$ is positive in the second quadrant (obtuse angle) so choose the positive square root.

Substitute the values for $\sin A$, $\sin B$, $\cos A$ and $\cos B$ into the formula and then simplify.

$\tan A = \dfrac{\sin A}{\cos A} = \dfrac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}$

$\tan B = \dfrac{\sin B}{\cos B} = \dfrac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}$

Remember $\operatorname{cosec} x = \dfrac{1}{\sin x}$

Exercise 4B

1 Without using your calculator, find the exact value of:

a $\cos 15°$ **b** $\sin 75°$ **c** $\sin(120° + 45°)$ **d** $\tan 165°$

2 Without using your calculator, find the exact value of:

a $\sin 30° \cos 60° + \cos 30° \sin 60°$

b $\cos 110° \cos 20° + \sin 110° \sin 20°$

c $\sin 33° \cos 27° + \cos 33° \sin 27°$

d $\cos \frac{\pi}{8} \cos \frac{\pi}{8} - \sin \frac{\pi}{8} \sin \frac{\pi}{8}$

e $\sin 60° \cos 15° - \cos 60° \sin 15°$

f $\cos 70° (\cos 50° - \tan 70° \sin 50°)$

g $\dfrac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°}$

h $\dfrac{1 - \tan 15°}{1 + \tan 15°}$

i $\dfrac{\tan \frac{7\pi}{12} - \tan \frac{\pi}{3}}{1 + \tan \frac{7\pi}{12} \tan \frac{\pi}{3}}$

j $\sqrt{3} \cos 15° - \sin 15°$

(E) **3** **a** Express $\tan(45° + 30°)$ in terms of $\tan 45°$ and $\tan 30°$ **(2 marks)**

b Hence show that $\tan 75° = 2 + \sqrt{3}$ **(2 marks)**

(P) **4** Given that $\cot A = \frac{1}{4}$ and $\cot(A + B) = 2$, find the value of $\cot B$.

(E/P) **5** **a** Using $\cos(\theta + \alpha) \equiv \cos\theta \cos\alpha - \sin\theta \sin\alpha$, or otherwise, show that $\cos 105° = \dfrac{\sqrt{2} - \sqrt{6}}{4}$ **(4 marks)**

b Hence, or otherwise, show that $\sec 105° = -\sqrt{a}(1 + \sqrt{b})$, where a and b are constants to be found. **(3 marks)**

(P) **6** Given that $\sin A = \frac{4}{5}$ and $\sin B = \frac{1}{2}$, where A and B are both acute angles, calculate the exact value of:

a $\sin(A + B)$ **b** $\cos(A - B)$ **c** $\sec(A - B)$

(P) **7** Given that $\cos A = -\frac{4}{5}$, and A is an obtuse angle measured in radians, find the exact value of:

a $\sin A$ **b** $\cos(\pi + A)$ **c** $\sin\left(\frac{\pi}{3} + A\right)$ **d** $\tan\left(\frac{\pi}{4} + A\right)$

(P) **8** Given that $\sin A = \frac{8}{17}$, where A is acute, and $\cos B = -\frac{4}{5}$, where B is obtuse, calculate the exact value of:

a $\sin(A - B)$ **b** $\cos(A - B)$ **c** $\cot(A - B)$

(P) **9** Given that $\tan A = \frac{7}{24}$, where A is reflex, and $\sin B = \frac{5}{13}$, where B is obtuse, calculate the exact value of:

a $\sin(A + B)$ **b** $\tan(A - B)$ **c** $\operatorname{cosec}(A + B)$

(P) **10** Given that $\tan A = \frac{1}{5}$ and $\tan B = \frac{2}{3}$, calculate, without using your calculator, the value of $A + B$ in degrees, where:

a A and B are both acute

b A is reflex and B is acute.

4.3 Double-angle formulae

You can use the addition formulae to derive the following double-angle formulae.

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

Example 7

Use the double-angle formulae to write each of the following as a single trigonometric ratio:

a $\cos^2 50° - \sin^2 50°$ **b** $\dfrac{2\tan\left(\frac{\pi}{6}\right)}{1 - \tan^2\left(\frac{\pi}{6}\right)}$ **c** $\dfrac{4\sin 70°}{\sec 70°}$

a $\cos^2 50° - \sin^2 50° = \cos(2 \times 50°)$
$= \cos 100°$

Use $\cos 2A \equiv \cos^2 A - \sin^2 A$ in reverse, with $A = 50°$

b $\dfrac{2\tan\left(\frac{\pi}{6}\right)}{1 - \tan^2\left(\frac{\pi}{6}\right)} = \tan\left(2 \times \frac{\pi}{6}\right)$

Use $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$ in reverse, with $A = \dfrac{\pi}{6}$

$= \tan\left(\frac{\pi}{3}\right)$

c $\dfrac{4\sin 70°}{\sec 70°} = 4\sin 70° \cos 70°$

$\sec x = \dfrac{1}{\cos x}$ so $\cos x = \dfrac{1}{\sec x}$

$= 2(2\sin 70° \cos 70°)$

Recognise this is a multiple of $2\sin A \cos A$.

$= 2\sin(2 \times 70°) = 2\sin 140°$

Use $\sin 2A \equiv 2\sin A \cos A$ in reverse with $A = 70°$

Example 8

Given that $x = 3\sin\theta$ and $y = 3 - 4\cos 2\theta$, eliminate θ and express y in terms of x.

The equations can be written as

$\sin\theta = \dfrac{x}{3}$ $\cos 2\theta = \dfrac{3 - y}{4}$

Watch out Be careful with this manipulation. Many errors can occur in the early part of a solution.

As $\cos 2\theta \equiv 1 - 2\sin^2\theta$ for all values of θ,

$\dfrac{3 - y}{4} = 1 - 2\left(\dfrac{x}{3}\right)^2$

θ has been eliminated from this equation. We still need to solve for y.

So $\dfrac{y}{4} = 2\left(\dfrac{x}{3}\right)^2 - \dfrac{1}{4}$

or $y = 8\left(\dfrac{x}{3}\right)^2 - 1$

The final answer should be in the form $y = \ldots$

Example 9

Given that $\cos x = \frac{3}{4}$, and that $180° < x < 360°$, find the exact value of:

a $\sin 2x$ **b** $\tan 2x$

a $\sin^2 A = 1 - \cos^2 A$

$= 1 - \left(\frac{3}{4}\right)^2$

$= \frac{7}{16}$

$180° < A < 360°$, so $\sin A = -\frac{\sqrt{7}}{4}$

$\sin 2x = 2 \sin x \cos x$

$= 2\left(-\frac{\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$

Use $\sin^2 A + \cos^2 A = 1$ to determine $\sin A$.

$\sin A$ is negative in the third and fourth quadrants, so choose the negative square root.

b $\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{\sqrt{7}}{4}}{\frac{3}{4}}$

$= -\frac{\sqrt{7}}{3}$

Find $\tan x$ in simplified surd form, then substitute this value into the double-angle formula for $\tan 2x$.

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} = \frac{-\frac{2\sqrt{7}}{3}}{1 - \frac{7}{9}}$

$= -\frac{2\sqrt{7}}{3} \times \frac{9}{2}$

$= -3\sqrt{7}$

Make sure you square all of $\tan x$ when working out $\tan^2 x$:

$\left(-\frac{\sqrt{7}}{3}\right)^2 = \frac{7}{9}$

Exercise 4C

(P) **1** Use the expansion of $\sin(A + B)$ to show that $\sin 2A \equiv 2 \sin A \cos A$

Hint Set $B = A$

(P) **2** **a** Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, show that $\cos 2A \equiv \cos^2 A - \sin^2 A$

b Hence show that:

i $\cos 2A \equiv 2\cos^2 A - 1$

ii $\cos 2A \equiv 1 - 2\sin^2 A$

Problem-solving Use $\sin^2 A + \cos^2 A \equiv 1$

(P) **3** Use the expansion of $\tan(A + B)$ to express $\tan 2A$ in terms of $\tan A$.

(P) **4** Write each of the following expressions as a single trigonometric ratio:

a $2\sin 10° \cos 10°$ **b** $1 - 2\sin^2 25°$ **c** $\cos^2 40° - \sin^2 40°$

d $\dfrac{2\tan 5°}{1 - \tan^2 5°}$ **e** $\dfrac{1}{2\sin(24.5°)\cos(24.5°)}$ **f** $6\cos^2 30° - 3$

g $\dfrac{\sin 8°}{\sec 8°}$ **h** $\cos^2\left(\dfrac{\pi}{16}\right) - \sin^2\left(\dfrac{\pi}{16}\right)$

(P) **5** Without using your calculator, find the exact values of:

a $2\sin 22.5° \cos 22.5°$ **b** $2\cos^2 15° - 1$

c $(\sin 75° - \cos 75°)^2$ **d** $\dfrac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$

(E/P) **6** **a** Show that $(\sin A + \cos A)^2 \equiv 1 + \sin 2A$ **(3 marks)**

b Hence find the exact value of $\left(\sin\dfrac{\pi}{8} + \cos\dfrac{\pi}{8}\right)^2$ **(2 marks)**

7 Write the following in their simplest form, involving only one trigonometric function:

a $\cos^2 3\theta - \sin^2 3\theta$ **b** $6\sin 2\theta \cos 2\theta$ **c** $\dfrac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$

d $2 - 4\sin^2\left(\dfrac{\theta}{2}\right)$ **e** $\sqrt{1 + \cos 2\theta}$ **f** $\sin^2\theta \cos^2\theta$

g $4\sin\theta \cos\theta \cos 2\theta$ **h** $\dfrac{\tan\theta}{\sec^2\theta - 2}$ **i** $\sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta$

(P) **8** Given that $p = 2\cos\theta$ and $q = \cos 2\theta$, express q in terms of p.

(P) **9** Eliminate θ from the following pairs of equations:

a $x = \cos^2\theta$, $y = 1 - \cos 2\theta$ **b** $x = \tan\theta$, $y = \cot 2\theta$

c $x = \sin\theta$, $y = \sin 2\theta$ **d** $x = 3\cos 2\theta + 1$, $y = 2\sin\theta$

(P) **10** Given that $\cos x = \frac{1}{4}$, find the exact value of $\cos 2x$.

(P) **11** Find the possible values of $\sin\theta$ when $\cos 2\theta = \frac{23}{25}$

(P) **12** Given that $\tan\theta = \frac{3}{4}$, and that θ is acute,

a find the exact value of: **i** $\tan 2\theta$ **ii** $\sin 2\theta$ **iii** $\cos 2\theta$

b deduce the value of $\sin 4\theta$.

(P) **13** Given that $\cos A = -\frac{1}{3}$, and that A is obtuse,

a find the exact value of: **i** $\cos 2A$ **ii** $\sin A$ **iii** $\operatorname{cosec} 2A$

b show that $\tan 2A = \frac{4\sqrt{2}}{7}$

(E/P) **14** Given that $\pi < \theta < \frac{3\pi}{2}$, find the value of $\tan\left(\frac{\theta}{2}\right)$ when $\tan\theta = \frac{3}{4}$ **(4 marks)**

(E/P) **15** Given that $\cos x + \sin x = m$ and $\cos x - \sin x = n$, where m and n are constants, write down, in terms of m and n, the value of $\cos 2x$. **(4 marks)**

(E/P) **16** In $\triangle PQR$, $PQ = 3$ cm, $PR = 6$ cm, $QR = 5$ cm and $\angle QPR = 2\theta$

a Use the cosine rule to show that $\cos 2\theta = \frac{5}{9}$ **(3 marks)**

b Hence find the exact value of $\sin\theta$. **(2 marks)**

(E/P) **17** The line l, with equation $y = \frac{3}{4}x$, bisects the angle between the x-axis and the line $y = mx$, $m > 0$

Given that the scales on each axis are the same, and that l makes an angle θ with the x-axis,

a write down the value of $\tan\theta$ **(1 mark)**

b show that $m = \frac{24}{7}$ **(3 marks)**

(E/P) **18** **a** Use the identity $\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$ to show that $\cos 2A \equiv 2\cos^2 A - 1$ **(2 marks)**

The curves C_1 and C_2 have equations

C_1: $y = 4\cos 2x$
C_2: $y = 6\cos^2 x - 3\sin 2x$

b Show that the x-coordinates of the points where C_1 and C_2 intersect satisfy the equation

$\cos 2x + 3\sin 2x - 3 = 0$ **(3 marks)**

(P) **19** Use the fact that $\tan 2A \equiv \frac{\sin 2A}{\cos 2A}$ to derive the formula for $\tan 2A$ in terms of $\tan A$.

Hint Use the identities for $\sin 2A$ and $\cos 2A$ and then divide both the numerator and denominator by $\cos^2 A$.

4.4 Solving trigonometric equations

You can use the addition and the double-angle formulae to help you solve trigonometric equations.

Example 10

Solve $4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$ in the range $0° \leqslant \theta \leqslant 360°$. Round your answer to 1 decimal place.

$4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$

$4\cos\theta\cos 30° + 4\sin\theta\sin 30° = 8\sqrt{2}\sin\theta$ — Use the formula for $\cos(A - B)$

$4\cos\theta\left(\frac{\sqrt{3}}{2}\right) + 4\sin\theta\left(\frac{1}{2}\right) = 8\sqrt{2}\sin\theta$ — Substitute $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$

$2\sqrt{3}\cos\theta + 2\sin\theta = 8\sqrt{2}\sin\theta$

$2\sqrt{3}\cos\theta = (8\sqrt{2} - 2)\sin\theta$ — Gather cosine terms on the LHS and sine terms on the RHS of the equation.

$\frac{2\sqrt{3}}{8\sqrt{2} - 2} = \tan\theta$ — Divide both sides by $\cos\theta$ and by $(8\sqrt{2} - 2)$

$\tan\theta = 0.3719...$

$\theta = 20.4°, 200.4°$ — Use a CAST diagram or a sketch graph to find all the solutions in the given range.

Example 11

Solve $3\cos 2x - \cos x + 2 = 0$ for $0° \leqslant x \leqslant 360°$

Using a double angle formula for $\cos 2x$

$$3\cos 2x - \cos x + 2 = 0$$

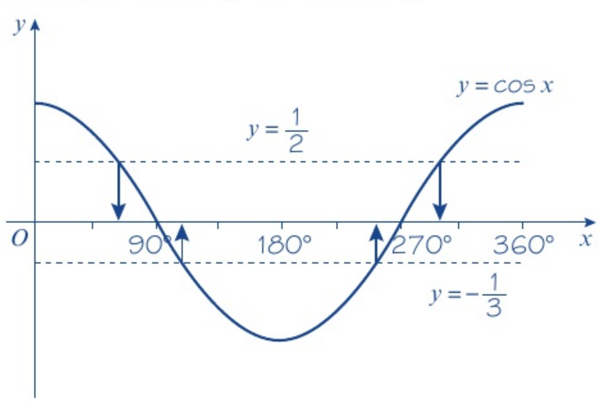
becomes

$$3(2\cos^2 x - 1) - \cos x + 2 = 0$$
$$6\cos^2 x - 3 - \cos x + 2 = 0$$
$$6\cos^2 x - \cos x - 1 = 0$$

This quadratic equation factorises: $6X^2 - X - 1 = (3X + 1)(2X - 1)$

So $(3\cos x + 1)(2\cos x - 1) = 0$

Solving: $\cos x = -\frac{1}{3}$ or $\cos x = \frac{1}{2}$



$\cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ $\quad\quad$ $\cos^{-1}\left(\frac{1}{2}\right) = 60°$

So $x = 60°, 109.5°, 250.5°, 300°$

Problem-solving

Choose the double-angle formula for $\cos 2x$ which only involves $\cos x$:

$\cos 2x \equiv 2\cos^2 x - 1$

This will give you a quadratic equation in $\cos x$.

Example 12

Solve $2\tan 2y \tan y = 3$ for $0 \leqslant y \leqslant 2\pi$. Give your answers to 2 decimal places.

$2\tan 2y \tan y = 3$

$2\left(\frac{2\tan y}{1-\tan^2 y}\right)\tan y = 3$ — Use the double-angle identity for tan.

$\frac{4\tan^2 y}{1-\tan^2 y} = 3$

$4\tan^2 y = 3 - 3\tan^2 y$ — This is a quadratic equation in $\tan y$. Because there is a $\tan^2 y$ term but no $\tan y$ term you can solve it directly.

$7\tan^2 y = 3$

$\tan^2 y = \frac{3}{7}$

$\tan y = \pm\sqrt{\frac{3}{7}}$

$y = 0.58, 2.56, 3.72, 5.70$

Watch out Remember to include the positive and negative square roots.

Example 13

a By expanding $\sin(2A + A)$ show that $\sin 3A \equiv 3\sin A - 4\sin^3 A$

b Hence, or otherwise, for $0 < \theta < 2\pi$, solve $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$ giving your answers in terms of π.

a $\text{LHS} \equiv \sin 3A \equiv \sin(2A + A)$

$\equiv \sin 2A\cos A + \cos 2A\sin A$ — Use the addition formula for $\sin(A + B)$

$\equiv (2\sin A\cos A)\cos A + (1 - 2\sin^2 A)\sin A$ — Subsitute for $\sin 2A$ and $\cos 2A$. As the answer is in terms of $\sin A$, $\cos 2A \equiv 1 - 2\sin^2 A$ is the best identity to use.

$\equiv 2\sin A\cos^2 A + \sin A - 2\sin^3 A$

$\equiv 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$ — Use $\sin^2 A + \cos^2 A \equiv 1$ to substitute for $\cos^2 A$.

$\equiv 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$

$\equiv 3\sin A - 4\sin^3 A \equiv \text{RHS}$

b $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$

$16\sin^3\theta - 12\sin\theta = 2\sqrt{3}$

$-4\sin 3\theta = 2\sqrt{3}$

$\sin 3\theta = -\frac{\sqrt{3}}{2}$

$3\theta = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \frac{17\pi}{3}$ — Use a CAST diagram or a sketch graph to find all answers for 3θ. $0 < \theta < 2\pi$ so $0 < 3\theta < 6\pi$

$\theta = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$

Problem-solving The question says 'hence' so look for an opportunity to use the identity you proved in part **a**. You need to multiply both sides of the identity by −4.

Exercise 4D

(P) **1** Solve, in the interval $0° \leqslant \theta < 360°$, the following equations. Give your answers to 1 d.p.

a $3\cos\theta = 2\sin(\theta + 60°)$ **b** $\sin(\theta + 30°) + 2\sin\theta = 0$

c $\cos(\theta + 25°) + \sin(\theta + 65°) = 1$ **d** $\cos\theta = \cos(\theta + 60°)$

(E/P) **2** **a** Show that $\sin\left(\theta + \frac{\pi}{4}\right) \equiv \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$ **(2 marks)**

b Hence, or otherwise, solve the equation $\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) = \frac{1}{\sqrt{2}}$, $0 \leqslant \theta \leqslant 2\pi$ **(4 marks)**

c Use your answer to part **b** to write down the solutions to $\sin\theta + \cos\theta = 1$ over the same interval. **(2 marks)**

(P) **3** **a** Solve the equation $\cos\theta\cos 30° - \sin\theta\sin 30° = 0.5$ for $0° \leqslant \theta \leqslant 360°$

b Hence write down, in the same interval, the solutions of $\sqrt{3}\cos\theta - \sin\theta = 1$

(P) **4** **a** Given that $3\sin(x - y) - \sin(x + y) = 0$, show that $\tan x = 2\tan y$

b Solve $3\sin(x - 45°) - \sin(x + 45°) = 0$ for $0° \leqslant x \leqslant 360°$

(P) **5** Solve the following equations, in the intervals given:

a $\sin 2\theta = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$ **b** $\cos 2\theta = 1 - \cos\theta$, $-180° < \theta \leqslant 180°$

c $3\cos 2\theta = 2\cos^2\theta$, $0° \leqslant \theta < 360°$ **d** $\sin 4\theta = \cos 2\theta$, $0 \leqslant \theta \leqslant \pi$

e $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0$, $0° \leqslant \theta < 720°$ **f** $\cos^2\theta - \sin 2\theta = \sin^2\theta$, $0 \leqslant \theta \leqslant \pi$

g $2\sin\theta = \sec\theta$, $0 \leqslant \theta \leqslant 2\pi$ **h** $2\sin 2\theta = 3\tan\theta$, $0° \leqslant \theta < 360°$

i $2\tan\theta = \sqrt{3}(1 - \tan\theta)(1 + \tan\theta)$, $0 \leqslant \theta \leqslant 2\pi$ **j** $\sin^2\theta = 2\sin 2\theta$, $-180° < \theta < 180°$

k $4\tan\theta = \tan 2\theta$, $0° \leqslant \theta \leqslant 360°$

(E/P) **6** In $\triangle ABC$, $AB = 4$ cm, $AC = 5$ cm, $\angle ABC = 2\theta$ and $\angle ACB = \theta$
Find the value of θ, giving your answer, in degrees, to 1 decimal place. **(4 marks)**

(E/P) **7** **a** Show that $5\sin 2\theta + 4\sin\theta = 0$ can be written in the form $a\sin\theta(b\cos\theta + c) = 0$, stating the values of a, b and c. **(2 marks)**

b Hence solve, for $0° \leqslant \theta < 360°$, the equation $5\sin 2\theta + 4\sin\theta = 0$ **(4 marks)**

(E/P) **8** **a** Given that $\sin 2\theta + \cos 2\theta = 1$, show that $2\sin\theta(\cos\theta - \sin\theta) = 0$ **(2 marks)**

b Hence, or otherwise, solve the equation $\sin 2\theta + \cos 2\theta = 1$ for $0° \leqslant \theta < 360°$ **(4 marks)**

(E/P) **9** **a** Prove that $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$ **(4 marks)**

b Use the result to solve, for $0 \leqslant \theta < \pi$, the equation $\cos 2\theta - \sin 2\theta = \frac{1}{\sqrt{2}}$
Give your answers in terms of π. **(3 marks)**

(P) **10 a** Show that:

i $\sin\theta \equiv \dfrac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$ **ii** $\cos\theta \equiv \dfrac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$

b By writing the following equations as quadratics in $\tan\left(\frac{\theta}{2}\right)$, solve, in the interval $0° \leqslant \theta \leqslant 360°$:

i $\sin\theta + 2\cos\theta = 1$ **ii** $3\cos\theta - 4\sin\theta = 2$

(E/P) **11 a** Show that $3\cos^2 x - \sin^2 x \equiv 1 + 2\cos 2x$ **(3 marks)**

b Hence sketch, for $-\pi \leqslant x \leqslant \pi$, the graph of $y = 3\cos^2 x - \sin^2 x$, showing the coordinates of points where the curve meets the axes. **(3 marks)**

(E/P) **12 a** Express $2\cos^2\left(\frac{\theta}{2}\right) - 4\sin^2\left(\frac{\theta}{2}\right)$ in the form $a\cos\theta + b$, where a and b are constants. **(4 marks)**

b Hence solve $2\cos^2\left(\frac{\theta}{2}\right) - 4\sin^2\left(\frac{\theta}{2}\right) = -3$ in the interval $0° \leqslant \theta < 360°$ **(3 marks)**

(E/P) **13 a** Use the identity $\sin^2 A + \cos^2 A \equiv 1$ to show that $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$ **(5 marks)**

b Deduce that $\sin^4 A + \cos^4 A \equiv \frac{1}{4}(3 + \cos 4A)$ **(3 marks)**

c Hence solve $8\sin^4\theta + 8\cos^4\theta = 7$, for $0 < \theta < \pi$ **(3 marks)**

Hint Start by squaring $(\sin^2 A + \cos^2 A)$

(E/P) **14 a** By writing 3θ as $2\theta + \theta$, show that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$ **(4 marks)**

b Hence, or otherwise, for $0 < \theta < \pi$, solve $6\cos\theta - 8\cos^3\theta + 1 = 0$, giving your answer in terms of π. **(5 marks)**

4.5 Simplifying $a\cos x \pm b\sin x$

You can use the addition formulae to simplify some trigonometric expressions:

- **For positive values of a and b,**
  - **$a\sin x \pm b\cos x$ can be expressed in the form $R\sin(x \pm \alpha)$**
  - **$a\cos x \pm b\sin x$ can be expressed in the form $R\cos(x \mp \alpha)$**

  with $R > 0$ and $0° < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

  where $R\cos\alpha = a$, $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$

Notation The symbol $\mp$ means that $a\cos x + b\sin x$ will be written in the form $R\cos(x - \alpha)$, and $a\cos x - b\sin x$ will be written in the form $R\cos(x + \alpha)$.

Use the addition formulae to expand $\sin(x \pm \alpha)$ or $\cos(x \mp \alpha)$, then equate coefficients.

Example 14

Show that you can express $3\sin x + 4\cos x$ in the form:

a $R\sin(x + \alpha)$

b $R\cos(x - \alpha)$

where $R > 0$, $0° < \alpha < 90°$, $0° < \beta < 90°$, giving your values of R, α and β to 1 decimal place when appropriate.

a $R\sin(x + \alpha) \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

Use $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$ and multiply through by R.

Let $3\sin x + 4\cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

So $R\cos\alpha = 3$ and $R\sin\alpha = 4$

Equate the coefficients of the $\sin x$ and $\cos x$ terms.

$\dfrac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \dfrac{4}{3}$

Divide the equations to eliminate R and use $\tan^{-1}$ to find α.

$\alpha = \tan^{-1}\left(\dfrac{4}{3}\right)$

So $\alpha = 53.1°$ (1 d.p.)

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$

Square and add the equations to eliminate α and find R^2.

$R^2(\cos^2\alpha + \sin^2\alpha) = 25$

$R^2 = 25$, so $R = 5$

Use $\sin^2\alpha + \cos^2\alpha \equiv 1$

$3\sin x + 4\cos x \equiv 5\sin(x + 53.1°)$

b $R\cos(x - \beta) \equiv R\cos x\cos\beta + R\sin x\sin\beta$

Use $\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$ and multiply through by R.

Let $3\sin x + 4\cos x \equiv R\cos x\cos\beta + R\sin x\sin\beta$

So $R\cos\beta = 4$ and $R\sin\beta = 3$

Equate the coefficients of the $\cos x$ and $\sin x$ terms.

$\dfrac{R\sin\beta}{R\cos\beta} = \tan\beta = \dfrac{3}{4}$

Divide the equations to eliminate R.

So $\beta = 36.9°$ (1 d.p.)

$R^2\cos^2\beta + R^2\sin^2\beta = 3^2 + 4^2$

Square and add the equations to eliminate α and find R^2.

$R^2(\cos^2\beta + \sin^2\beta) = 25$

$R^2 = 25$, so $R = 5$

Remember $\sin^2\alpha + \cos^2\alpha \equiv 1$

$3\sin x + 4\cos x \equiv 5\cos(x - 36.9°)$

Online Explore how you can transform the graphs of $y = \sin x$ and $y = \cos x$ to obtain the graph of $y = 3\sin x + 4\cos x$ using technology.

Example 15

a Show that you can express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$, $0 < \alpha < \frac{\pi}{2}$

b Hence sketch the graph of $y = \sin x - \sqrt{3}\cos$

a Set $\sin x - \sqrt{3}\cos x \equiv R\sin(x - \alpha)$

$\sin x - \sqrt{3}\cos x \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

Expand $\sin(x - \alpha)$ and multiply by R.

So $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$

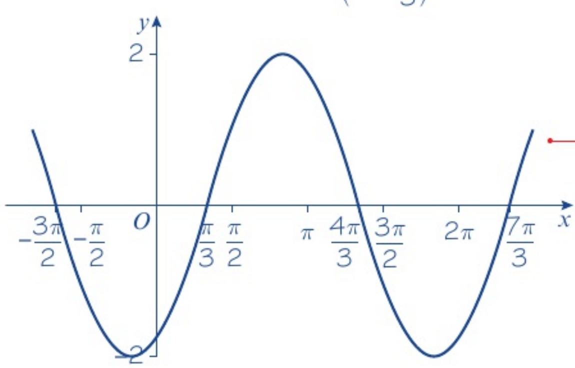
Equate the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.

Dividing, $\tan\alpha = \sqrt{3}$, so $\alpha = \frac{\pi}{3}$

Squaring and adding: $R = 2$

So $\sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \frac{\pi}{3}\right)$

b $y = \sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \frac{\pi}{3}\right)$



You can sketch $y = 2\sin\left(x - \frac{\pi}{3}\right)$ by translating $y = \sin x$ by $\frac{\pi}{3}$ to the right and then stretching by a scale factor of 2 in the y-direction.

Example 16

a Express $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$, $0° < \alpha < 90°$

b Hence solve, for $0° < \theta < 360°$, the equation $2\cos\theta + 5\sin\theta = 3$

a Set $2\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 2$ and $R\sin\alpha = 5$

Equate the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.

Dividing $\tan\alpha = \frac{5}{2}$, so $\alpha = 68.2°$

Squaring and adding: $R = \sqrt{29}$

So $2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2°)$

b $\sqrt{29}\cos(\theta - 68.2°) = 3$

Use the result from part **a**:
$2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2°)$

So $\cos(\theta - 68.2°) = \frac{3}{\sqrt{29}}$

Divide both sides by $\sqrt{29}$.

$\cos^{-1}\left(\frac{3}{\sqrt{29}}\right) = 56.1...°$

As $0° < \theta < 360°$, the interval for $(\theta - 68.2°)$ is $-68.2° < \theta - 68.2° < 291.8°$

$\frac{3}{\sqrt{29}}$ is positive, so solutions for $\theta - 68.2°$ are in the 1st and 4th quadrants.

So $\theta - 68.2° = -56.1...°, 56.1...°$

$\theta = 12.1°, 124.3°$ (to the nearest 0.1°)

Example 17

$f(\theta) = 12\cos\theta + 5\sin\theta$

a Write $f(\theta)$ in the form $R\cos(\theta - \alpha)$.

b Find the maximum value of $f(\theta)$ and the smallest positive value of θ at which it occurs.

Online Use technology to explore maximums and minimums of curves in the form $R\cos(\theta - \alpha)$.

a Set $12\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha)$

So $12\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$R = 13$ and $\tan\alpha = \frac{5}{12} \Rightarrow \alpha = 22.6°$

So $12\cos\theta + 5\sin\theta \equiv 13\cos(\theta - 22.6°)$

Equate $\sin x$ and $\cos x$ terms and then solve for R and α.

b The maximum value of $13\cos(\theta - 22.6°)$ is 13.

The maximum value of $\cos x$ is 1 so the maximum value of $\cos(\theta - 22.6°)$ is also 1.

This occurs when $\cos(\theta - 22.6°) = 1$

$\theta - 22.6° = \ldots, -360°, 0°, 360°, \ldots$

The smallest positive value of θ is 22.6°

Solve the equation to find the smallest positive value of θ.

Exercise 4E

Unless otherwise stated, give all angles to 1 decimal place and write non-integer values of R in surd form.

1 Given that $5\sin\theta + 12\cos\theta \equiv R\sin(\theta + \alpha)$, find the value of R, $R > 0$, and the value of $\tan\alpha$.

2 Given that $\sqrt{3}\sin\theta + \sqrt{6}\cos\theta \equiv 3\cos(\theta - \alpha)$, where $0° < \alpha < 90°$, find the value of α.

3 Given that $2\sin\theta - \sqrt{5}\cos\theta \equiv -3\cos(\theta + \alpha)$, where $0° < \alpha < 90°$, find the value of α.

4 **a** Show that $\cos\theta - \sqrt{3}\sin\theta$ can be written in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

b Hence sketch the graph of $y = \cos\theta - \sqrt{3}\sin\theta$, $0 < \theta < \frac{\pi}{2}$, giving the coordinates of points of intersection with the axes.

(P) 5 **a** Express $7\cos\theta - 24\sin\theta$ in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0° < \alpha < 90°$

b The graph of $y = 7\cos\theta - 24\sin\theta$ meets the y-axis at P. State the coordinates of P.

c Write down the maximum and minimum values of $7\cos\theta - 24\sin\theta$

d Deduce the number of solutions, in the interval $0° < \theta < 360°$, of the following equations:

i $7\cos\theta - 24\sin\theta = 15$ **ii** $7\cos\theta - 24\sin\theta = 26$ **iii** $7\cos\theta - 24\sin\theta = -25$

(E) 6 $f(\theta) = \sin\theta + 3\cos\theta$

Given $f(\theta) = R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$

a Find the value of R and the value of α. **(4 marks)**

b Hence, or otherwise, solve $f(\theta) = 2$ for $0° \leqslant \theta < 360°$ **(3 marks)**

E **7** **a** Express $\cos 2\theta - 2\sin 2\theta$ in the form $R\cos(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$
Give the value of α to 3 decimal places. **(4 marks)**

b Hence, or otherwise, solve for $0 \leqslant \theta < \pi$, $\cos 2\theta - 2\sin 2\theta = -1.5$, rounding your answers to 2 decimal places. **(4 marks)**

P **8** Solve the following equations, in the intervals given in brackets:

a $6\sin x + 8\cos x = 5\sqrt{3}$, [0°, 360°]

b $2\cos 3\theta - 3\sin 3\theta = -1$, [0°, 90°]

c $8\cos\theta + 15\sin\theta = 10$, [0°, 360°]

d $5\sin\frac{x}{2} - 12\cos\frac{x}{2} = -6.5$, [−360°, 360°]

E/P **9** **a** Express $3\sin 3\theta - 4\cos 3\theta$ in the form $R\sin(3\theta - \alpha)$, with $R > 0$ and $0° < \alpha < 90°$ **(3 marks)**

b Hence write down the minimum value of $3\sin 3\theta - 4\cos 3\theta$ and the value of θ at which it occurs. **(3 marks)**

c Solve, for $0° \leqslant \theta < 180°$, the equation $3\sin 3\theta - 4\cos 3\theta = 1$ **(3 marks)**

E/P **10** **a** Express $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$ in the form $a\sin 2\theta + b\cos 2\theta + c$, where a, b and c are constants to be found. **(3 marks)**

b Hence find the maximum and minimum values of $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$ **(4 marks)**

c Solve $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta = -1$ for $0° \leqslant \theta < 180°$, rounding your answers to 1 decimal place. **(4 marks)**

P **11** A class were asked to solve $3\cos\theta = 2 - \sin\theta$ for $0° \leqslant \theta < 360°$. One student expressed the equation in the form $R\cos(\theta - \alpha) = 2$, with $R > 0$ and $0° < \alpha < 90°$, and correctly solved the equation.

a Find the values of R and α and hence find her solutions.

Another student decided to square both sides of the equation and then form a quadratic equation in $\sin\theta$.

b Show that the correct quadratic equation is $10\sin^2\theta - 4\sin\theta - 5 = 0$

c Solve this equation for $0° \leqslant \theta < 360°$

d Explain why not all of the answers satisfy $3\cos\theta = 2 - \sin\theta$

E/P **12** **a** Given $\cot\theta + 2 = \operatorname{cosec}\theta$, show that $2\sin\theta + \cos\theta = 1$ **(4 marks)**

b Solve $\cot\theta + 2 = \operatorname{cosec}\theta$ for $0° \leqslant \theta < 360°$ **(3 marks)**

E/P **13** **a** Given $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$, show that $\cos\theta + \sqrt{3}\sin\theta = 2$ **(4 marks)**

b Solve $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$ for $0 \leqslant \theta \leqslant 2\pi$ **(2 marks)**

E/P **14** **a** Express $9\cos\theta + 40\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$
Give the value of α to 3 decimal places. **(4 marks)**

b $g(\theta) = \dfrac{18}{50 + 9\cos\theta + 40\sin\theta}$, $0° \leqslant \theta \leqslant 360°$

Calculate:

i the minimum value of $g(\theta)$ **(2 marks)**

ii the smallest positive value of θ at which the minimum occurs. **(2 marks)**

E/P **15** $p(\theta) = 12\cos 2\theta - 5\sin 2\theta$

Given that $p(\theta) = R\cos(2\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$,

a find the value of R and the value of α. **(3 marks)**

b Hence solve the equation $12\cos 2\theta - 5\sin 2\theta = -6.5$ for $0° \leqslant \theta < 180°$ **(5 marks)**

c Express $24\cos^2\theta - 10\sin\theta\cos\theta$ in the form $a\cos 2\theta + b\sin 2\theta + c$, where a, b and c are constants to be found. **(3 marks)**

d Hence, or otherwise, find the minimum value of $24\cos^2\theta - 10\sin\theta\cos\theta$ **(2 marks)**

4.6 Proving trigonometric identities

You can use known trigonometric identities to prove other identities.

Example 18

a Show that $2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos\theta \equiv \frac{1}{2}\sin 2\theta$

b Show that $1 + \cos 4\theta \equiv 2\cos^2 2\theta$

a $\sin 2A \equiv 2\sin A\cos A$

$\sin\theta \equiv 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ — Substitute $A = \frac{\theta}{2}$ into the formula for $\sin 2A$.

Problem-solving

Always be aware that the addition formulae can be altered by making a substitution.

$\text{LHS} \equiv 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos\theta$

$\equiv \sin\theta\cos\theta$ — Use the above result for $2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$\equiv \frac{1}{2}\sin 2\theta$ — Remember $\sin 2\theta \equiv 2\sin\theta\cos\theta$

$\equiv \text{RHS}$

b $\text{LHS} \equiv 1 + \cos 4\theta$

$\equiv 1 + 2\cos^2 2\theta - 1$ — Use $\cos 2A \equiv 2\cos^2 A - 1$ with $A = 2\theta$

$\equiv 2\cos^2 2\theta$

$\equiv \text{RHS}$

Example 19

Prove the identity $\tan 2\theta \equiv \dfrac{2}{\cot\theta - \tan\theta}$

LHS $\equiv \tan 2\theta \equiv \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Divide the numerator and denominator by $\tan\theta$.

So $\tan 2\theta \equiv \dfrac{2}{\dfrac{1}{\tan\theta} - \tan\theta}$

$\equiv \dfrac{2}{\cot\theta - \tan\theta}$

Problem-solving

Dividing the numerator and denominator by a common term can be helpful when trying to rearrange an expression into a required form.

Example 20

Prove that $\sqrt{3}\cos 4\theta + \sin 4\theta \equiv 2\cos\left(4\theta - \frac{\pi}{6}\right)$

RHS $\equiv 2\cos\left(4\theta - \frac{\pi}{6}\right)$

$\equiv 2\cos 4\theta\cos\left(\frac{\pi}{6}\right) + 2\sin 4\theta\sin\left(\frac{\pi}{6}\right)$ — Use the addition formulae.

$\equiv 2\cos 4\theta\left(\frac{\sqrt{3}}{2}\right) + 2\sin 4\theta\left(\frac{1}{2}\right)$ — Write the exact values of $\cos\left(\frac{\pi}{6}\right)$ and $\sin\left(\frac{\pi}{6}\right)$

$\equiv \sqrt{3}\cos 4\theta + \sin 4\theta \equiv$ LHS

Problem-solving

Sometimes it is easier to begin with the RHS of the identity.

Exercise 4F

(P) **1** Prove the following identities:

a $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \cos A - \sin A$

b $\dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A} \equiv 2\operatorname{cosec} 2A \sin(B - A)$

c $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

d $\dfrac{\sec^2\theta}{1 - \tan^2\theta} \equiv \sec 2\theta$

e $2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \equiv \sin 2\theta$

f $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} \equiv 2$

g $\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv 2\sin\theta$

h $\dfrac{\sec\theta - 1}{\sec\theta + 1} \equiv \tan^2\left(\dfrac{\theta}{2}\right)$

i $\tan\left(\dfrac{\pi}{4} - x\right) \equiv \dfrac{1 - \sin 2x}{\cos 2x}$

(P) **2** Prove the identities:

a $\sin(A + 60°) + \sin(A - 60°) \equiv \sin A$

b $\frac{\cos A}{\sin B} - \frac{\sin A}{\cos B} \equiv \frac{\cos(A+B)}{\sin B \cos B}$

c $\frac{\sin(x+y)}{\cos x \cos y} \equiv \tan x + \tan y$

d $\frac{\cos(x+y)}{\sin x \sin y} + 1 \equiv \cot x \cot y$

e $\cos\left(\theta + \frac{\pi}{3}\right) + \sqrt{3}\sin\theta \equiv \sin\left(\theta + \frac{\pi}{6}\right)$

f $\cot(A+B) \equiv \frac{\cot A \cot B - 1}{\cot A + \cot B}$

g $\sin^2(45° + \theta) + \sin^2(45° - \theta) \equiv 1$

h $\cos(A+B)\cos(A-B) \equiv \cos^2 A - \sin^2 B$

(E/P) **3** **a** Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$ **(3 marks)**

b Hence find the value of $\tan 75° + \cot 75°$ **(2 marks)**

(E/P) **4** **a** Show that $\sin 3\theta \equiv 3\sin\theta\cos^2\theta - \sin^3\theta$ **(3 marks)**

b Show that $\cos 3\theta \equiv \cos^3\theta - 3\sin^2\theta\cos\theta$ **(3 marks)**

c Hence, or otherwise, show that $\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ **(4 marks)**

d Given that θ is acute and that $\cos\theta = \frac{1}{3}$, show that $\tan 3\theta = \frac{10\sqrt{2}}{23}$ **(3 marks)**

5 **a** Using $\cos 2A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$, show that:

i $\cos^2\left(\frac{x}{2}\right) \equiv \frac{1 + \cos x}{2}$ **ii** $\sin^2\left(\frac{x}{2}\right) \equiv \frac{1 - \cos x}{2}$

b Given that $\cos\theta = 0.6$, and that θ is acute, write down the values of:

i $\cos\left(\frac{\theta}{2}\right)$ **ii** $\sin\left(\frac{\theta}{2}\right)$ **iii** $\tan\left(\frac{\theta}{2}\right)$

c Show that $\cos^4\left(\frac{A}{2}\right) \equiv \frac{1}{8}(3 + 4\cos A + \cos 2A)$

(E/P) **6** Show that $\cos^4\theta \equiv \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$. You must show each stage of your working. **(6 marks)**

(E/P) **7** Prove that $\sin^2(x+y) - \sin^2(x-y) \equiv \sin 2x \sin 2y$ **(5 marks)**

(E/P) **8** Prove that $\cos 2\theta - \sqrt{3}\sin 2\theta \equiv 2\cos\left(2\theta + \frac{\pi}{3}\right)$ **(4 marks)**

(E/P) **9** Prove that $4\cos\left(2\theta - \frac{\pi}{6}\right) \equiv 2\sqrt{3} - 4\sqrt{3}\sin^2\theta + 4\sin\theta\cos\theta$ **(4 marks)**

(P) **10** Show that:

a $\cos\theta + \sin\theta \equiv \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

b $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv 2\sin\left(2\theta - \frac{\pi}{6}\right)$

Challenge

1 a Show that $\cos(A + B) - \cos(A - B) \equiv -2\sin A \sin B$

b Hence show that $\cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

c Express $3\sin x \sin 7x$ as the difference of cosines.

2 a Prove that $\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

b Hence, or otherwise, show that $2\sin\left(\frac{11\pi}{24}\right)\cos\left(\frac{5\pi}{24}\right) = \frac{\sqrt{3}+\sqrt{2}}{2}$

Chapter review 4

(P) **1** Without using a calculator, find the value of:

a $\sin 40° \cos 10° - \cos 40° \sin 10°$ **b** $\frac{1}{\sqrt{2}}\cos 15° - \frac{1}{\sqrt{2}}\sin 15°$ **c** $\frac{1 - \tan 15°}{1 + \tan 15°}$

(P) **2** Given that $\sin x = \frac{1}{\sqrt{5}}$ where x is acute and that $\cos(x - y) = \sin y$, show that $\tan y = \frac{\sqrt{5}+1}{2}$

(P) **3** The lines l_1 and l_2, with equations $y = 2x$ and $3y = x - 1$ respectively, are drawn on the same set of axes. Given that the scales are the same on both axes and that the angles l_1 and l_2 make with the positive x-axis are A and B respectively,

a write down the value of $\tan A$ and the value of $\tan B$

b without using your calculator, work out the acute angle between l_1 and l_2.

(P) **4** In $\triangle ABC$, $AB = 5$ cm and $AC = 4$ cm, $\angle ABC = (\theta - 30°)$ and $\angle ACB = (\theta + 30°)$. Using the sine rule, show that $\tan\theta = 3\sqrt{3}$

(P) **5** The first three terms of an arithmetic series are $\sqrt{3}\cos\theta$, $\sin(\theta - 30°)$ and $\sin\theta$, where θ is acute. Find the value of θ.

(P) **6** Two of the angles, A and B, in $\triangle ABC$ are such that $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$

a Find the exact value of: **i** $\sin(A + B)$ **ii** $\tan 2B$.

b By writing C as $180° - (A + B)$, show that $\cos C = -\frac{33}{65}$

(P) **7** The angles x and y are acute angles such that $\sin x = \frac{2}{\sqrt{5}}$ and $\cos y = \frac{3}{\sqrt{10}}$

a Show that $\cos 2x = -\frac{3}{5}$

b Find the value of $\cos 2y$.

c Show without using your calculator, that:

i $\tan(x + y) = 7$ **ii** $x - y = \frac{\pi}{4}$

(P) **8** Given that $\sin x \cos y = \frac{1}{2}$ and $\cos x \sin y = \frac{1}{3}$,

a show that $\sin(x + y) = 5\sin(x - y)$.

Given also that $\tan y = k$, express in terms of k:

b $\tan x$

c $\tan 2x$

(E/P) **9** **a** Given that $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$, show that $\tan 2\theta = \dfrac{1}{\sqrt{3}}$ **(2 marks)**

b Hence solve, for $0 \leqslant \theta \leqslant \pi$, the equation $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$ **(4 marks)**

(E/P) **10** **a** Show that $\cos 2\theta = 5\sin\theta$ may be written in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants to be found. **(3 marks)**

b Hence solve, for $-\pi \leqslant \theta \leqslant \pi$, the equation $\cos 2\theta = 5\sin\theta$ **(4 marks)**

(E/P) **11** **a** Given that $2\sin x = \cos(x - 60°)$, show that $\tan x = \dfrac{1}{4 - \sqrt{3}}$ **(4 marks)**

b Hence solve, for $0° \leqslant x \leqslant 360°$, $2\sin x = \cos(x - 60°)$, giving your answers to 1 decimal place. **(2 marks)**

(E/P) **12** **a** Given that $4\sin(x + 70°) = \cos(x + 20°)$, show that $\tan x = -\frac{3}{5}\tan 70°$ **(4 marks)**

b Hence solve, for $0° \leqslant x \leqslant 180°$, $4\sin(x + 70°) = \cos(x + 20°)$, giving your answers to 1 decimal place. **(3 marks)**

(P) **13** **a** Given that α is acute and $\tan\alpha = \frac{3}{4}$, prove that

$$3\sin(\theta + \alpha) + 4\cos(\theta + \alpha) \equiv 5\cos\theta$$

b Given that $\sin x = 0.6$ and $\cos x = -0.8$, evaluate $\cos(x + 270°)$ and $\cos(x + 540°)$

(E/P) **14** **a** Prove, by counter-example, that the statement

$$\sec(A + B) \equiv \sec A + \sec B, \text{ for all } A \text{ and } B$$

is false. **(2 marks)**

b Prove that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$, $\theta \neq \dfrac{n\pi}{2}$, $n \in \mathbb{Z}$ **(4 marks)**

(P) **15** Using $\tan 2\theta \equiv \dfrac{2\tan\theta}{1 - \tan^2\theta}$ with an appropriate value of θ,

a show that $\tan\left(\dfrac{\pi}{8}\right) = \sqrt{2} - 1$

b Use the result in **a** to find the exact value of $\tan\left(\dfrac{3\pi}{8}\right)$

(E/P) **16** **a** Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$, with $R > 0$ and $0° < \alpha < 90°$ **(4 marks)**

b Hence sketch the graph of $y = \sin x - \sqrt{3}\cos x$, for $-360° \leqslant x \leqslant 360°$, giving the coordinates of all points of intersection with the axes. **(4 marks)**

(E/P) **17** Given that $7\cos 2\theta + 24\sin 2\theta \equiv R\cos(2\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, find:

a the value of R and the value of α, to 2 decimal places **(4 marks)**

b the maximum value of $14\cos^2\theta + 48\sin\theta\cos\theta$ **(1 mark)**

c Solve the equation $7\cos 2\theta + 24\sin 2\theta = 12.5$, for $0 \leqslant \theta \leqslant \pi$, giving your answers to 2 decimal places. **(5 marks)**

18 a Express $1.5\sin 2x + 2\cos 2x$ in the form $R\sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving your values of R and α to 3 decimal places where appropriate. **(4 marks)**

b Express $3\sin x\cos x + 4\cos^2 x$ in the form $a\sin 2x + b\cos 2x + c$, where a, b and c are constants to be found. **(3 marks)**

c Hence, using your answer to part **a**, deduce the maximum value of $3\sin x\cos x + 4\cos^2 x$ **(1 mark)**

(E/P) **19 a** Given that $\sin^2\left(\frac{\theta}{2}\right) = 2\sin\theta$, show that $\sqrt{17}\sin(\theta + \alpha) = 1$ and state the value of α, where $0 \leqslant \alpha \leqslant \frac{\pi}{2}$ **(3 marks)**

b Hence, or otherwise, solve $\sin^2\left(\frac{\theta}{2}\right) = 2\sin\theta$ for $0° \leqslant \theta \leqslant 360°$ **(4 marks)**

(E/P) **20 a** Given that $2\cos\theta = 1 + 3\sin\theta$, show that $R\cos(\theta + \alpha) = 1$, where R and α are constants to be found, and $0 \leqslant \alpha \leqslant \frac{\pi}{2}$ **(2 marks)**

b Hence, or otherwise, solve $2\cos\theta = 1 + 3\sin\theta$ for $0° \leqslant \theta \leqslant 360°$ **(4 marks)**

(P) **21** Using known trigonometric identities, prove the following:

a $\sec\theta\,\mathrm{cosec}\,\theta \equiv 2\,\mathrm{cosec}\,2\theta$

b $\tan\left(\frac{\pi}{4} + x\right) - \tan\left(\frac{\pi}{4} - x\right) \equiv 2\tan 2x$

c $\sin(x + y)\sin(x - y) \equiv \cos^2 y - \cos^2 x$

d $1 + 2\cos 2\theta + \cos 4\theta \equiv 4\cos^2\theta\cos 2\theta$

(E/P) **22 a** Use the double-angle formulae to prove that $\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \tan^2 x$ **(4 marks)**

b Hence find, for $-\pi \leqslant x \leqslant \pi$, all the solutions of $\frac{1 - \cos 2x}{1 + \cos 2x} = 3$, leaving your answers in terms of π. **(2 marks)**

(E/P) **23 a** Prove that $\cos^4 2\theta - \sin^4 2\theta \equiv \cos 4\theta$ **(4 marks)**

b Hence find, for $0° \leqslant \theta \leqslant 180°$, all the solutions of $\cos^4 2\theta - \sin^4 2\theta = \frac{1}{2}$ **(2 marks)**

(E/P) **24 a** Prove that $\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$ **(4 marks)**

b Verify that $\theta = 180°$ is a solution of the equation $\sin 2\theta = 2 - 2\cos 2\theta$ **(1 mark)**

c Using the result in part **a**, or otherwise, find the two other solutions, $0° < \theta < 360°$, of the equation $\sin 2\theta = 2 - 2\cos 2\theta$ **(3 marks)**

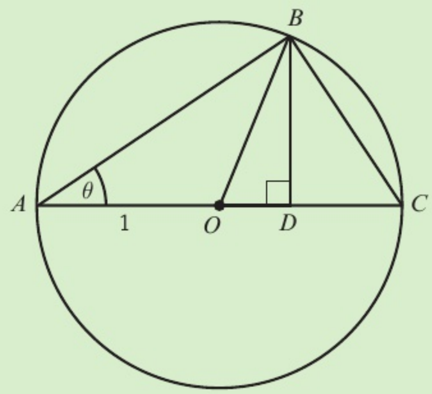
Challenge

1 Prove the identities:

a $\dfrac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv -\cot\theta$

b $\cos x + 2\cos 3x + \cos 5x \equiv 4\cos^2 x \cos 3x$

2 The points A, B and C lie on a circle with centre O and radius 1. AC is a diameter of the circle and point D lies on OC such that $\angle ODB = 90°$



Use this construction to prove:

a $\sin 2\theta \equiv 2\sin\theta\cos\theta$ **b** $\cos 2\theta \equiv 2\cos^2\theta - 1$

Hint Find expressions for $\angle BOD$ and AB, then consider the lengths OD and DB.

Summary of key points

1 The **addition** (or compound-angle) formulae are:

- $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$ $\sin(A - B) \equiv \sin A\cos B - \cos A\sin B$
- $\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$ $\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

2 The **double-angle** formulae are:

- $\sin 2A \equiv 2\sin A\cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

3 For positive values of a and b,

- $a\sin x \pm b\cos x$ can be expressed in the form $R\sin(x \pm \alpha)$
- $a\cos x \pm b\sin x$ can be expressed in the form $R\cos(x \mp \alpha)$

with $R > 0$ and $0° < \alpha < 90°$ $\left(\text{or } \dfrac{\pi}{2}\right)$

where $R\cos\alpha = a$, $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$

Review exercise 1

E **1** Express $\frac{4x}{x^2 - 2x - 3} + \frac{1}{x^2 + x}$ as a single fraction in its simplest form. **(4)**

← Pure 3 Section 1.1

E/P **2** $f(x) = 1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}, x \neq -2$

a Show that $f(x) = \frac{x^2 + x + 1}{(x+2)^2}, x \neq -2$ **(2)**

b Show that $x^2 + x + 1 > 0$ for all values of $x, x \neq -2$ **(2)**

c Show that $f(x) > 0$ for all values of $x, x \neq -2$ **(2)**

← Pure 3 Section 1.1

E **3** Given that $\frac{3x^2 + 6x - 2}{x^2 + 4} \equiv d + \frac{ex + f}{x^2 + 4}$

find the values of d, e and f. **(4)**

← Pure 3 Section 1.2

E **4** Solve the inequality $|4x + 3| > 7 - 2x$ **(3)**

← Pure 3 Section 2.1

E/P **5** The function p(x) is defined by

$$p:x \mapsto \begin{cases} 4x + 5, & x < -2 \\ -x^2 + 4, & x \geqslant -2 \end{cases}$$

a Sketch p(x), stating its range. **(3)**

b Find the exact values of a such that $p(a) = -20$ **(4)**

← Pure 3 Section 2.2

E/P **6** The functions p and q are defined by

$p(x) = \frac{1}{x+4}, x \in \mathbb{R}, x \neq -4$

$q(x) = 2x - 5, x \in \mathbb{R}$

a Find an expression for qp(x) in the form $\frac{ax + b}{cx + d}$ **(3)**

b Solve $qp(x) = 15$ **(3)**

Let $r(x) = qp(x)$

c Find $r^{-1}(x)$, stating its domain. **(3)**

← Pure 3 Sections 2.3, 2.4

E/P **7** The functions f and g are defined by:

$f:x \mapsto \frac{x+2}{x}, x \in \mathbb{R}, x \neq 0$

$g:x \mapsto \ln(2x - 5), x \in \mathbb{R}, x > \frac{5}{2}$

a Sketch the graph of f. **(3)**

b Show that $f^2(x) = \frac{3x + 2}{x + 2}$ **(3)**

c Find the exact value of $gf\left(\frac{1}{4}\right)$ **(2)**

d Find $g^{-1}(x)$, stating its domain. **(3)**

← Pure 3 Sections 2.3, 2.4

E/P **8** The functions p and q are defined by:

$p(x) = 3x + b, x \in \mathbb{R}$

$q(x) = 1 - 2x, x \in \mathbb{R}$

Given that $pq(x) = qp(x)$,

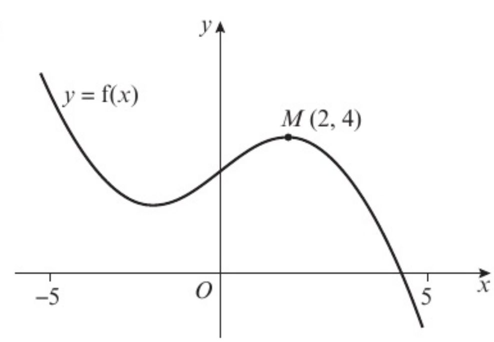
a show that $b = -\frac{2}{3}$ **(3)**

b find $p^{-1}(x)$ and $q^{-1}(x)$ **(3)**

c show that $p^{-1}q^{-1}(x) = q^{-1}p^{-1}(x) = \frac{ax + b}{c}$, where a, b and c are integers to be found. **(4)**

← Pure 3 Sections 2.3, 2.4

E **9**



The figure shows the graph of $y = f(x), -5 \leqslant x \leqslant 5$

The point $M(2, 4)$ is the maximum turning point of the graph.

Sketch, on separate diagrams, the graphs of:

a $y = \mathrm{f}(x) + 3$ **(2)**

b $y = |\mathrm{f}(x)|$ **(2)**

c $y = \mathrm{f}(|x|)$ **(2)**

Show on each graph the coordinates of any maximum turning points.

← Pure 3 Sections 2.5, 2.6

E/P **10** The function h is defined by

$\mathrm{h}: x \mapsto 2(x+3)^2 - 8,\ x \in \mathbb{R}$

a Draw a sketch of $y = \mathrm{h}(x)$, labelling the turning points and the x- and y-intercepts. **(4)**

b Write down the coordinates of the turning points on the graphs with equations:

i $y = 3\mathrm{h}(x+2)$ **(2)**

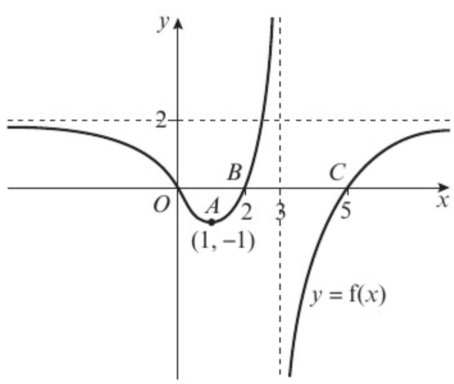
ii $y = \mathrm{h}(-x)$ **(2)**

iii $y = |\mathrm{h}(x)|$ **(2)**

c Sketch the curve with equation $y = \mathrm{h}(-|x|)$. On your sketch, show the coordinates of all turning points and all x- and y-intercepts. **(4)**

← Pure 3 Sections 2.5, 2.6

E **11**



The diagram shows a sketch of the graph of $y = \mathrm{f}(x)$.

The curve has a minimum at the point $A(1, -1)$, passes through the x-axis at the origin, and the points $B(2, 0)$ and $C(5, 0)$; the asymptotes have equations $x = 3$ and $y = 2$.

a Sketch, on separate axes, the graphs of:

i $y = |\mathrm{f}(x)|$ **(2)**

ii $y = -\mathrm{f}(x+1)$ **(2)**

iii $y = \mathrm{f}(-2x)$ **(2)**

in each case, showing the images of the points A, B and C.

b State the number of solutions to each equation:

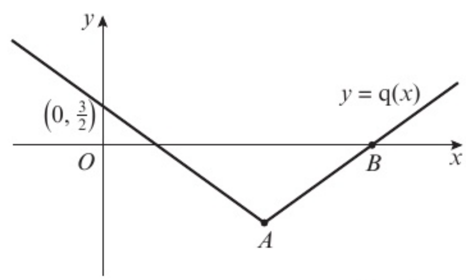
i $3|\mathrm{f}(x)| = 2$ **(2)**

ii $2|\mathrm{f}(x)| = 3$ **(2)**

← Pure 3 Sections 2.6, 2.7

E/P **12** The diagram shows a sketch of part of the graph $y = \mathrm{q}(x)$, where

$\mathrm{q}(x) = \frac{1}{2}|x + b| - 3,\ b < 0$



The graph cuts the y-axis at $\left(0, \frac{3}{2}\right)$.

a Find the value of b. **(2)**

b Find the coordinates of A and B. **(3)**

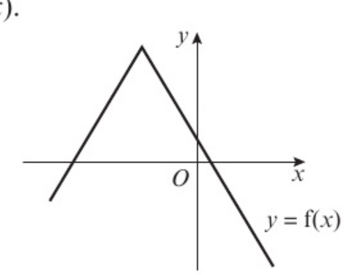
c Solve $\mathrm{q}(x) = -\frac{1}{3}x + 5$ **(5)**

← Pure 3 Section 2.7

E/P **13** The function f is defined by

$\mathrm{f}(x) = -\frac{5}{3}|x + 4| + 8,\ x \in \mathbb{R}$

The diagram shows a sketch of the graph $y = \mathrm{f}(x)$.



a State the range of f. **(1)**

b Give a reason why $\mathrm{f}^{-1}(x)$ does not exist. **(1)**

c Solve the inequality $\mathrm{f}(x) > \frac{2}{3}x + 4$ **(5)**

d State the range of values of k for which the equation $f(x) = \frac{5}{3}x + k$ has no solutions. **(2)**

← Pure 3 Section 2.7

E/P **14 a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 4 - 2\operatorname{cosec} x$.
Mark any asymptotes on your graph. **(3)**

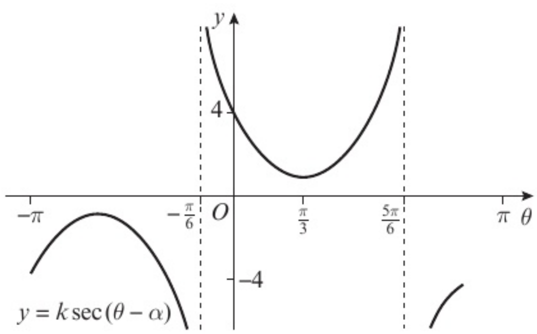
b Hence deduce the range of values of k for which the equation $4 - 2\operatorname{cosec} x = k$ has no solutions. **(2)**

← Pure 3 Sections 3.1, 3.2

E/P **15** The diagram shows the graph of

$$y = k\sec(\theta - \alpha)$$

The curve crosses the y-axis at the point (0, 4), and the θ-coordinate of its minimum point is $\frac{\pi}{3}$



a State, as a multiple of π, the value of α. **(1)**

b Find the value of k. **(2)**

c Find the exact values of θ at the points where the graph crosses the line $y = -2\sqrt{2}$ **(3)**

← Pure 3 Section 3.2

E/P **16 a** Show that

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} \equiv 2\sec x$$ **(4)**

b Hence solve, in the interval $0 \leqslant x \leqslant 4\pi$,
$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} = -2\sqrt{2}$$ **(4)**

← Pure 3 Section 3.3

E/P **17 a** Prove that

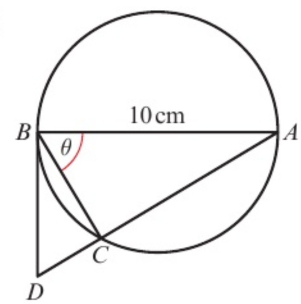
$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 2\operatorname{cosec} 2\theta, \ \theta \neq 90n°$$ **(3)**

b Sketch the graph of $y = 2\operatorname{cosec} 2\theta$ for $0° < \theta < 360°$. **(3)**

c Solve, for $0° < \theta < 360°$, the equation $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 3$, giving your answer to 1 decimal place. **(4)**

← Pure 3 Section 3.3

E/P **18**



In the diagram, $AB = 10$ cm is the diameter of the circle and BD is the tangent to the circle at B. The chord AC is extended to meet this tangent at D and $\angle ABC = \theta$

a Show that $BD = 10\cot\theta$ **(4)**

b Given that $BD = \frac{10}{\sqrt{3}}$ cm, calculate the exact length of DC. **(3)**

← Pure 3 Section 3.4

E/P **19 a** Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \tan^2\theta = \sec^2\theta$ **(2)**

b Solve, for $0° \leqslant \theta < 360°$, the equation

$$2\tan^2\theta + \sec\theta = 1$$

giving your answers to 1 decimal place. **(6)**

← Pure 3 Section 3.3

E/P **20** Given that $a = \operatorname{cosec} x$ and $b = 2\sin x$,

a express a in terms of b **(2)**

b find the value of $\frac{4 - b^2}{a^2 - 1}$ in terms of b. **(2)**

← Pure 3 Section 3.4

E/P **21** Given that

$$y = \arcsin x,\ -1 \leqslant x \leqslant 1,\ -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

a express $\arccos x$ in terms of y. **(2)**

b Hence find, in terms of π, the value of $\arcsin x + \arccos x$ **(1)**

← Pure 3 Section 3.5

E **22** **a** Prove that, for $x \geqslant 1$,

$$\arccos \frac{1}{x} = \arcsin \frac{\sqrt{x^2-1}}{x}$$ **(4)**

b Explain why this identity is not true for $0 \leqslant x < 1$ **(2)**

← Pure 3 Section 3.5

E **23** **a** Sketch the graph of $y = 2\arccos x - \frac{\pi}{2}$, showing clearly the exact endpoints of the curve. **(4)**

b Find the exact coordinates of the point where the curve crosses the x-axis. **(3)**

← Pure 3 Section 3.5

E **24** Given that $\tan\left(x + \frac{\pi}{6}\right) = \frac{1}{6}$, show that

$$\tan x = \frac{72 - 111\sqrt{3}}{321}$$ **(5)**

← Pure 3 Section 4.1

E/P **25** Given that $\sin(x + 30°) = 2\sin(x - 60°)$

a show that $\tan x = 8 + 5\sqrt{3}$ **(4)**

b Hence, express $\tan(x + 60°)$ in the form $a + b\sqrt{3}$ **(3)**

← Pure 3 Section 4.1

E/P **26** **a** Use $\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$, or otherwise, to show that

$$\sin 165° = \frac{\sqrt{6} - \sqrt{2}}{4}$$ **(4)**

b Hence, or otherwise, show that $\operatorname{cosec} 165° = \sqrt{a} + \sqrt{b}$, where a and b are constants to be found. **(3)**

← Pure 3 ections 4.1, 4.2

E/P **27** Given that $\cos A = \frac{3}{4}$ where $270° < A < 360°$,

a find the exact value of $\sin 2A$ **(3)**

b show that $\tan 2A = -3\sqrt{7}$ **(3)**

← Pure 3 Section 4.3

E/P **28** Solve, in the interval $-180° \leqslant x \leqslant 180°$, the equations

a $\cos 2x + \sin x = 1$ **(3)**

b $\sin x(\cos x + \operatorname{cosec} x) = 2\cos^2 x$ **(3)**

giving your answers to 1 decimal place.

← Pure 3 Section 4.4

E **29** $f(x) = 3\sin x + 2\cos x$

Given $f(x) = R\sin(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$,

a find the value of R and the value of α. **(4)**

b Hence, find the greatest value of $(3\sin x + 2\cos x)^4$ **(2)**

c Hence, or otherwise, solve for $0 \leqslant \theta < 2\pi$, $f(x) = 1$, rounding your answers to 3 decimal places. **(3)**

← Pure 3 Section 4.5

E **30** **a** Prove that

$$\cot\theta - \tan\theta \equiv 2\cot 2\theta,\ \theta \neq \frac{n\pi}{2}$$ **(3)**

b Solve, for $-\pi < \theta < \pi$, the equation

$$\cot\theta - \tan\theta = 5$$

giving your answers to 3 significant figures. **(3)**

← Pure 3 Sections 3.3, 4.6

E **31** **a** By writing $\cos 3\theta$ as $\cos(2\theta + \theta)$, show that

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$$ **(4)**

b Given that $\cos\theta = \frac{\sqrt{2}}{3}$, find the exact value of $\sec 3\theta$. Give your answer in the form $k\sqrt{2}$ where k is a rational constant to be found. **(3)**

← Pure 3 Sections 3.3, 4.1

E **32** Show that $\sin^4\theta \equiv \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$

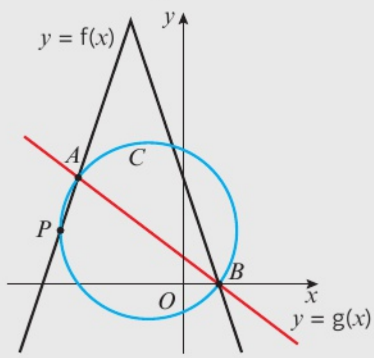
You must show each stage of your working. **(6)**

← Pure 3 Section 4.6

Challenge

SKILLS INNOVATION

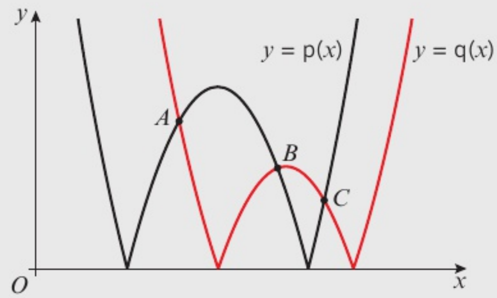
1 The functions f and g are defined by
$f(x) = -3|x + 3| + 15, x \in \mathbb{R}$
$g(x) = -\frac{3}{4}x + \frac{3}{2}, x \in \mathbb{R}$
The diagram shows a sketch of the graphs $y = f(x)$ and $y = g(x)$, which intersect at points A and B. M is the **midpoint** of AB. The circle C, with centre M, passes through points A and B, and meets $y = f(x)$ at point P as shown in the diagram.



a Find the equation of the circle.
b Find the area of the triangle APB.

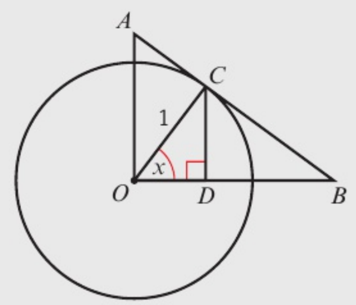
← Pure 3 Sections 2.2, 2.6

2 The diagram shows a sketch of the functions $p(x) = |x^2 - 8x + 12|$ and $q(x) = |x^2 - 11x + 28|$



Find the exact values of the x-coordinates of the points A, B and C. ← Pure 3 Section 2.5

3 The diagram shows a circle, centre O. The radius of the circle, OC, is 1, and $\angle CDO = 90°$



Given that $\angle COD = x$, express the following lengths as single trigonometric functions of x

a CD **b** OD **c** OA
d AC **e** CB **f** OB

← Pure 3 Section 3.1

5 EXPONENTIALS AND LOGARITHMS

3.1
3.2
3.3

Learning objectives

After completing this unit you should be able to:

- Sketch graphs of the form $y = a^x$, $y = e^x$, $y = e^{ax+b} + c$, and transformations of these graphs → pages 103–105
- Differentiate e^x and understand why this result is important → pages 105–108
- Describe and use the natural logarithm function → pages 108–110
- Use logarithms to estimate the values of constants in non-linear models → pages 110–116
- Use and interpret models that use exponential functions → pages 116–118

Prior knowledge check

1 Given that $x = 3$ and $y = -1$, evaluate these expressions without using a calculator:

a 5^x **b** 3^y **c** 2^{2x-1} **d** 7^{1-y} **e** 11^{x+3y}

← International GCSE Mathematics

2 Simplify these expressions, writing each answer as a single power:

a $6^8 \div 6^2$ **b** $y^3 \times (y^9)^2$ **c** $\frac{2^5 \times 2^9}{2^8}$ **d** $\sqrt{x^8}$

← International GCSE Mathematics

3 Plot the following data on a scatter graph and draw a line of best fit.

x	1.2	2.1	3.5	4	5.8
y	5.8	7.4	9.4	10.3	12.8

Determine the gradient and y-intercept of your line of best fit, giving your answers to 1 decimal place.

← International GCSE Mathematics

Radioactive atoms contain an excess of energy in their nucleus (i.e. more energy than is needed). To become stable, they release this excess energy as alpha, beta or gamma radiation. The time it takes a radioactive atom to decrease to half its original value is called the half-life. This is an exponential decay.

5.1 Exponential functions

Functions of the form $f(x) = a^x$, where a is a constant, are called **exponential** functions. You should become familiar with these functions and the shapes of their graphs.

For example, look at a table of values of $y = 2^x$

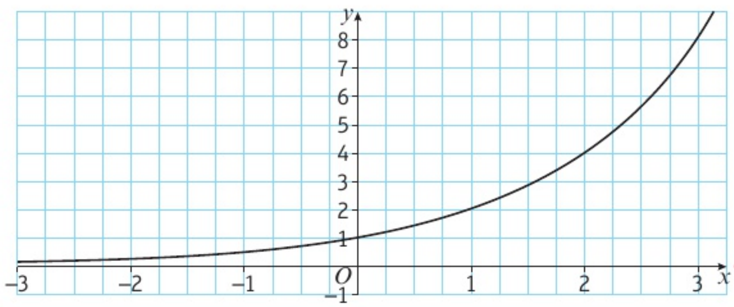
x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notation In the expression 2^x, x can be called an **index**, a **power** or an **exponent**

The value of 2^x tends toward 0 as x decreases, and grows without limit as x increases.

Links Recall that $2^0 = 1$ and that $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ ← Pure 1 Section 1.4

The graph of $y = 2^x$ is a smooth curve that looks like this:



The x-axis is an asymptote to the curve.

Example 1

a On the same axes, sketch the graphs of $y = 3^x$, $y = 2^x$ and $y = 1.5^x$

b On another set of axes, sketch the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$

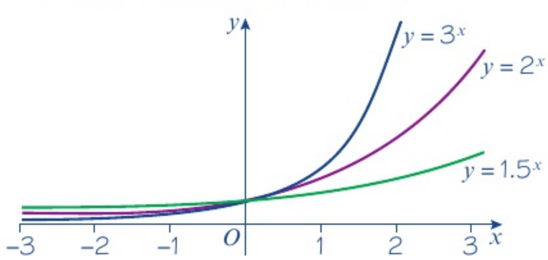
a For all three graphs, $y = 1$ when $x = 0$

$a^0 = 1$

When $x > 0$, $3^x > 2^x > 1.5^x$

When $x < 0$, $3^x < 2^x < 1.5^x$

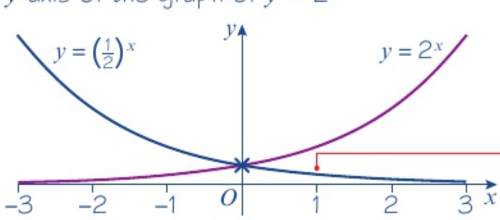
Work out the relative positions of the three graphs.



Whenever $a > 1$, $f(x) = a^x$ is an increasing function. In this case, the value of a^x grows without limit as x **increases**, and tends toward 0 as x **decreases**.

b The graph of $y = \left(\frac{1}{2}\right)^x$ is a reflection in the y-axis of the graph of $y = 2^x$

Since $\frac{1}{2} = 2^{-1}$, $y = \left(\frac{1}{2}\right)^x$ is the same as $y = (2^{-1})^x = 2^{-x}$



Whenever $0 < a < 1$, $f(x) = a^x$ is a decreasing function. In this case, the value of a^x tends toward 0 as x **increases**, and grows without limit as x **decreases**.

Example 2

Sketch the graph of $y = \left(\frac{1}{2}\right)^{x-3}$ and give the coordinates of the point where the graph crosses the y-axis.

If $f(x) = \left(\frac{1}{2}\right)^x$ then $y = f(x - 3)$

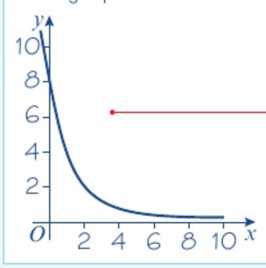
The graph is a translation of the graph $y = \left(\frac{1}{2}\right)^x$ by the vector $\binom{3}{0}$

The graph crosses the y-axis when $x = 0$

$y = \left(\frac{1}{2}\right)^{0-3}$

$y = 8$

The graph crosses the y-axis at (0, 8)



Problem-solving

If you have to sketch the graph of an unfamiliar function, try writing it as a transformation of a familiar function. ← Pure 1 Section 4.4

You can also consider this graph as a stretch of the graph $y = \left(\frac{1}{2}\right)^x$

$$y = \left(\frac{1}{2}\right)^{x-3}$$
$$= \left(\frac{1}{2}\right)^x \times \left(\frac{1}{2}\right)^{-3}$$
$$= \left(\frac{1}{2}\right)^x \times 8$$
$$= 8\left(\frac{1}{2}\right)^x = 8f(x)$$

So the graph of $y = \left(\frac{1}{2}\right)^{x-3}$ is a vertical stretch of the graph of $y = \left(\frac{1}{2}\right)^x$ with scale factor 8.

Exercise 5A

SKILLS INTERPRETATION

1 **a** Draw an accurate graph of $y = (1.7)^x$ for $-4 \leqslant x \leqslant 4$

b Use your graph to solve the equation $(1.7)^x = 4$

2 **a** Draw an accurate graph of $y = (0.6)^x$ for $-4 \leqslant x \leqslant 4$

b Use your graph to solve the equation $(0.6)^x = 2$

3 Sketch the graph of $y = 1^x$

(P) 4 For each of these statements, decide whether it is true or false, justifying your answer or offering a counter-example.

a The graph of $y = a^x$ passes through (0, 1) for all positive real numbers a.

b The function $f(x) = a^x$ is always an increasing function for $a > 0$

c The graph of $y = a^x$, where a is a positive real number, never crosses the x-axis.

5 The function $f(x)$ is defined as $f(x) = 3^x$, $x \in \mathbb{R}$. On the same axes, sketch the graphs of:

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = f(x) - 4$ **d** $y = f\left(\frac{1}{2}x\right)$

Write down the coordinates of the point where each graph crosses the y-axis, and give the equations of any asymptotes.

(P) 6 The graph of $y = ka^x$ passes through the points (1, 6) and (4, 48). Find the values of the constants k and a.

Hint Substitute the coordinates into $y = ka^x$ to create two simultaneous equations. Use division to eliminate one of the two unknowns.

7 The graph of $y = pq^x$ passes through the points $(-3, 150)$ and $(2, 0.048)$

a By drawing a sketch or otherwise, explain why $0 < q < 1$

b Find the values of the constants p and q.

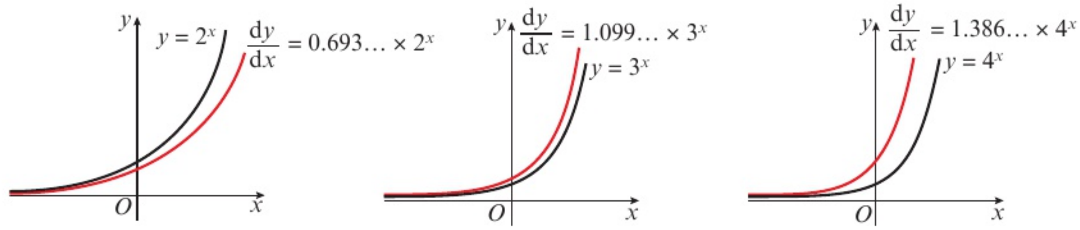
Challenge

SKILLS CREATIVITY

Sketch the graph of $y = 2^{x-2} + 5$, giving the coordinates of the point where the graph crosses the y-axis.

5.2 $y = e^{ax+b} + c$

Exponential functions of the form $f(x) = a^x$ have a special mathematical property. The graphs of their gradient functions are a similar shape to the graphs of the functions themselves.



In each case $f'(x) = kf(x)$, where k is a constant. As the value of a increases, so does the value of k.

Something unique happens between $a = 2$ and $a = 3$. There is going to be a value of a where the gradient function is exactly the same as the original function. This occurs when a is approximately equal to 2.71828. The exact value is represented by the letter e. Like π, e is both an important mathematical constant and an irrational number.

Function	Gradient function
$f(x) = 1^x$	$f'(x) = 0 \times 1^x$
$f(x) = 2^x$	$f'(x) = 0.693\ldots \times 2^x$
$f(x) = 3^x$	$f'(x) = 1.099\ldots \times 3^x$
$f(x) = 4^x$	$f'(x) = 1.386\ldots \times 4^x$

Online Explore the relationship between exponential functions and their **derivatives** using technology.

- **For all real values of x:**
  - **If $f(x) = e^x$ then $f'(x) = e^x$**
  - **If $y = e^x$ then $\frac{dy}{dx} = e^x$**

A similar result holds for functions such as e^{5x}, e^{-x} and $e^{\frac{1}{2}x}$

- **For all real values of x and for any constant k:**
  - **If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$**
  - **If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$**

Example 3

Differentiate with respect to x.

a e^{4x} **b** $e^{-\frac{1}{2}x}$ **c** $3e^{2x}$

a $y = e^{4x}$

$\frac{dy}{dx} = 4e^{4x}$

Use the rule for differentiating e^{kx} with $k = 4$

b $y = e^{-\frac{1}{2}x}$

$\frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}$

c $y = 3e^{2x}$

$\frac{dy}{dx} = 2 \times 3e^{2x} = 6e^{2x}$

To differentiate ae^{kx}, multiply the whole function by k. The derivative is kae^{kx}

Example 4

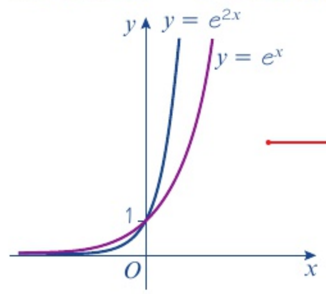
Sketch the graphs of the following equations. Give the coordinates of any points where the graphs cross the axes, and state the equations of any asymptotes.

a $y = e^{2x}$ **b** $y = 10e^{-x}$ **c** $y = 3 + 4e^{\frac{1}{2}x}$

a $y = e^{2x}$

When $x = 0$, $y = e^{2\times 0} = 1$ so the graph crosses the y-axis at (0, 1).

The x-axis ($y = 0$) is an asymptote.



The graph of $y = e^x$ has been shown in purple on this sketch.

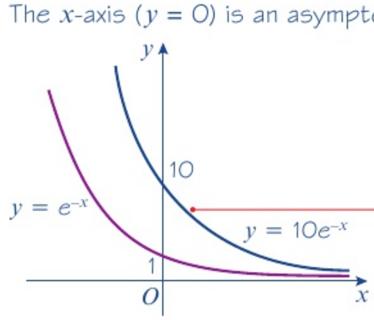
This is a stretch of the graph of $y = e^x$, **parallel** to the x-axis and with scale factor $\frac{1}{2}$

← Pure 1 Section 4.6

b $y = 10e^{-x}$

When $x = 0$, $y = 10e^{-0}$. So the graph crosses the y-axis at (0, 10).

The x-axis ($y = 0$) is an asymptote.



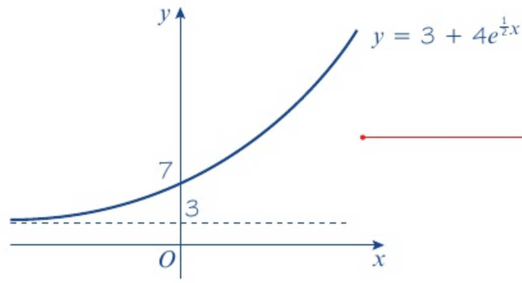
Negative powers of e^x, such as e^{-x} or e^{-4x}, give rise to decreasing functions.

The graph of $y = e^x$ has been reflected in the y-axis and stretched parallel to the y-axis with scale factor 10.

c $y = 3 + 4e^{\frac{1}{2}x}$

When $x = 0$, $y = 3 + 4e^{\frac{1}{2}\times 0} = 7$ so the graph crosses the y-axis at (0, 7).

The line $y = 3$ is an asymptote.



The graph of $y = e^{\frac{1}{2}x}$ has been stretched parallel to the y-axis with scale factor 4 and then translated by $\begin{pmatrix}0\\3\end{pmatrix}$

Problem-solving

If you have to sketch a transformed graph with an asymptote, it is often easier to sketch the asymptote first.

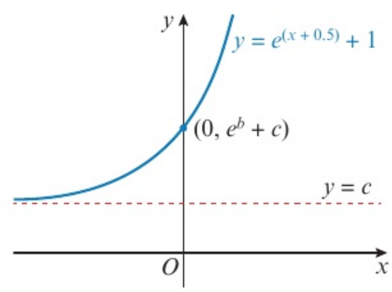
Online Use technology to draw transformations of $y = e^x$

We can develop Example 4c above into a general case:

$$y = e^{ax+b} + c$$

A little calculation will show that the asymptote is $y = c$

This will help to sketch the curve.



Exercise 5B

SKILLS INTERPRETATION

1 Use a calculator to find the value of e^x to 5 decimal places when:

a $x = 1$ **b** $x = 4$ **c** $x = -10$ **d** $x = 0.2$

2 a Draw an accurate graph of $y = e^x$ for $-4 \leqslant x \leqslant 4$

b By drawing **appropriate** tangent lines, estimate the gradient at $x = 1$ and $x = 3$

c Compare your answers to the actual values of e and e^3.

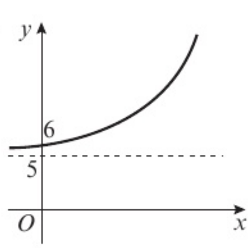
3 Sketch the graphs of:

a $y = e^{x+1}$ **b** $y = 4e^{-2x}$ **c** $y = 2e^x - 3$

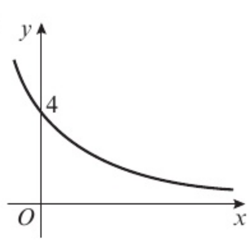
d $y = 4 - e^x$ **e** $y = 6 + 10e^{\frac{1}{2}x}$ **f** $y = 100e^{-x} + 10$

4 Each of the sketch graphs below is of the form $y = Ae^{bx} + C$, where A, b and C are constants. Find the values of A and C for each graph, and state whether b is positive or negative.

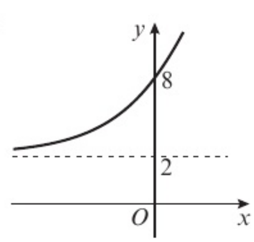
a



b



c



Hint You do not have enough information to work out the value of b, so simply state whether it is positive or negative.

5 Rearrange $f(x) = e^{3x+2}$ into the form $f(x) = Ae^{bx}$, where A and b are constants whose values are to be found. Hence, or otherwise, sketch the graph of $y = f(x)$.

Hint $e^{m+n} = e^m \times e^n$

6 Differentiate the following with respect to x:

a e^{6x} **b** $e^{-\frac{1}{3}x}$ **c** $7e^{2x}$

d $5e^{0.4x}$ **e** $e^{3x} + 2e^x$ **f** $e^x(e^x + 1)$

Hint For part **f**, start by expanding the bracket.

7 Find the gradient of the curve with equation $y = e^{3x}$ at the point where:

a $x = 2$ **b** $x = 0$ **c** $x = -0.5$

8 The function f is defined as $f(x) = e^{0.2x}$, $x \in \mathbb{R}$. Show that the tangent to the curve at the point (5, e) goes through the origin.

5.3 Natural logarithms

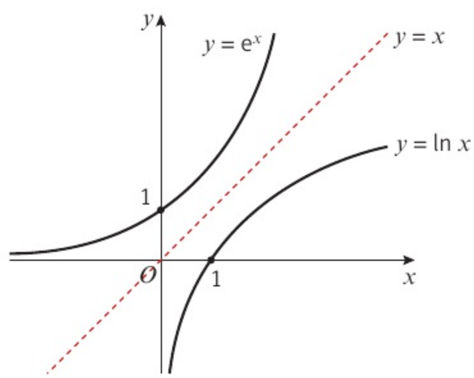
■ **The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$**

The graph of $y = \ln x$ passes through (1, 0) and does not cross the y-axis.

The y-axis is an asymptote of the graph $y = \ln x$.
This means that $\ln x$ is defined only for positive values of x.

As x increases, $\ln x$ grows without limit, but relatively slowly.

You can also use the fact that **logarithms** are the inverses of exponential functions to solve equations involving powers and logarithms.



■ $e^{\ln x} = \ln(e^x) = x$

Notation $\ln x = \log_e x$

Example 5

Solve these equations, giving your answers in exact form.

a $e^x = 5$ **b** $\ln x = 3$

a When $e^x = 5$

$\ln(e^x) = \ln 5$

$x = \ln 5$

b When $\ln x = 3$

$e^{\ln x} = e^3$

$x = e^3$

The inverse operation of raising e to the power x is taking natural logarithms (logarithms to the base e) and vice versa.

You can write the natural logarithm on both sides. $\ln(e^x) = x$

Leave your answer as a logarithm or a power of e so that it is exact.

Example 6

Solve these equations, giving your answers in exact form.

a $e^{2x+3} = 7$ **b** $2\ln x + 1 = 5$ **c** $e^{2x} + 5e^x = 14$

a $e^{2x+3} = 7$

$2x + 3 = \ln 7$

$2x = \ln 7 - 3$

$x = \frac{1}{2}\ln 7 - \frac{3}{2}$

Take natural logarithms of both sides and use the fact that the inverse of e^x is $\ln x$.

b $2\ln x + 1 = 5$

$2\ln x = 4$

$\ln x = 2$

$x = e^2$

Rearrange to make $\ln x$ the subject.

The inverse of $\ln x$ is e^x

c $e^{2x} + 5e^x = 14$

$e^{2x} + 5e^x - 14 = 0$

$(e^x + 7)(e^x - 2) = 0$

$e^x = -7$ or $e^x = 2$

$e^x = 2$

$x = \ln 2$

$e^{2x} = (e^x)^2$, so this is a quadratic function of e^x Start by setting the equation equal to 0 and factorise. You could also use the substitution $u = e^x$ and write the equation as $u^2 + 5u - 14 = 0$

Watch out e^x is always positive, so you can't have $e^x = -7$. You need to discard this solution.

Exercise 5C

1 Solve these equations, giving your answers in exact form.

a $e^x = 6$ **b** $e^{2x} = 11$ **c** $e^{-x+3} = 20$

d $3e^{4x} = 1$ **e** $e^{2x+6} = 3$ **f** $e^{5-x} = 19$

2 Solve these equations, giving your answers in exact form.

a $\ln x = 2$ **b** $\ln(4x) = 1$ **c** $\ln(2x + 3) = 4$

d $2\ln(6x - 2) = 5$ **e** $\ln(18 - x) = \frac{1}{2}$ **f** $\ln(x^2 - 7x + 11) = 0$

3 Solve these equations, giving your answers in exact form.

a $e^{2x} - 8e^x + 12 = 0$ **b** $e^{4x} - 3e^{2x} = -2$

c $(\ln x)^2 + 2\ln x - 15 = 0$ **d** $e^x - 5 + 4e^{-x} = 0$

e $3e^{2x} + 5 = 16e^x$ **f** $(\ln x)^2 = 4(\ln x + 3)$

Hint All of the equations in question 3 are quadratic equations in a function of x.

Hint First in part **d** multiply each term by e^x

E/P **4** Find the exact solutions to the equation $e^x + 12e^{-x} = 7$ **(4 marks)**

5 Solve these equations, giving your answers in exact form.

a $\ln(8x - 3) = 2$ **b** $e^{5(x-8)} = 3$ **c** $e^{10x} - 8e^{5x} + 7 = 0$ **d** $(\ln x - 1)^2 = 4$

E/P **6** Solve $3^x e^{4x-1} = 5$, giving your answer in the form $\frac{a + \ln b}{c + \ln d}$ **(5 marks)**

Hint Take natural logarithms of both sides and then apply the laws of logarithms.

P **7** Officials are testing athletes for banned medicines at a sporting event. They model the concentration of a particular substance in an athlete's bloodstream using the equation $D = 6e^{\frac{-t}{10}}$ where D is the concentration of the substance in mg/l, and t is the time in hours since the athlete took the substance.

a Interpret the meaning of the constant 6 in this model.

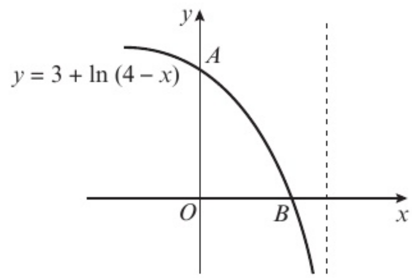
b Find the concentration of the substance in the bloodstream after 2 hours.

c It is impossible to detect this substance in the bloodstream if the concentration is lower than 3 mg/l. Show that this happens after $t = -10 \ln\left(\frac{1}{2}\right)$ and convert this result into hours and minutes.

E/P **8** The graph of $y = 3 + \ln(4 - x)$ is shown to the right.

a State the exact coordinates of point A. **(1 mark)**

b Calculate the exact coordinates of point B. **(3 marks)**



Challenge

The graph of the function $g(x) = Ae^{Bx} + C$ passes through (0, 5) and (6, 10). Given that the line $y = 2$ is an asymptote to the graph, show that $B = \frac{1}{6}\ln\left(\frac{8}{3}\right)$

5.4 Logarithms and non-linear data

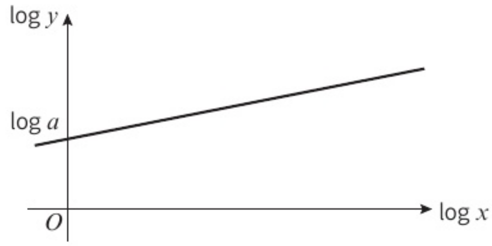
Logarithms can also be used to manage and explore non-linear **trends** in data.

Case 1: $y = ax^n$

Start with a non-linear relationship —— $y = ax^n$
Take logs of both sides ($\log = \log_{10}$) —— $\log y = \log ax^n$
Use the multiplication law —— $\log y = \log a + \log x^n$
Use the power law —— $\log y = \log a + n \log x$
Compare this equation to the common form of a straight line, $Y = MX + C$

$\log y$ variable	=	n constant (gradient)	$\log x$ variable	+	$\log a$ constant (intercept)
Y variable	=	M constant (gradient)	X variable	+	C constant (intercept)

- If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$



Example 7

The table below gives the rank (by size) and population of the UK's largest cities and districts (London is ranked number 1 but has been excluded as an **outlier**).

City	Birmingham	Leeds	Glasgow	Sheffield	Bradford
Rank, R	2	3	4	5	6
Population, P (2 s.f.)	1 000 000	730 000	620 000	530 000	480 000

The relationship between the rank and population can be modelled by the formula

$P = aR^n$ where a and n are constants.

a Draw a table giving values of $\log R$ and $\log P$ to 2 decimal places.

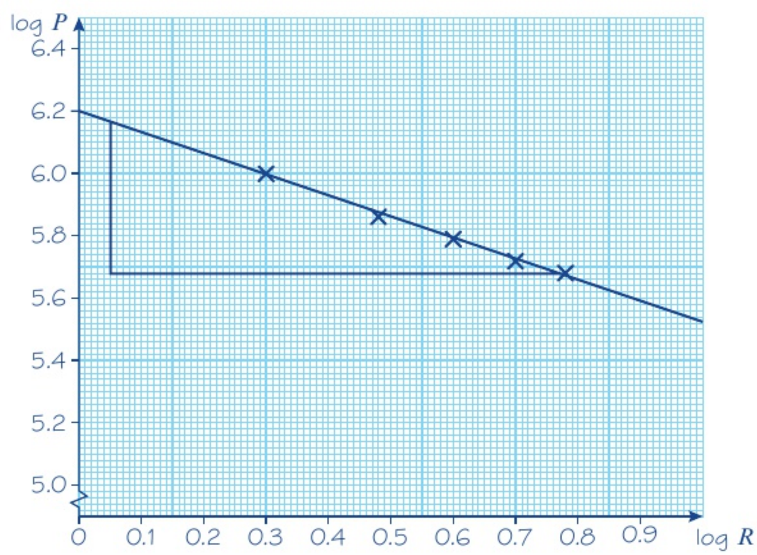
b Plot a graph of $\log R$ against $\log P$ using the values from your table and draw a line of best fit.

c Use your graph to estimate the values of a and n to 2 significant figures.

a

$\log R$	0.30	0.48	0.60	0.70	0.78
$\log P$	6	5.86	5.79	5.72	5.68

b



c $P = aR^n$

$\log P = \log aR^n$

$= \log a + \log R^n$

$= \log a + n \log R$

so the gradient is n and the intercept is $\log a$

Reading the gradient from the graph,

$$n = \frac{5.68 - 6.16}{0.77 - 0.05} = \frac{-0.48}{0.72} = -0.67$$

Reading the intercept from the graph,

$\log a = 6.2$

$a = 10^{6.2} = 1\,600\,000$ (2 s.f.)

> Start with the formula given in the question. Take logs of both sides and use the laws of logarithms to rearrange it into a linear relationship between $\log P$ and $\log R$.

> The gradient of the line of best fit will give you your value for n.

> The vertical intercept will give you the value of $\log a$. You need to raise 10 to this power to find the value of a.

Case 2: $y = ab^x$

Start with a non-linear relationship — $y = ab^x$

Take logs of both sides ($\log = \log_{10}$) — $\log y = \log ab^x$

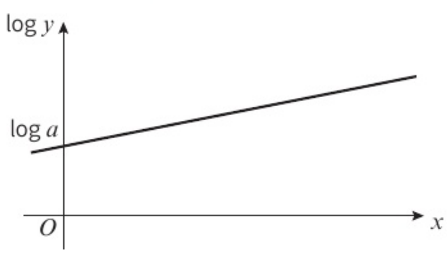
Use the multiplication law — $\log y = \log a + \log b^x$

Use the power law — $\log y = \log a + x \log b$

Compare this equation to the common form of a straight line, $Y = MX + C$

$\log y$ variable	=	$\log b$ constant (gradient)	x variable	+	$\log a$ constant (intercept)
Y variable	=	M constant (gradient)	X variable	+	C constant (intercept)

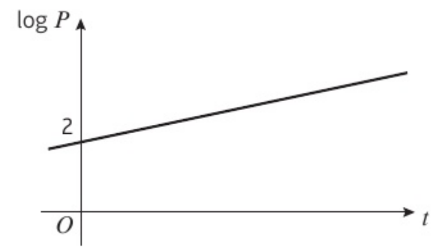
- **If $y = ab^x$ then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$**



> **Watch out** For $y = ab^x$ you need to plot $\log y$ against x to obtain a linear graph. If you plot $\log y$ against $\log x$ you will **not** get a linear relationship.

Example 8

The graph represents the growth of a population of bacteria, P, over t hours. The graph has a gradient of 0.6 and meets the vertical axis at (0, 2) as shown.



A scientist suggests that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a Write down an equation for the line.

b Using your answer to part **a** or otherwise, find the values of a and b, giving them to 3 significant figures where necessary.

c Interpret the meaning of the constant a in this model.

a $\log P = 0.6t + 2$

b $P = 10^{0.6t + 2}$

$= 10^{0.6t} \times 10^2$

$= 10^2 \times (10^{0.6})^t$

$= 100 \times 3.98^t$

$a = 100$, $b = 3.98$ (3 s.f.)

c The value of a gives the initial size of the bacteria population.

$\log P = (\text{gradient}) \times t + (y\text{-intercept})$

Rewrite the logarithm as a power. An alternative method would be to start with $P = ab^t$ and take logs of both sides, as in Example 7.

Rearrange the equation into the form ab^t. You can use $x^{mn} = (x^m)^n$ to write $10^{0.6t}$ in the form b^t

Exercise 5D

SKILLS INTERPRETATION

1 Two variables, S and x, satisfy the formula $S = 4 \times 7^x$

a Show that $\log S = \log 4 + x \log 7$

b The straight line graph of $\log S$ against x is plotted. Write down the gradient and the value of the intercept on the vertical axis.

2 Two variables, A and x, satisfy the formula $A = 6x^4$

a Show that $\log A = \log 6 + 4 \log x$

b The straight line graph of $\log A$ against $\log x$ is plotted. Write down the gradient and the value of the intercept on the vertical axis.

3 The data below follows a trend of the form $y = ax^n$, where a and n are constants.

x	3	5	8	10	15
y	16.3	33.3	64.3	87.9	155.1

a Copy and complete the table of values of $\log x$ and $\log y$, giving your answers to 2 decimal places.

$\log x$	0.48	0.70	0.90	1	1.18
$\log y$	1.21				2.19

b Plot a graph of $\log y$ against $\log x$ and draw in a line of best fit.

c Use your graph to estimate the values of a and n to 1 decimal place.

4 The data below follows a trend of the form $y = ab^x$, where a and b are constants.

x	2	3	5	6.5	9
y	124.8	424.4	4097.0	30 763.6	655 743.5

a Copy and complete the table of values of x and $\log y$, giving your answers to 2 decimal places.

x	2	3	5	6.5	9
$\log y$	2.10				

b Plot a graph of $\log y$ against x and draw in a line of best fit.

c Use your graph to estimate the values of a and b to 1 decimal place.

(E) **5** Kleiber's law is an empirical law in biology which connects the mass of an animal, m, to its resting metabolic rate, R. The law follows the form $R = am^b$, where a and b are constants.

The table below contains data on five animals.

Animal	Mouse	Guinea pig	Rabbit	Goat	Cow
Mass, m (kg)	0.030	0.408	4.19	34.6	650
Metabolic rate, R (kcal per day)	4.2	32.3	195	760	7637

a Copy and complete this table giving values of $\log R$ and $\log m$ to 2 decimal places. **(1 mark)**

$\log m$	−1.52				
$\log R$	0.62	1.51	2.29	2.88	3.88

b Plot a graph of $\log R$ against $\log m$ using the values from your table and draw in a line of best fit. **(2 marks)**

c Use your graph to estimate the values of a and b to 2 significant figures. **(4 marks)**

d Using your values of a and b, estimate the resting metabolic rate of a human male with a mass of 80 kg. **(1 mark)**

6 Zipf's law is an empirical law which relates how frequently a word is used, f, to its ranking in a list of the most common words of a language, R. The law follows the form $f = AR^b$, where A and b are constants to be found.

The table below contains data on four words.

Word	'the'	'it'	'well'	'detail'
Rank, R	1	10	100	1000
Frequency per 100 000 words, f	4897	861	92	9

a Copy and complete this table giving values of $\log f$ to 2 decimal places.

$\log R$	0	1	2	3
$\log f$	3.69			

b Plot a graph of $\log f$ against $\log R$ using the values from your table and draw in a line of best fit.

c Use your graph to estimate the value of A to 2 significant figures and the value of b to 1 significant figure.

d The word 'when' is the 57th most commonly used word in the English language. A series of three novels contains 455 125 words. Use your values of A and b to estimate the number of times the word 'when' appears in the trilogy.

P **7** The table below shows the population of Mozambique between 1960 and 2010.

Year	1960	1970	1980	1990	2000	2010
Population, *P* (millions)	7.6	9.5	12.1	13.6	18.3	23.4

This data can be modelled using an exponential function of the form $P = ab^t$, where t is the time in years since 1960 and a and b are constants.

a Copy and complete the table below.

Time in years since 1960, *t*	0	10	20	30	40	50
log *P*	0.88					

b Show that $P = ab^t$ can be rearranged into the form $\log P = \log a + t \log b$

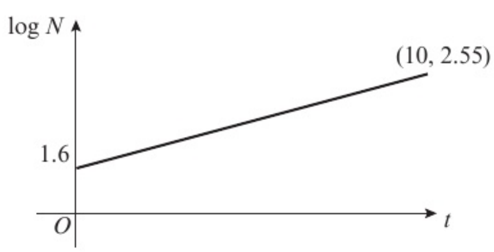
c Plot a graph of $\log P$ against t using the values from your table and draw in a line of best fit.

d Use your graph to estimate the values of a and b.

e Explain why an exponential model is often appropriate for modelling population growth.

Hint For part **e**, think about the relationship between P and $\frac{dP}{dt}$

E/P **8** A scientist is modelling the number of people, N, who have fallen sick with a virus after t days.



From looking at this graph, the scientist suggests that the number of sick people can be modelled by the equation $N = ab^t$, where a and b are constants to be found.

The graph passes through the points (0, 1.6) and (10, 2.55).

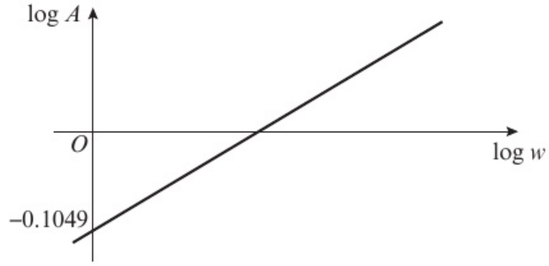
a Write down the equation of the line. **(2 marks)**

b Using your answer to part **a** or otherwise, find the values of a and b, giving them to 2 significant figures. **(4 marks)**

c Interpret the meaning of the constant a in this model. **(1 mark)**

d Use your model to predict the number of sick people to the nearest 100 after 30 days. Give one reason why this might be an overestimate. **(2 marks)**

(P) **9** A student is investigating a family of similar shapes. She measures the width, w, and the area, A, of each shape. She suspects there is a formula of the form $A = pw^q$, so she plots the logarithms of her results.



The graph has a gradient of 2 and passes through −0.1049 on the vertical axis.

a Write down an equation for the line.

b Starting with your answer to part **a**, or otherwise, find the exact value of q and the value of p to 4 decimal places.

c Suggest the name of the family of shapes that the student is investigating, and justify your answer.

Hint Multiply p by 4 and think about another name for 'half the width'.

5.5 Exponential modelling

You can use e^x to **model** situations such as population growth, where the rate of **increase** is proportional to the size of the population at any given moment. Similarly, e^{-x} can be used to model situations such as radioactive decay (the process of being destroyed by radioactivity), where the rate of **decrease** is proportional to the number of atoms remaining.

Example 9

The density of a pesticide (a chemical used for killing insects) in a given section of field, P mg/m², can be modelled by the equation

$$P = 160e^{-0.006t}$$

where t is the time in days since the pesticide was first applied.

a Use this model to estimate the density of pesticide after 15 days.

b Interpret the meaning of the value 160 in this model.

c Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.

d Interpret the significance of the sign of your answer to part **c**.

e Sketch the graph of P against t.

a After 15 days, $t = 15$

$P = 160e^{-0.006 \times 15}$

$= 146.2$ mg/m^2

Substitute $t = 15$ into the model.

Online Work this out in one go using the e^x button on your calculator.

b When $t = 0$, $P = 160e^0 = 160$, so 160 mg/m^2 is the initial density of pesticide in the field.

Notation The value given by a model when $t = 0$ is called the **initial value**.

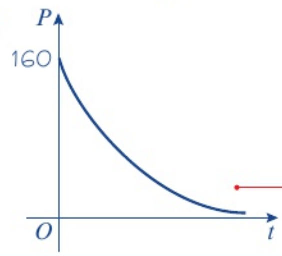
c $P = 160e^{-0.006t}$

$\frac{dP}{dt} = -0.96e^{-0.006t}$, so $k = -0.96$

If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

d As k is negative, the density of pesticide is decreasing (there is exponential decay).

e



Use your answers to parts **a** and **d** to help you draw the graph. To check what happens to P in the **long term**, substitute in a very large value of t.

Exercise 5E SKILLS INTERPRETATION

1 The value of a car is modelled by the formula

$$V = 20\,000e^{-\frac{t}{12}}$$

where V is the value in euros and t is its age in years from new.

a State its value when new.

b Find its value (to the nearest euro) after 4 years.

c Sketch the graph of V against t.

(P) 2 The population of a country is modelled using the formula

$$P = 20 + 10e^{\frac{t}{50}}$$

where P is the population in thousands and t is the time in years after the year 2000.

a State the population in the year 2000.

b Use the model to predict the population in the year 2030.

c Sketch the graph of P against t for the years 2000 to 2100.

d Do you think that it would be valid to use this model to predict the population in the year 2500? Explain your answer.

(P) 3 The number of people infected with a disease is modelled by the formula

$$N = 300 - 100e^{-0.5t}$$

where N is the number of people infected with the disease and t is the time in years after it was first seen.

a How many people were initially diagnosed with the disease?
b What is the long term prediction of how this disease will spread?
c Sketch the graph of N against t, for $t > 0$

(P) **4** The number of rabbits, R, in a population after m months is modelled by the formula

$R = 12e^{0.2m}$

a Use this model to estimate the number of rabbits after
i 1 month **ii** 1 year
b Interpret the meaning of the constant 12 in this model.
c Show that after 6 months, the rabbit population is increasing by almost 8 rabbits per month.
d Suggest one reason why this model will stop giving valid results for large enough values of t.

Problem-solving
Your answer to part **b** must refer to the context of the model.

(E/P) **5** On Earth, the atmospheric pressure, p, in bars can be modelled approximately by the formula $p = e^{-0.13h}$ where h is the height above sea level in kilometres.

a Use this model to estimate the pressure at the top of Mount Rainier, which has an altitude (height above sea level) of 4.394 km. **(1 mark)**
b Demonstrate that $\frac{dp}{dh} = kp$, where k is a constant to be found. **(2 marks)**
c Interpret the significance of the sign of k in part **b**. **(1 mark)**
d This model predicts that the atmospheric pressure will change by s% for every kilometre gained in height. Calculate the value of s. **(3 marks)**

(E/P) **6** Fadi has bought a car for 20 000 Dirhams. He wants to model the value, V Dirhams, of his car after t years. His friend suggests two models:

Model 1: $V = 20\,000e^{-0.24t}$
Model 2: $V = 19\,000e^{-0.255t} + 1000$

a Use both models to predict the value of the car after one year. Compare your results. **(2 marks)**
b Use both models to predict the value of the car after ten years. Compare your results. **(2 marks)**
c Sketch a graph of V against t for both models. **(2 marks)**
d Interpret the meaning of the 1000 in Model 2, and suggest why this might make Model 2 more realistic. **(1 mark)**

Chapter review 5

1 Sketch each of the following graphs, labelling all intersections and asymptotes.

a $y = 2^{-x}$ **b** $y = 5e^x - 1$ **c** $y = \ln x$

Hint Recall that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

(P) **2** **a** Express $\ln(p^2q)$ in terms of $\ln p$ and $\ln q$
b Given that $\ln(pq) = 5$ and $\ln(p^2q) = 9$, find the values of $\ln p$ and $\ln q$

3 Differentiate each of the following expressions with respect to x.

a e^{-x} **b** e^{11x} **c** $6e^{5x}$

4 Solve the following equations, giving exact solutions.

a $\ln(2x - 5) = 8$ **b** $e^{4x} = 5$ **c** $24 - e^{-2x} = 10$

d $\ln x + \ln(x - 3) = 0$ **e** $e^x + e^{-x} = 2$ **f** $\ln 2 + \ln x = 4$

P **5** The price of a computer system can be modelled by the formula

$$P = 100 + 850\,e^{-\frac{t}{2}}$$

where P is the price of the system in euros and t is the age of the computer in years after being purchased.

a Calculate the price of the system when new.

b Calculate its price after 3 years, giving your answer to the nearest euro.

c When will it be worth less than €200?

d Find its price as $t \to \infty$.

e Sketch the graph showing P against t.

f Comment on the appropriateness of this model.

P **6** The points P and Q lie on the curve with equation $y = e^{\frac{1}{2}x}$

The x-coordinates of P and Q are $\ln 4$ and $\ln 16$ respectively.

a Find an equation for the line PQ.

b Show that this line passes through the origin O.

c Calculate the length, to 3 significant figures, of the line segment PQ.

E/P **7** The temperature, T °C, of a cup of tea is given by $T = 55e^{-\frac{t}{8}} + 20$, $t \geqslant 0$, where t is the time in minutes since measurements began.

a Briefly explain why $t \geqslant 0$ **(1 mark)**

b State the starting temperature of the cup of tea. **(1 mark)**

c Find the time at which the temperature of the tea is 50 °C, giving your answer to the nearest minute. **(3 marks)**

d By sketching a graph or otherwise, explain why the temperature of the tea will never fall below 20 °C. **(2 marks)**

E **8** The table below gives the surface area, S, and the volume, V of five different spheres, rounded to 1 decimal place.

S	18.1	50.3	113.1	221.7	314.2
V	7.2	33.5	113.1	310.3	523.6

Given that $S = aV^b$, where a and b are constants,

a show that $\log S = \log a + b \log V$ **(2 marks)**

b Copy and complete the table of values of $\log S$ and $\log V$, giving your answers to 2 decimal places. **(1 mark)**

log S					
log V	0.86				

c Plot a graph of $\log V$ against $\log S$ and draw in a line of best fit. **(2 marks)**

d Use your graph to confirm that $b = 1.5$ and estimate the value of a to 1 significant figure. **(4 marks)**

E/P **9** A student is asked to solve the equation

$$\log_2 x - \tfrac{1}{2}\log_2\sqrt{x+1} = 1$$

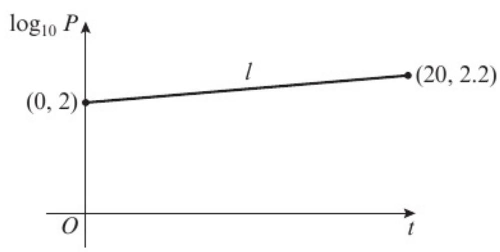
The student's attempt is shown below.

$$\log_2 x - \log_2\sqrt{x+1} = 1$$
$$x - \sqrt{x+1} = 2^1$$
$$x - 2 = \sqrt{x+1}$$
$$(x-2)^2 = x + 1$$
$$x^2 - 5x + 3 = 0$$
$$x = \frac{5+\sqrt{13}}{2} \quad x = \frac{5-\sqrt{13}}{2}$$

a Identify the error made by the student. **(1 mark)**

b Solve the equation correctly. **(3 marks)**

10 The population, P, of a colony of endangered Sumatran ground-cuckoos can be modelled by the equation $P = ab^t$ where a and b are constants and t is the time, in months, since the population was first recorded.



The line l shows the relationship between t and $\log_{10} P$ for the population over a period of 20 years.

a Write down the equation of line l. **(3 marks)**

b Work out the value of a and interpret this value in the context of the model. **(3 marks)**

c Work out the value of b, giving your answer correct to 3 decimal places. **(2 marks)**

d Find the population predicted by the model when $t = 30$. **(1 mark)**

Challenge

SKILLS PROBLEM-SOLVING

Find a formula to describe the relationship between the data in this table.

x	1	2	3	4
y	5.22	4.698	4.2282	3.805 38

Hint Sketch the graphs of $\log y$ against $\log x$, and $\log y$ against x. This will help you determine if the relationship is of the form $y = ax^n$ or $y = ab^x$

Summary of key points

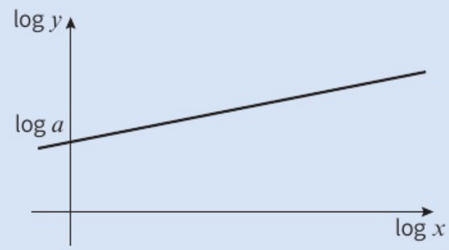
1 For all real values of x:

- If $f(x) = e^x$ then $f'(x) = e^x$
- If $y = e^x$ then $\frac{dy}{dx} = e^x$

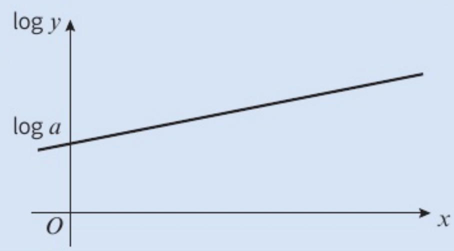
2 For all real values of x and for any constant k:

- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

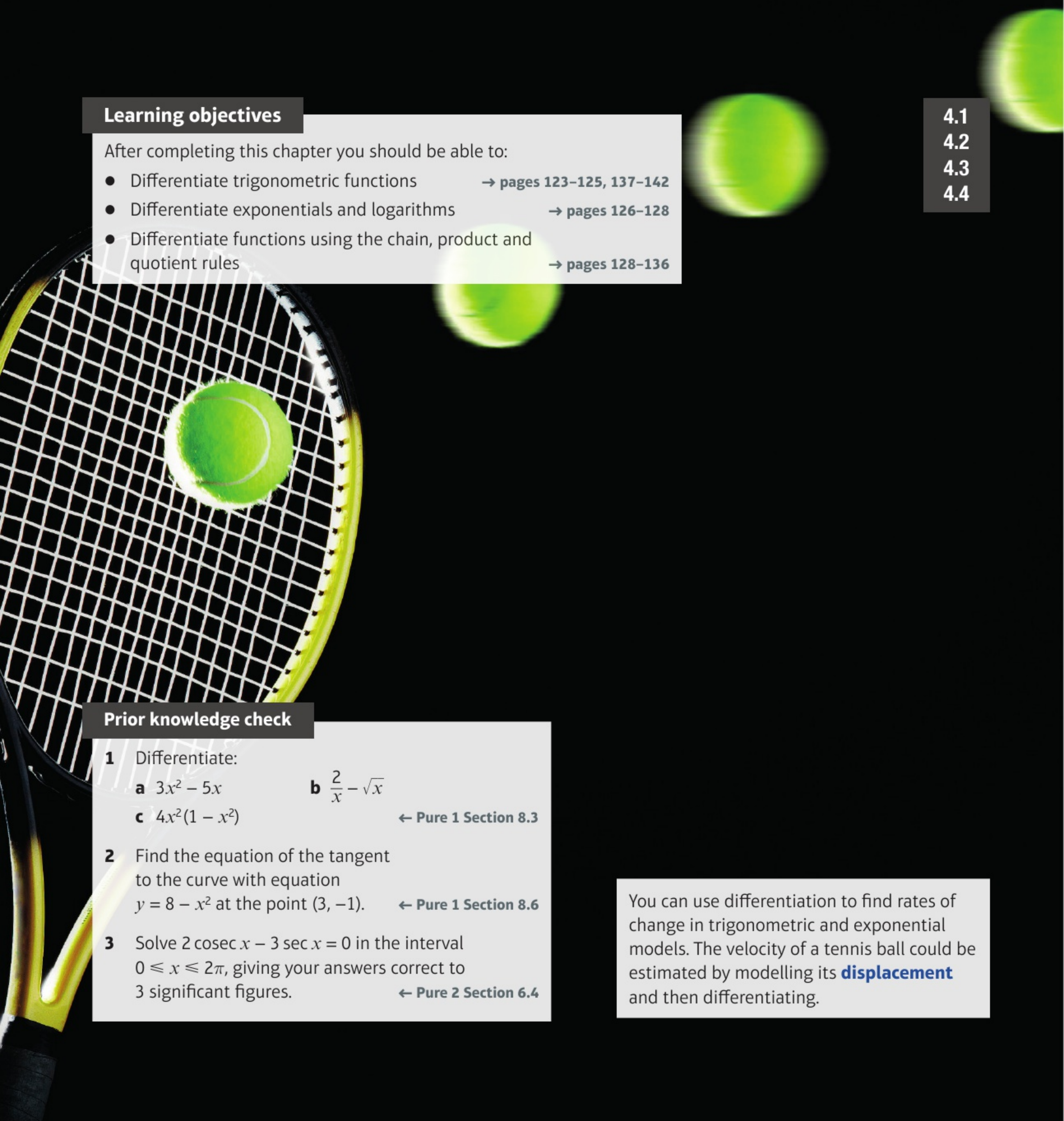
3 If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$



4 If $y = ab^x$ then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$



6 DIFFERENTIATION



Learning objectives

After completing this chapter you should be able to:

- Differentiate trigonometric functions → pages 123–125, 137–142
- Differentiate exponentials and logarithms → pages 126–128
- Differentiate functions using the chain, product and quotient rules → pages 128–136

4.1
4.2
4.3
4.4

Prior knowledge check

1 Differentiate:

a $3x^2 - 5x$ **b** $\frac{2}{x} - \sqrt{x}$

c $4x^2(1 - x^2)$ ← Pure 1 Section 8.3

2 Find the equation of the tangent to the curve with equation $y = 8 - x^2$ at the point $(3, -1)$. ← Pure 1 Section 8.6

3 Solve $2\,\text{cosec}\,x - 3\sec x = 0$ in the interval $0 \leqslant x \leqslant 2\pi$, giving your answers correct to 3 significant figures. ← Pure 2 Section 6.4

You can use differentiation to find rates of change in trigonometric and exponential models. The velocity of a tennis ball could be estimated by modelling its **displacement** and then differentiating.

6.1 Differentiating sin x and cos x

To differentiate sin x and cos x **from first principles**, we can use the following small angle approximations for sin and cos when the angle is measured in **radians**:

- $\sin x \approx x$
- $\cos x \approx 1 - \frac{1}{2}x^2$

This means that $\lim\limits_{h\to 0} \frac{\sin h}{h} = \lim\limits_{h\to 0} \frac{h}{h} = 1$, and

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{1 - \frac{1}{2}h^2 - 1}{h} = \lim_{h\to 0} \left(-\frac{1}{2}h\right) = 0$$

You will need to use these two limits when you differentiate sin and cos from first principles, but note that this technique is not required by the examination syllabus.

Watch out You will always need to use radians when differentiating trigonometric functions.

Example 1

SKILLS ANALYSIS

Prove, from first principles, that the **derivative** of sin x is cos x

You may assume that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

Let $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\left(\frac{\cos h - 1}{h}\right)\sin x + \left(\frac{\sin h}{h}\right)\cos x\right)$$

Since $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$

the expression inside the limit tends to

$(0 \times \sin x + 1 \times \cos x)$

So $\lim\limits_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$

Hence the derivative of sin x is cos x

Problem-solving Use the rule for differentiating from first principles. This is provided in the formula booklet. If you don't want to use limit notation, you could write an expression for the gradient of the chord joining $(x, \sin x)$ to $(x + h, \sin(x + h))$ and show that as $h \to 0$ the gradient of the chord tends to cos x ← Pure 1 Section 8.2

Use the formula for $\sin(A + B)$ to expand $\sin(x + h)$, then write the resulting expression in terms of $\frac{\cos h - 1}{h}$ and $\frac{\sin h}{h}$ ← Pure 3 Section 4.1

Make sure you state where you are using the two limits given in the question.

Write down what you have proved.

- If $y = \sin kx$, then $\frac{dy}{dx} = k\cos kx$

You can use a similar technique to find the derivative of cos x

- If $y = \cos kx$, then $\frac{dy}{dx} = -k\sin kx$

Online Explore the relationship between sin and cos and their derivatives using technology.

Example 2

Find $\frac{dy}{dx}$ given that:

a $y = \sin 2x$ **b** $y = \cos 5x$ **c** $y = 3\cos x + 2\sin 4x$

a $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x$

Use the standard result for $\sin kx$ with $k = 2$

b $y = \cos 5x$

$\frac{dy}{dx} = -5\sin 5x$

Use the standard result for $\cos kx$ with $k = 5$

c $y = 3\cos x + 2\sin 4x$

$\frac{dy}{dx} = 3 \times (-\sin x) + 2 \times (4\cos 4x)$

$= -3\sin x + 8\cos 4x$

Differentiate each term separately.

Example 3

A curve has equation $y = \frac{1}{2}x - \cos 2x$. Find the stationary points on the curve in the interval $0 \leqslant x \leqslant \pi$

$\frac{dy}{dx} = \frac{1}{2} - (-2\sin 2x) = \frac{1}{2} + 2\sin 2x$

Start by differentiating $\frac{1}{2}x - \cos 2x$

Let $\frac{dy}{dx} = 0$ and solve for x:

Stationary points occur when $\frac{dy}{dx} = 0$

← Pure 2 Section 7.2

$\frac{1}{2} + 2\sin 2x = 0$

$2\sin 2x = -\frac{1}{2}$

$\sin 2x = -\frac{1}{4}$

$2x = 3.394..., 6.030...$

$x = 1.70, 3.02$ (3 s.f.)

$0 \leqslant x \leqslant \pi$ so the range for $2x$ is $0 \leqslant 2x \leqslant 2\pi$

Watch out Whenever you are using calculus, you must work in **radians**.

When $x = 1.70$:

$y = \frac{1}{2}(1.70) - \cos(2 \times 1.70) = 1.82$ (3 s.f.)

When $x = 3.02$:

$y = \frac{1}{2}(3.02) - \cos(2 \times 3.02) = 0.539$ (3 s.f.)

Substitute x values into $y = \frac{1}{2}x - \cos 2x$ to find the corresponding y values.

The stationary points of $y = \frac{1}{2}x - \cos 2x$ in the interval $0 \leqslant x \leqslant \pi$ are (1.70, 1.82) and (3.02, 0.539).

Exercise 6A

SKILLS PROBLEM-SOLVING

1 Differentiate:

a $y = 2\cos x$ **b** $y = 2\sin \frac{1}{2}x$ **c** $y = \sin 8x$ **d** $y = 6\sin \frac{2}{3}x$

2 Find $f'(x)$ given that:

a $f(x) = 2\cos x$ **b** $f(x) = 6\cos \frac{5}{6}x$ **c** $f(x) = 4\cos \frac{1}{2}x$ **d** $f(x) = 3\cos 2x$

3 Find $\frac{dy}{dx}$ given that:

a $y = \sin 2x + \cos 3x$ **b** $y = 2\cos 4x - 4\cos x + 2\cos 7x$

c $y = x^2 + 4\cos 3x$ **d** $y = \frac{1 + 2x\sin 5x}{x}$

4 A curve has equation $y = x - \sin 3x$. Find the stationary points of the curve in the interval $0 \leqslant x \leqslant \pi$

5 Find the gradient of the curve $y = 2\sin 4x - 4\cos 2x$ at the point where $x = \frac{\pi}{2}$

(P) **6** A curve has the equation $y = 2\sin 2x + \cos 2x$. Find the stationary points of the curve in the interval $0 \leqslant x \leqslant \pi$

(E/P) **7** A curve has the equation $y = \sin 5x + \cos 3x$. Find the equation of the tangent to the curve at the point $(\pi, -1)$. **(4 marks)**

(E/P) **8** A curve has the equation $y = 2x^2 - \sin x$. Show that the equation of the **normal** to the curve at the point with x-coordinate π is:

$$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$$ **(7 marks)**

(E) **9** Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume the formula for $\sin(A + B)$ and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5 marks)

Challenge

SKILLS CREATIVITY

Prove, from first principles, that the derivative of $\sin kx$ is $k\cos kx$

You may assume the formula for $\sin(A + B)$ and that as $h \to 0$, $\frac{\sin kh}{h} \to k$ and $\frac{\cos kh - 1}{h} \to 0$

6.2 Differentiating exponentials and logarithms

You need to be able to differentiate expressions involving exponentials and logarithms.

- If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$
- If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

Watch out For any real constant, k, $\ln kx = \ln k + \ln x$. Since $\ln k$ is also a constant, the derivative of $\ln kx$ is $\frac{1}{x}$

You can use the derivative of e^{kx} to find the derivative of a^{kx} where a is any positive real number.

Example 4

Show that the derivative of a^x is $a^x \ln a$

Online Explore the function a^x and its derivative using technology.

Let $y = a^x$

$= e^{\ln(a^x)}$

$= e^{x \ln a}$

You could also use the laws of logs like this:
$\ln y = \ln a^x = x \ln a \Rightarrow y = e^{x \ln a}$
← Pure 2 Section 3.3

$\frac{dy}{dx} = \ln a \, e^{x \ln a}$

$\ln a$ is just a constant so use the standard result for the derivative of e^{kx} with $k = \ln a$

$= \ln a \, e^{\ln(a^x)}$

$= a^x \ln a$

- If $y = a^{kx}$, where k is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx} k \ln a$

Example 5

Find $\frac{dy}{dx}$ given that:

a $y = e^{3x} + 2^{3x}$ **b** $y = \ln(x^3) + \ln 7x$ **c** $y = \frac{2 - 3e^{7x}}{4e^{3x}}$

a $y = e^{3x} + 2^{3x}$

$\frac{dy}{dx} = 3e^{3x} + 2^{3x}(3 \ln 2)$

Differentiate each term separately using the standard results for e^{kx} with $k = 3$, and a^{kx} with $a = 2$ and $k = 3$

b $y = \ln(x^3) + \ln 7x$

$= 3 \ln x + \ln 7 + \ln x = 4 \ln x + \ln 7$

Rewrite y using the laws of logs.

$\frac{dy}{dx} = 4 \times \frac{1}{x} + 0 = \frac{4}{x}$

Use the standard result for $\ln x$. $\ln 7$ is a constant, so it disappears when you differentiate.

c $y = \frac{2 - 3e^{7x}}{4e^{3x}}$

$= \frac{1}{2}e^{-3x} - \frac{3}{4}e^{4x}$

Divide each term in the numerator by the denominator.

$\frac{dy}{dx} = \frac{1}{2} \times (-3e^{-3x}) - \frac{3}{4} \times 4e^{4x}$

$= -\frac{3}{2}e^{-3x} - 3e^{4x}$

Differentiate each term separately using the standard result for e^{kx}

Exercise 6B SKILLS ANALYSIS

1 a Find $\frac{dy}{dx}$ for each of the following:

a $y = 4e^{7x}$ **b** $y = 3^x$ **c** $y = \left(\frac{1}{2}\right)^x$ **d** $y = \ln 5x$

e $y = 4\left(\frac{1}{3}\right)^x$ **f** $y = \ln(2x^3)$ **g** $y = e^{3x} - e^{-3x}$ **h** $y = \frac{(1 + e^x)^2}{e^x}$

2 Find $f'(x)$ given that:

a $f(x) = 3^{4x}$ **b** $f(x) = \left(\frac{3}{2}\right)^{2x}$

c $f(x) = 2^{4x} + 4^{2x}$ **d** $f(x) = \frac{2^{7x} + 8^x}{4^{2x}}$

Hint In parts **c** and **d**, rewrite the terms so that they all have the same base and hence can be simplified.

3 Find the gradient of the curve $y = (e^{2x} - e^{-2x})^2$ at the point where $x = \ln 3$

(E) **4** Find the equation of the tangent to the curve $y = 2^x + 2^{-x}$ at the point $\left(2, \frac{17}{4}\right)$ **(6 marks)**

(E/P) **5** A curve has the equation $y = e^{2x} - \ln x$. Show that the equation of the tangent at the point with x-coordinate 1 is:

$y = (2e^2 - 1)x - e^2 + 1$ **(6 marks)**

6 A particular radioactive isotope has an activity, R millicuries at time t days, given by the equation $R = 200 \times 0.9^t$. Find the value of $\frac{dR}{dt}$ when $t = 8$

(P) **7** The population of Cambridge was 37 000 in 1900, and was about 109 000 in 2000. Given that the population, P, at a time t years after 1900 can be modelled using the equation $P = P_0k^t$

a find the values of P_0 and k

b evaluate $\frac{dP}{dt}$ in the year 2000

c interpret your answer to part **b** in the context of the model.

(P) **8** A student is attempting to differentiate $\ln kx$. The student writes:

$y = \ln kx$, so $\frac{dy}{dx} = k \ln kx$

Explain the mistake made by the student and state the correct derivative.

(E/P) **9** Prove that the derivative of a^{kx} is $a^{kx} k \ln a$. You may assume that the derivative of e^{kx} is ke^{kx}. **(4 marks)**

(E/P) **10** $f(x) = e^{2x} - \ln(x^2) + 4$, $x > 0$

a Find $f'(x)$. **(3 marks)**

The curve with equation $y = f(x)$ has a gradient of 2 at point P. The x-coordinate of P is a.

b Show that $a(e^{2a} - 1) = 1$ **(2 marks)**

E/P **11** A curve C has equation:

$$y = 5\sin 3x + 2\cos 3x, -\pi \leqslant x \leqslant \pi$$

a Show that the point P (0, 2) lies on C. **(1 mark)**

b Find an equation of the normal to the curve C at P. **(5 marks)**

E/P **12** The point P lies on the curve with equation $y = 2(3^{4x})$. The x-coordinate of P is 1. Find an equation of the normal to the curve at the point P in the form $y = ax + b$, where a and b are constants to be found in exact form. **(5 marks)**

Challenge

SKILLS CREATIVITY

A curve C has the equation $y = e^{4x} - 5x$. Find the equation of the tangent to C that is parallel to the line $y = 3x + 4$

6.3 The chain rule

You can use the chain rule to differentiate composite functions, or functions of another function.

- **The chain rule is:**

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where y is a function of u, and u is another function of x.

Example 6 SKILLS INTERPRETATION

Given that $y = (3x^4 + x)^5$, find $\frac{dy}{dx}$ using the chain rule.

Let $u = 3x^4 + x$:

$$\frac{du}{dx} = 12x^3 + 1$$

Differentiate u with respect to x to get $\frac{du}{dx}$

$$y = u^5$$

$$\frac{dy}{du} = 5u^4$$

Substitute u into the equation for y and differentiate with respect to u to get $\frac{dy}{du}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5u^4(12x^3 + 1)$$

$$= 5(3x^4 + x)^4(12x^3 + 1)$$

Use $u = 3x^4 + x$ to write your final answer in terms of x only.

Example 7

Given that $y = \sin^4 x$, find $\frac{dy}{dx}$

$y = \sin^4 x = (\sin x)^4$

Let $u = \sin x$:

$$\frac{du}{dx} = \cos x$$

Differentiate u with respect to x to get $\frac{du}{dx}$

$$y = u^4$$
$$\frac{dy}{du} = 4u^3$$

Substitute u into the equation for y and differentiate with respect to u to get $\frac{dy}{du}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 4u^3(\cos x)$$
$$= 4\sin^3 x \cos x$$

Substitute $u = \sin x$ back into $\frac{dy}{dx}$ to get an answer in terms of x only.

You can write the chain rule using function notation:

- **The chain rule enables you to differentiate a function of a function. In general,**
  - **if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1} f'(x)$**
  - **if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$**

Example 8

Given that $y = \sqrt{5x^2 + 1}$, find $\frac{dy}{dx}$ at (4, 9).

$y = \sqrt{5x^2 + 1}$

This is $y = (f(x))^n$ with $f(x) = 5x^2 + 1$ and $n = \frac{1}{2}$

So $\frac{dy}{dx} = \frac{1}{2}(f(x))^{-\frac{1}{2}} f'(x)$

Let $f(x) = 5x^2 + 1$

Then $f'(x) = 10x$

Using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(5x^2 + 1)^{-\frac{1}{2}} \times 10x$$
$$= 5x(5x^2 + 1)^{-\frac{1}{2}}$$

At (4, 9), $\frac{dy}{dx} = 5(4)(5(4)^2 + 1)^{-\frac{1}{2}} = \frac{20}{9}$

Substitute $x = 4$ into $\frac{dy}{dx}$ to find the required value.

The following particular case of the chain rule is useful for differentiating functions that are in the form $x = f(y)$

- $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Example 9

Find the value of $\frac{dy}{dx}$ at the point (2, 1) on the curve with equation $y^3 + y = x$

$\frac{dx}{dy} = 3y^2 + 1$ — Start with $x = y^3 + y$ and differentiate with respect to y.

$\therefore \frac{dy}{dx} = \frac{1}{3y^2 + 1}$ — Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$= \frac{1}{4}$ — Substitute $y = 1$

Exercise 6C SKILLS ANALYSIS

1 Differentiate:

a $(1 + 2x)^4$
b $(3 - 2x^2)^{-5}$
c $(3 + 4x)^{\frac{1}{2}}$
d $(6x + x^2)^7$
e $\frac{1}{3 + 2x}$
f $\sqrt{7 - x}$
g $4(2 + 8x)^4$
h $3(8 - x)^{-6}$

2 Differentiate:

a $e^{\cos x}$
b $\cos(2x - 1)$
c $\sqrt{\ln x}$
d $(\sin x + \cos x)^5$
e $\sin(3x^2 - 2x + 1)$
f $\ln(\sin x)$
g $2e^{\cos 4x}$
h $\cos(e^{2x} + 3)$

3 Given that $y = \frac{1}{(4x + 1)^2}$, find the value of $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$

(E) 4 A curve C has equation $y = (5 - 2x)^3$. Find the tangent to the curve at the point P with x-coordinate 1. **(7 marks)**

(E) 5 Given that $y = (1 + \ln 4x)^{\frac{3}{2}}$, find the value of $\frac{dy}{dx}$ at $x = \frac{1}{4}e^3$ **(5 marks)**

(P) 6 Find $\frac{dy}{dx}$ for the following curves, giving your answers in terms of y:

a $x = y^2 + y$
b $x = e^y + 4y$
c $x = \sin 2y$
d $4x = \ln y + y^3$

(P) **7** Find the value of $\frac{dy}{dx}$ at the point (8, 2) on the curve with equation $3y^2 - 2y = x$

Problem-solving

Your expression for $\frac{dy}{dx}$ will be in terms of y.
Remember to substitute the y-coordinate into the expression to find the gradient.

(P) **8** Find the value of $\frac{dy}{dx}$ at the point $\left(\frac{5}{2}, 4\right)$ on the curve with equation $y^{\frac{1}{2}} + y^{-\frac{1}{2}} = x$

9 **a** Differentiate $e^y = x$ with respect to y.

b Hence, prove that if $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

(E/P) **10** The curve C has equation $x = 4\cos 2y$

a Show that the point $Q\left(2, \frac{\pi}{6}\right)$ lies on C. **(1 mark)**

b Show that $\frac{dy}{dx} = -\frac{1}{4\sqrt{3}}$ at Q. **(4 marks)**

c Find an equation of the normal to C at Q. Give your answer in the form $ax + by + c = 0$, where a, b and c are exact constants. **(4 marks)**

11 Differentiate:

a $\sin^2 3x$ **b** $e^{(x+1)^2}$ **c** $\ln(\cos x)^2$

d $\frac{1}{3 + \cos 2x}$ **e** $\sin\left(\frac{1}{x}\right)$

(E/P) **12** The curve C has equation $y = \frac{4}{(2-4x)^2}$, $x \neq \frac{1}{2}$

The point A on C has x-coordinate 3.
Find an equation of the normal to C at A in the form $ax + by + c = 0$, where a, b and c are integers. **(7 marks)**

(E/P) **13** Find the exact value of the gradient of the curve with equation $y = 3^{x^3}$ at the point with coordinates (1, 3). **(4 marks)**

Challenge

SKILLS INNOVATION

Find $\frac{dy}{dx}$ given that:

a $y = \sqrt{\sin\sqrt{x}}$ **b** $\ln y = \sin^3(3x + 4)$

6.4 The product rule

You need to be able to differentiate the product of two functions.

- If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

  where u and v are functions of x.

The product rule in function notation is:

- If $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + h(x)g'(x)$

Watch out Make sure you can spot the difference between a product of two functions and a function of a function. A product is two separate functions multiplied together.

Example 10

Given that $f(x) = x^2\sqrt{3x - 1}$, find $f'(x)$.

Let $u = x^2$ and $v = \sqrt{3x-1} = (3x-1)^{\frac{1}{2}}$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3 \times \frac{1}{2}(3x-1)^{-\frac{1}{2}}$

Write out your functions u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$ before substituting into the product rule. Use the chain rule to differentiate $(3x-1)^{\frac{1}{2}}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$f'(x) = x^2 \times \frac{3}{2}(3x-1)^{-\frac{1}{2}} + \sqrt{3x-1} \times 2x$

Substitute u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

$= \frac{3x^2 + 12x^2 - 4x}{2\sqrt{3x-1}}$

$= \frac{15x^2 - 4x}{2\sqrt{3x-1}}$

$= \frac{x(15x-4)}{2\sqrt{3x-1}}$

Example 11

Given that $y = e^{4x}\sin^2 3x$, show that $\frac{dy}{dx} = e^{4x}\sin 3x\,(A\cos 3x + B\sin 3x)$, where A and B are constants to be determined.

Let $u = e^{4x}$ and $v = \sin^2 3x = (\sin 3x)^2$

$\frac{du}{dx} = 4e^{4x}$ and $\frac{dv}{dx} = 2(\sin 3x) \times (3\cos 3x)$

Write out u and v and find $\frac{du}{dx}$ and $\frac{dv}{dx}$

Use the chain rule to find $\frac{dv}{dx}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Write out the product rule before substituting.

$\frac{dy}{dx} = e^{4x} \times (6\sin 3x\cos 3x) + \sin^2 3x \times 4e^{4x}$

$= 6e^{4x}\sin 3x\cos 3x + 4e^{4x}\sin^2 3x$

$= e^{4x}\sin 3x\,(6\cos 3x + 4\sin 3x)$

This is in the required form with $A = 6$ and $B = 4$

Problem-solving Write the value of any constants you have determined at the end of your working. You can use this to check that your answer is in the required form.

Exercise 6D SKILLS ANALYSIS

1 Differentiate:

a $x(1 + 3x)^5$ b $2x(1 + 3x^2)^3$ c $x^3(2x + 6)^4$ d $3x^2(5x - 1)^{-1}$

2 Differentiate:

a $e^{-2x}(2x - 1)^5$ b $\sin 2x \cos 3x$ c $e^x \sin x$ d $\sin(5x)\ln(\cos x)$

3 a Find the value of $\frac{dy}{dx}$ at the point (1, 8) on the curve with equation $y = x^2(3x - 1)^3$

b Find the value of $\frac{dy}{dx}$ at the point (4, 36) on the curve with equation $y = 3x(2x + 1)^{\frac{1}{2}}$

c Find the value of $\frac{dy}{dx}$ at the point $\left(2, \frac{1}{5}\right)$ on the curve with equation $y = (x - 1)(2x + 1)^{-1}$

4 Find the stationary points of the curve C with the equation $y = (x - 2)^2(2x + 3)$

5 A curve C has equation $y = \left(x - \frac{\pi}{2}\right)^5 \sin 2x$, $0 < x < \pi$. Find the gradient of the curve at the point with x-coordinate $\frac{\pi}{4}$

E/P 6 A curve C has equation $y = x^2\cos(x^2)$. Find the equation of the tangent to the curve C at the point $P\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where a, b and c are exact constants. **(7 marks)**

E/P 7 Given that $y = 3x^2(5x - 3)^3$, show that

$$\frac{dy}{dx} = Ax(5x - 3)^n(Bx + C)$$

where n, A, B and C are constants to be determined. **(4 marks)**

E 8 A curve C has equation $y = (x + 3)^2 e^{3x}$

a Find $\frac{dy}{dx}$, using the product rule for **differentiation**. **(3 marks)**

b Find the gradient of C at the point where $x = 2$ **(3 marks)**

E 9 Differentiate with respect to x:

a $(2\sin x - 3\cos x)\ln 3x$ **(3 marks)**

b $x^4 e^{7x-3}$ **(3 marks)**

E 10 Find the value of $\frac{dy}{dx}$ at the point where $x = 1$ on the curve with equation

$$y = x^5\sqrt{10x + 6}$$ **(6 marks)**

Challenge

SKILLS ANALYSIS

Find $\frac{dy}{dx}$ for the following functions:

a $y = e^x \sin^2 x \cos x$ **b** $y = x(4x - 3)^6(1 - 4x)^9$

6.5 The quotient rule

You need to be able to differentiate the **quotient** of two functions.

- If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where u and v are functions of x.

The quotient rule in function notation is:

- If $f(x) = \frac{g(x)}{h(x)}$ then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

Watch out There is a minus sign in the numerator, so the order of the functions is important.

Example 12

Given that $y = \frac{x}{2x + 5}$, find $\frac{dy}{dx}$

Let $u = x$ and $v = 2x + 5$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(2x + 5) \times 1 - x \times 2}{(2x + 5)^2}$

$= \frac{5}{(2x + 5)^2}$

Let u be the numerator and let v be the denominator.

Recognise that y is a quotient and use the quotient rule.

Simplify the numerator of the fraction.

Example 13

A curve C with equation $y = \frac{\sin x}{e^{2x}}$, $0 < x < \pi$, has a stationary point at P. Find the coordinates of P. Give your answer to 3 significant figures.

Online Explore the graph of this function using technology.

Let $u = \sin x$ and $v = e^{2x}$

$\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = 2e^{2x}$

Write out u and v and find $\frac{du}{dx}$ and $\frac{dv}{dx}$ before using the quotient rule.

Using the quotient rule,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{e^{2x}\cos x - \sin x(2e^{2x})}{(e^{2x})^2}$$

$$= \frac{e^{2x}\cos x - 2e^{2x}\sin x}{e^{4x}}$$

$$= \frac{e^{2x}(\cos x - 2\sin x)}{e^{4x}}$$

$$= e^{-2x}(\cos x - 2\sin x)$$

When $\frac{dy}{dx} = 0$:

$e^{-2x}(\cos x - 2\sin x) = 0$

$e^{-2x} = 0$ or $\cos x - 2\sin x = 0$

$e^{-2x} = 0$ has no solution.

$\cos x - 2\sin x = 0$

$\cos x = 2\sin x$

$\frac{1}{2} = \tan x$

$x = 0.464$ (3 s.f.)

$y = \frac{\sin x}{e^{2x}}$

$= \frac{\sin(0.464)}{e^{2\times 0.464}} = 0.177$ (3 s.f.)

So the coordinates of P are (0.464, 0.177).

Write out the rule before substituting.

Simplify your expression for $\frac{dy}{dx}$ as much as possible.

P is a stationary point so $\frac{dy}{dx} = 0$

Problem-solving

If the product of two factors is equal to 0 then one of the factors must be equal to 0.

This is the only solution in the range $0 < x < \pi$

Substitute x into y to find the y-coordinate of the stationary point.

Exercise 6E

SKILLS PROBLEM-SOLVING

1 Differentiate with respect to x:

a $\frac{5x}{x+1}$ b $\frac{2x}{3x-2}$ c $\frac{x+3}{2x+1}$ d $\frac{3x^2}{(2x-1)^2}$ e $\frac{6x}{(5x+3)^{\frac{1}{2}}}$

2 Differentiate with respect to x:

a $\frac{e^{4x}}{\cos x}$ b $\frac{\ln x}{x+1}$ c $\frac{e^{-2x}+e^{2x}}{\ln x}$ d $\frac{(e^x+3)^3}{\cos x}$ e $\frac{\sin^2 x}{\ln x}$

3 Find the value of $\frac{dy}{dx}$ at the point $\left(1, \frac{1}{4}\right)$ on the curve with equation $y = \frac{x}{3x+1}$

4 Find the value of $\frac{dy}{dx}$ at the point (12, 3) on the curve with equation $y = \frac{x+3}{(2x+1)^{\frac{1}{2}}}$

5 Find the stationary points of the curve C with equation $y = \frac{e^{2x+3}}{x}$, $x \neq 0$

E **6** Find the equation of the tangent to the curve $y = \frac{e^{\frac{1}{3}x}}{x}$ at the point $\left(3, \frac{1}{3}e\right)$ **(7 marks)**

7 Find the exact value of $\frac{dy}{dx}$ at the point $x = \frac{\pi}{9}$ on the curve with equation $y = \frac{\ln x}{\sin 3x}$

E/P **8** The curve C has equation $x = \frac{e^y}{3 + 2y}$

a Find the coordinates of the point P where the curve cuts the x-axis. **(1 mark)**

b Find an equation of the normal to the curve at P, giving your answer in the form $y = mx + c$, where m and c are integers to be found. **(6 marks)**

E **9** Differentiate $\frac{x^4}{\cos 3x}$ with respect to x. **(4 marks)**

E/P **10** A curve C has equation $y = \frac{e^{2x}}{(x-2)^2}$, $x \neq 2$

a Show that

$$\frac{dy}{dx} = \frac{Ae^{2x}(Bx - C)}{(x-2)^3}$$

where A, B and C are integers to be found. **(4 marks)**

b Find the equation of the tangent of C at the point $x = 1$ **(3 marks)**

E/P **11** Given that

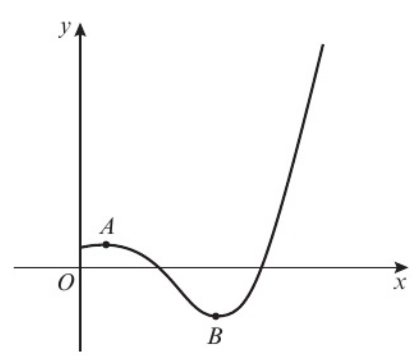
$$f(x) = \frac{2x}{x+5} + \frac{6x}{x^2 + 7x + 10}, \ x > 0$$

a show that $f(x) = \frac{2x}{x+2}$ **(4 marks)**

b Hence find $f'(3)$. **(3 marks)**

E/P **12** The diagram shows a sketch of the curve with equation $y = f(x)$, where

$$f(x) = \frac{2\cos 2x}{e^{2-x}}, \ 0 < x < \pi$$



The curve has a maximum turning point at A and a minimum turning point at B as shown in the diagram.

a Show that the x-coordinates of point A and point B are solutions to the equation $\tan 2x = \frac{1}{2}$ **(4 marks)**

b Find the range of $f(x)$. **(2 marks)**

6.6 Differentiating trigonometric functions

You can combine all the aforementioned rules and apply them to trigonometric functions to obtain standard results.

Example 14

If $y = \tan x$, find $\frac{dy}{dx}$

$y = \tan x = \frac{\sin x}{\cos x}$

Let $u = \sin x$ and $v = \cos x$

$\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\cos x \times \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

You can write $\tan x$ as $\frac{\sin x}{\cos x}$ and then use the quotient rule.

Use the identity $\cos^2 x + \sin^2 x \equiv 1$

You can generalise this method to differentiate $\tan kx$:

- **If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$**

Example 15

Differentiate: **a** $y = x\tan 2x$ **b** $y = \tan^4 x$

a $y = x\tan 2x$

$\frac{dy}{dx} = x \times 2\sec^2 2x + \tan 2x$

$= 2x\sec^2 2x + \tan 2x$

b $y = \tan^4 x = (\tan x)^4$

$\frac{dy}{dx} = 4(\tan x)^3(\sec^2 x)$

$= 4\tan^3 x \sec^2 x$

This is a product. Use $u = x$ and $v = \tan 2x$, together with the product rule.

Use the chain rule with $u = \tan x$

Example 16

SKILLS ANALYSIS

Show that if $y = \operatorname{cosec} x$, then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$

$y = \operatorname{cosec} x = \frac{1}{\sin x}$

Let $u = 1$ and $v = \sin x$

$\frac{du}{dx} = 0$ and $\frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x}$

$= -\frac{\cos x}{\sin^2 x}$

$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$

Use the quotient rule with $u = 1$ and $v = \sin x$

$u = 1$ is a constant so $\frac{du}{dx} = 0$

Rearrange your answer into the desired form using the definitions of cosec and cot.

← Pure 3 Section 3.1

You can use similar techniques to differentiate $\sec x$ and $\cot x$ giving you the following general results:

- **If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k \operatorname{cosec} kx \cot kx$**
- **If $y = \sec kx$, then $\frac{dy}{dx} = k \sec kx \tan kx$**
- **If $y = \cot kx$, then $\frac{dy}{dx} = -k \operatorname{cosec}^2 kx$**

Watch out While the standard results for tan, cosec, sec and cot are given in the formulae booklet, learning these results will enable you to differentiate a wide range of functions quickly and confidently.

Example 17

Differentiate: **a** $y = \frac{\operatorname{cosec} 2x}{x^2}$ **b** $y = \sec^3 x$

a $y = \frac{\operatorname{cosec} 2x}{x^2}$

So $\frac{dy}{dx} = \frac{x^2(-2\operatorname{cosec} 2x \cot 2x) - \operatorname{cosec} 2x \times 2x}{x^4}$

$= \frac{-2\operatorname{cosec} 2x(x \cot 2x + 1)}{x^3}$

Use the quotient rule with $u = \operatorname{cosec} 2x$ and $v = x^2$

b $y = \sec^3 x = (\sec x)^3$

$\frac{dy}{dx} = 3(\sec x)^2 (\sec x \tan x)$

$= 3\sec^3 x \tan x$

Use the chain rule with $u = \sec x$

You can use the rule $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ to differentiate arcsin x, arccos x and arctan x

Example 18 SKILLS ANALYSIS

Show that the derivative of arcsin x is $\frac{1}{\sqrt{1-x^2}}$

Let $y = \arcsin x$

So $x = \sin y$

$\frac{dx}{dy} = \cos y$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\sin^2 y + \cos^2 y \equiv 1$

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

So $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

arcsin is the inverse function of sin, so if $y = \arcsin x$ then $x = \sin y$ ← Pure 3 Section 3.5

Differentiate x with respect to y.

Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$. This gives you an expression for $\frac{dy}{dx}$ in terms of y.

Problem-solving

Use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to write $\cos y$ in terms of $\sin y$. This will enable you to find an expression for $\frac{dy}{dx}$ in terms of x.

Since $x = \sin y$, $x^2 = \sin^2 y$

You can use similar techniques to differentiate arccos x and arctan x giving you the following results:

- **If $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$**
- **If $y = \arccos x$, then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$**
- **If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$**

Example 19

Given $y = \arcsin x^2$, find $\frac{dy}{dx}$

Let $t = x^2$, then $y = \arcsin t$

Then $\frac{dt}{dx} = 2x$ $\quad \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2x}{\sqrt{1-x^4}}$

Substitute $t = x^2$ to get arcsin x^2 in the form arcsin t and use the chain rule.

Problem-solving

You could also write $x^2 = \sin y$ and therefore $x = \sqrt{\sin y}$. Then you could use the chain rule to find $\frac{dy}{dx}$ in terms of y and use $\sin^2 x + \cos^2 x \equiv 1$ to write the answer in terms of x.

Example 20

Given that $y = \arctan\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$

$$y = \arctan\left(\frac{1-x}{1+x}\right)$$

Let $u = \left(\frac{1-x}{1+x}\right)$

$$\frac{du}{dx} = \frac{(1+x)\times(-1) - (1-x)\times 1}{(1+x)^2}$$

Use the quotient rule, and simplify your answer as much as possible.

$$= \frac{-1-x-1+x}{(1+x)^2} = -\frac{2}{(1+x)^2}$$

$$y = \arctan u$$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

Differentiate with respect to u using the standard result for $y = \arctan x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \times \left(-\frac{2}{(1+x)^2}\right) = -\frac{2}{(1+u^2)(1+x)^2}$$

Use the chain rule with your expressions for $\frac{dy}{du}$ and $\frac{du}{dx}$

$$= -\frac{2}{\left(1+\left(\frac{1-x}{1+x}\right)^2\right)(1+x)^2}$$

Substitute $u = \left(\frac{1-x}{1+x}\right)$ back into $\frac{dy}{dx}$, to get your answer in terms of x only.

$$= -\frac{2}{(1+x)^2 + (1-x)^2}$$

$$= -\frac{2}{1+2x+x^2+1-2x+x^2}$$

$$= -\frac{2}{2+2x^2}$$

$$= -\frac{1}{1+x^2}$$

Expand the brackets in the denominator and collect like terms to simplify your final answer as much as possible.

Exercise 6F

SKILLS PROBLEM-SOLVING

1 Differentiate with respect to x:

a $y = \tan 3x$ **b** $y = 4\tan^3 x$ **c** $y = \tan(x-1)$ **d** $y = x^2 \tan \frac{1}{2}x + \tan\left(x - \frac{1}{2}\right)$

2 Differentiate with respect to x:

a $\cot 4x$ **b** $\sec 5x$ **c** $\operatorname{cosec} 4x$ **d** $\sec^2 3x$

e $x \cot 3x$ **f** $\frac{\sec^2 x}{x}$ **g** $\operatorname{cosec}^3 2x$ **h** $\cot^2(2x-1)$

3 Find the function $f'(x)$ where $f(x)$ is:

a $(\sec x)^{\frac{1}{2}}$ **b** $\sqrt{\cot x}$ **c** $\operatorname{cosec}^2 x$ **d** $\tan^2 x$

e $\sec^3 x$ **f** $\cot^3 x$

4 Find $f'(x)$ where $f(x)$ is:

a $x^2 \sec 3x$ **b** $\frac{\tan 2x}{x}$ **c** $\frac{x^2}{\tan x}$ **d** $e^x \sec 3x$

e $\frac{\ln x}{\tan x}$ **f** $\frac{e^{\tan x}}{\cos x}$

E/P **5** The curve C has equation

$$y = \frac{1}{\cos x \sin x}, \ 0 < x \leqslant \pi$$

a Find $\frac{dy}{dx}$ **(4 marks)**

b Determine the number of stationary points of the curve C. **(2 marks)**

c Find the equation of the tangent at the point where $x = \frac{\pi}{3}$, giving your answer in the form $ax + by + c = 0$, where a, b and c are exact constants to be determined. **(3 marks)**

E/P **6** Show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$ **(5 marks)**

E/P **7** Show that if $y = \cot x$ then $\frac{dy}{dx} = -\text{cosec}^2 x$ **(5 marks)**

P **8** Assuming standard results for $\sin x$ and $\cos x$, prove that:

a the derivative of $\arccos x$ is $-\frac{1}{\sqrt{1 - x^2}}$

b the derivative of $\arctan x$ is $\frac{1}{1 + x^2}$

9 Differentiate with respect to x:

a $\arccos 2x$ **b** $\arctan\left(\frac{x}{2}\right)$ **c** $\arcsin 3x$

d $\text{arccot}\, x$ **e** $\text{arcsec}\, x$ **f** $\text{arccosec}\, x$

g $\arcsin\left(\frac{x}{x - 1}\right)$ **h** $\arccos x^2$ **i** $e^x \arccos x$

j $\arcsin x \cos x$ **k** $x^2 \arccos x$ **l** $e^{\arctan x}$

E/P **10** Given that the curve C has equation

$$y = \frac{\arctan 2x}{x}$$

a show that the value of $\frac{dy}{dx}$ when $x = \frac{\sqrt{3}}{2}$ is $\frac{3\sqrt{3} - 4\pi}{9}$ **(4 marks)**

b find the equation of the normal to the curve C at $x = \frac{\sqrt{3}}{2}$ **(3 marks)**

E/P **11** A curve C has equation $x = (\arccos y)^2$. Show that

$$\frac{dy}{dx} = -\frac{\sqrt{1 - \cos^2 \sqrt{x}}}{2\sqrt{x}}$$ **(5 marks)**

E/P **12** Given that $x = \operatorname{cosec} 5y$

a find $\frac{dy}{dx}$ in terms of y. **(2 marks)**

b Hence find $\frac{dy}{dx}$ in terms of x. **(4 marks)**

Chapter review 6

E **1** Differentiate with respect to x:

a $\ln x^2$ **(3 marks)**

b $x^2 \sin 3x$ **(4 marks)**

E/P **2** **a** Given that $2y = x - \sin x \cos x$, $0 < x < 2\pi$, show that $\frac{dy}{dx} = \sin^2 x$ **(4 marks)**

b Find the coordinates of the **points of inflection** of the curve. **(4 marks)**

E **3** Differentiate, with respect to x:

a $\frac{\sin x}{x}$, $x > 0$ **(4 marks)**

b $\ln \frac{1}{x^2 + 9}$ **(4 marks)**

E/P **4** $f(x) = \frac{x}{x^2 + 2}$, $x \in \mathbb{R}$

a Given that $f(x)$ is increasing on the interval $[-k, k]$, find the largest possible value of k. **(4 marks)**

b Find the exact coordinates of the points of inflection of $f(x)$. **(5 marks)**

E/P **5** The function f is defined for positive real values of x by:

$$f(x) = 12 \ln x + x^{\frac{3}{2}}$$

a Find the set of values of x for which $f(x)$ is an increasing function of x. **(4 marks)**

b Find the coordinates of the point of inflection of the function f. **(4 marks)**

E/P **6** Given that a curve has equation $y = \cos^2 x + \sin x$, $0 < x < 2\pi$, find the coordinates of the stationary points of the curve. **(6 marks)**

E/P **7** The maximum point on the curve with equation $y = x\sqrt{\sin x}$, $0 < x < \pi$, is the point A. Show that the x-coordinate of point A satisfies the equation $2 \tan x + x = 0$ **(5 marks)**

E **8** $f(x) = e^{0.5x} - x^2$, $x \in \mathbb{R}$

a Find $f'(x)$. **(3 marks)**

b By evaluating $f'(6)$ and $f'(7)$, show that the curve with equation $y = f(x)$ has a stationary point at $x = p$, where $6 < p < 7$ **(2 marks)**

E/P **9** $f(x) = e^{2x} \sin 2x, 0 < x < \pi$

a Use calculus to find the coordinates of the turning points on the graph of $y = f(x)$ **(6 marks)**

b Show that $f''(x) = 8e^{2x} \cos 2x$ **(4 marks)**

c Hence, or otherwise, determine which turning point is a maximum and which is a minimum. **(3 marks)**

d Find the points of inflection of $f(x)$. **(2 marks)**

E **10** The curve C has equation $y = 2e^x + 3x^2 + 2$. Find the equation of the normal to C at the point where the curve intercepts the y-axis. Give your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found. **(5 marks)**

E **11** The curve C has equation $y = f(x)$, where

$$f(x) = 3 \ln x + \frac{1}{x}, x > 0$$

The point P is a stationary point on C.

a Calculate the x-coordinate of P. **(4 marks)**

The point Q on C has x-coordinate 1.

b Find an equation for the normal to C at Q. **(4 marks)**

E **12** The curve C has equation $y = e^{2x} \cos x$

a Show that the turning points on C occur when $\tan x = 2$ **(4 marks)**

b Find an equation of the tangent to C at the point where $x = 0$ **(4 marks)**

E **13** Given that $x = y^2 \ln y, y > 0$

a find $\frac{dx}{dy}$ **(4 marks)**

b Use your answer to part **a** to find, in terms of e, the value of $\frac{dy}{dx}$ at $y = e$ **(2 marks)**

E **14** A curve has equation $f(x) = (x^3 - 2x)e^{-x}$

a Find $f'(x)$. **(4 marks)**

The normal to C at the origin O intersects C again at P.

b Show that the x-coordinate of P is the solution to the equation $2x^2 = e^x + 4$ **(6 marks)**

Challenge

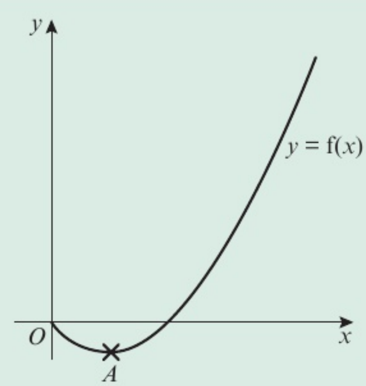
SKILLS CREATIVITY

The diagram shows part of the curve with equation $y = f(x)$ where $f(x) = x(1 + x) \ln x, x > 0$

The point A is the minimum point of the curve.

a Find $f'(x)$.

b Hence show that the x-coordinate of A is the solution to the equation $x = e^{-\frac{1+x}{1+2x}}$



Summary of key points

1 For small angles, measured in radians:
- $\sin x \approx x$
- $\cos x \approx 1 - \frac{1}{2}x^2$

2
- If $y = \sin kx$ then $\frac{dy}{dx} = k\cos kx$
- If $y = \cos kx$ then $\frac{dy}{dx} = -k\sin kx$

3
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$
- If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

4 If $y = a^{kx}$, where k is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx}k \ln a$

5 The **chain rule** is: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
where y is a function of u, and u is another function of x.

6 The chain rule enables you to differentiate a function of a function. In general,
- if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1} f'(x)$
- if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$

7 $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

8 The **product rule**:
- If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, where u and v are functions of x.
- If $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + h(x)g'(x)$

9 The **quotient rule**:
- If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where u and v are functions of x.
- If $f(x) = \frac{g(x)}{h(x)}$ then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

10
- If $y = \tan kx$ then $\dfrac{dy}{dx} = k\sec^2 kx$
- If $y = \operatorname{cosec} kx$ then $\dfrac{dy}{dx} = -k\operatorname{cosec} kx \cot kx$
- If $y = \sec kx$ then $\dfrac{dy}{dx} = k\sec kx \tan kx$
- If $y = \cot kx$ then $\dfrac{dy}{dx} = -k\operatorname{cosec}^2 kx$

11
- If $y = \arcsin x$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$
- If $y = \arccos x$ then $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$
- If $y = \arctan x$ then $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$

7 INTEGRATION

5.1
5.2

Learning objectives

After completing this chapter you should be able to:

- Integrate standard mathematical functions including trigonometric and exponential functions and use the reverse of the chain rule to integrate functions of the form $f(ax + b)$ → pages 147–151
- Use trigonometric identities in integration → pages 151–153
- Use the reverse of the chain rule to integrate more complex functions → pages 153–156

Prior knowledge check

1 Differentiate:

a $(2x - 7)^6$ **b** $\sin 5x$ **c** $e^{\frac{x}{3}}$

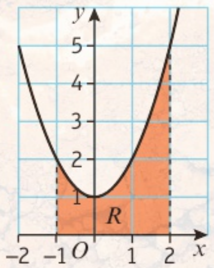
← Pure 3 Sections 6.1, 6.2, 6.3

2 Given $f(x) = 8x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

a find $\int f(x)\,dx$ ← Pure 1 Sections 9.1, 9.2

b find $\int_4^9 f(x)\,dx$ ← Pure 2 Section 8.1

3 Find the area of the region R bounded by the curve $y = x^2 + 1$, the x-axis and the lines $x = -1$ and $x = 2$



← Pure 2 Section 8.2

Archaeologists use carbon dating to estimate the age of fossilised plants and animals. This estimation is based on the principle of exponential decay.

7.1 Integrating standard functions

Integration is the inverse of differentiation. You can use your knowledge of derivatives to integrate familiar functions.

① $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

② $\int e^x \, dx = e^x + c$

③ $\int \frac{1}{x} \, dx = \ln|x| + c$

④ $\int \cos x \, dx = \sin x + c$

⑤ $\int \sin x \, dx = -\cos x + c$

⑥ $\int \sec^2 x \, dx = \tan x + c$

⑦ $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

⑧ $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

⑨ $\int \sec x \tan x \, dx = \sec x + c$

Watch out This is true for all values of n except −1.

Notation When finding $\int \frac{1}{x} \, dx$, it is usual to write the answer as $\ln|x| + c$. The modulus sign removes difficulties that could arise when evaluating the integral for negative values of x.

Links For example, if $y = \cos x$ then $\frac{dy}{dx} = -\sin x$

This means that $\int (-\sin x) \, dx = \cos x + c$ and hence $\int \sin x \, dx = -\cos x + c$

← Pure 3 Section 6.1

Example 1

Find the following integrals:

a $\int \left(2\cos x + \frac{3}{x} - \sqrt{x}\right) dx$ **b** $\int \left(\frac{\cos x}{\sin^2 x} - 2e^x\right) dx$

a $\int 2\cos x \, dx = 2\sin x + c$ — Integrate each term separately. Use ④

$\int \frac{3}{x} \, dx = 3\ln|x| + c$ — Use ③

$\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{2}{3}x^{\frac{3}{2}} + c$ — Use ①

So $\int \left(2\cos x + \frac{3}{x} - \sqrt{x}\right) dx$

$= 2\sin x + 3\ln|x| - \frac{2}{3}x^{\frac{3}{2}} + c$ — This is an indefinite integral so don't forget the $+ c$

b $\frac{\cos x}{\sin^2 x} = \frac{\cos x}{\sin x} \times \frac{1}{\sin x} = \cot x \operatorname{cosec} x$ — Look at the list of integrals of standard functions and express the **integrand** in terms of these standard functions.

$\int (\cot x \operatorname{cosec} x) \, dx = -\operatorname{cosec} x + c$ — Remember the minus sign.

$\int 2e^x \, dx = 2e^x + c$

So $\int \left(\frac{\cos x}{\sin^2 x} - 2e^x\right) dx$

$= -\operatorname{cosec} x - 2e^x + c$

Example 2

Given that a is a positive constant and $\int_a^{3a}\left(\frac{2x+1}{x}\right)dx = \ln 12$, find the exact value of a.

$\int_a^{3a}\left(\frac{2x+1}{x}\right)dx$

$= \int_a^{3a}\left(2 + \frac{1}{x}\right)dx$ — Separate the terms by dividing by x, then integrate term by term.

$= [2x + \ln x]_a^{3a}$ — Remember the limits are a and $3a$.

$= (6a + \ln 3a) - (2a + \ln a)$ — Substitute $3a$ and a into the integrated expression.

$= 4a + \ln\left(\frac{3a}{a}\right)$ — Use the laws of logarithms: $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$= 4a + \ln 3$

So, $4a + \ln 3 = \ln 12$

$4a = \ln 12 - \ln 3$

$4a = \ln 4$ — $\ln 12 - \ln 3 = \ln\left(\frac{12}{3}\right) = \ln 4$

$a = \frac{1}{4}\ln 4$

Problem-solving

Integrate as normal and write the limits as a and $3a$. Substitute these limits into your integral to get an expression in a and set this equal to ln 12. Solve the resulting equation to find the value of a.

Online Use your calculator to check your value of a using numerical integration.

Exercise 7A SKILLS PROBLEM-SOLVING

1 Integrate the following with respect to x:

a $3\sec^2 x + \frac{5}{x} + \frac{2}{x^2}$

b $5e^x - 4\sin x + 2x^3$

c $2(\sin x - \cos x + x)$

d $3\sec x \tan x - \frac{2}{x}$

e $5e^x + 4\cos x - \frac{2}{x^2}$

f $\frac{1}{2x} + 2\operatorname{cosec}^2 x$

g $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

h $e^x + \sin x + \cos x$

i $2\operatorname{cosec} x \cot x - \sec^2 x$

j $e^x + \frac{1}{x} - \operatorname{cosec}^2 x$

2 Find the following integrals:

a $\int\left(\frac{1}{\cos^2 x} + \frac{1}{x^2}\right)dx$

b $\int\left(\frac{\sin x}{\cos^2 x} + 2e^x\right)dx$

c $\int\left(\frac{1+\cos x}{\sin^2 x} + \frac{1+x}{x^2}\right)dx$

d $\int\left(\frac{1}{\sin^2 x} + \frac{1}{x}\right)dx$

e $\int \sin x(1 + \sec^2 x)\,dx$

f $\int \cos x(1 + \operatorname{cosec}^2 x)\,dx$

g $\int \operatorname{cosec}^2 x(1 + \tan^2 x)\,dx$

h $\int \sec^2 x(1 - \cot^2 x)\,dx$

i $\int \sec^2 x(1 + e^x\cos^2 x)\,dx$

j $\int\left(\frac{1+\sin x}{\cos^2 x} + \cos^2 x \sec x\right)dx$

3 Evaluate the following. Give your answers as exact values.

a $\int_3^7 2e^x\,dx$

b $\int_1^6\left(\frac{1+x}{x^3}\right)dx$

c $\int_{\frac{\pi}{2}}^{\pi} -5\sin x\,dx$

d $\int_{-\frac{\pi}{4}}^{0} \sec x(\sec x + \tan x)\,dx$

Watch out When applying limits to integrated trigonometric functions, always work in radians.

(E/P) **4** Given that a is a positive constant and $\int_a^{2a}\left(\frac{3x-1}{x}\right)dx = 6 + \ln\left(\frac{1}{2}\right)$ find the exact value of a. **(4 marks)**

(E/P) **5** Given that a is a positive constant and $\int_{\ln 1}^{\ln a}(e^x + e^{-x})\,dx = \frac{48}{7}$, find the exact value of a. **(4 marks)**

(E/P) **6** Given $\int_2^b (3e^x + 6e^{-2x})\,dx = 0$, find the value of b. **(4 marks)**

(E/P) **7** $f(x) = \frac{1}{8}x^{\frac{3}{2}} - \frac{4}{x}$, $x > 0$

a Solve the equation $f(x) = 0$ **(2 marks)**

b Find $\int f(x)\,dx$ **(2 marks)**

c Evaluate $\int_1^4 f(x)\,dx$, giving your answer in the form $p + q\ln r$, where p, q and r are rational numbers. **(3 marks)**

7.2 Integrating f($ax + b$)

If you know the integral of a function f(x) you can integrate a function of the form f($ax + b$) using the reverse of the chain rule for differentiation.

Example 3

Find the following integrals:

a $\int \cos(2x + 3)\,dx$ **b** $\int e^{4x+1}\,dx$ **c** $\int \sec^2 3x\,dx$

a Consider $y = \sin(2x + 3)$:

Integrating cos x gives sin x, so try sin ($2x + 3$)

$$\frac{dy}{dx} = \cos(2x + 3) \times 2$$

Use the chain rule. Remember to multiply by the derivative of $2x + 3$ which is 2.

This is 2 times the required expression so you need to divide sin ($2x + 3$) by 2.

So $\int \cos(2x + 3)\,dx = \frac{1}{2}\sin(2x + 3) + c$

b Consider $y = e^{4x+1}$:

The integral of e^x is e^x, so try e^{4x+1}

$$\frac{dy}{dx} = e^{4x+1} \times 4$$

This is 4 times the required expression so you divide by 4.

So $\int e^{4x+1}\,dx = \frac{1}{4}e^{4x+1} + c$

c Consider $y = \tan 3x$:

$$\frac{dy}{dx} = \sec^2 3x \times 3$$

Recall (6). Let $y = \tan 3x$ and differentiate using the chain rule. This is 3 times the required expression so you divide by 3.

So $\int \sec^2 3x\,dx = \frac{1}{3}\tan 3x + c$

In general:

$$\int f'(ax + b)\,dx = \frac{1}{a}f(ax + b) + c$$

Watch out You cannot use this method to integrate an expression such as $\cos(2x^2 + 3)$ since it is not in the form $f(ax + b)$.

Example 4

Find the following integrals:

a $\int\left(\frac{1}{3x+2}\right)dx$ **b** $\int(2x+3)^4\,dx$

a Consider $y = \ln(3x + 2)$:

$\frac{dy}{dx} = \frac{1}{3x+2} \times 3$

So $\int\left(\frac{1}{3x+2}\right)dx = \frac{1}{3}\ln|3x + 2| + c$

b Consider $y = (2x + 3)^5$:

$\frac{dy}{dx} = 5 \times (2x + 3)^4 \times 2$

$= 10 \times (2x + 3)^4$

So $\int(2x + 3)^4\,dx = \frac{1}{10}(2x + 3)^5 + c$

Integrating $\frac{1}{x}$ gives $\ln|x|$ so try $\ln(3x + 2)$

The 3 comes from the chain rule. It is 3 times the required expression, so divide by 3.

To integrate $(ax + b)^n$ try $(ax + b)^{n+1}$

The 5 comes from the exponent and the 2 comes from the chain rule.

This answer is 10 times the required expression, so divide by 10.

Exercise 7B

SKILLS ANALYSIS

1 Integrate the following with respect to x:

a $\sin(2x + 1)$ **b** $3e^{2x}$

c $4e^{x+5}$ **d** $\cos(1 - 2x)$

e $\operatorname{cosec}^2 3x$ **f** $\sec 4x \tan 4x$

g $3\sin\left(\frac{1}{2}x + 1\right)$ **h** $\sec^2(2 - x)$ **i** $\operatorname{cosec} 2x \cot 2x$ **j** $\cos 3x - \sin 3x$

Hint For part **a**, consider $y = \cos(2x + 1)$. You do not need to write out this step once you are confident with using this method.

2 Find the following integrals:

a $\int(e^{2x} - \frac{1}{2}\sin(2x - 1))\,dx$ **b** $\int(e^x + 1)^2\,dx$

c $\int\sec^2 2x(1 + \sin 2x)\,dx$ **d** $\int\left(\frac{3 - 2\cos\frac{x}{2}x}{\sin^2\frac{x}{2}x}\right)dx$

e $\int(e^{3-x} + \sin(3 - x) + \cos(3 - x))\,dx$

3 Integrate the following with respect to x:

a $\frac{1}{2x+1}$ **b** $\frac{1}{(2x+1)^2}$ **c** $(2x + 1)^2$ **d** $\frac{3}{4x-1}$

e $\frac{3}{1-4x}$ **f** $\frac{3}{(1-4x)^2}$ **g** $(3x + 2)^5$ **h** $\frac{3}{(1-2x)^3}$

4 Find the following integrals:

a $\int\left(3\sin(2x+1)+\frac{4}{2x+1}\right)\mathrm{d}x$

b $\int(\mathrm{e}^{5x}+(1-x)^5)\,\mathrm{d}x$

c $\int\left(\frac{1}{\sin^2 2x}+\frac{1}{1+2x}+\frac{1}{(1+2x)^2}\right)\mathrm{d}x$

d $\int\left((3x+2)^2+\frac{1}{(3x+2)^2}\right)\mathrm{d}x$

5 Evaluate:

a $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos(\pi-2x)\,\mathrm{d}x$

b $\int_{\frac{1}{2}}^{1}\frac{12}{(3-2x)^4}\,\mathrm{d}x$

c $\int_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}\sec^2(\pi-3x)\,\mathrm{d}x$

d $\int_{2}^{3}\frac{5}{7-2x}\,\mathrm{d}x$

E/P 6 Given $\int_3^b (2x-6)^2\,\mathrm{d}x = 36$, find the value of b. **(4 marks)**

E/P 7 Given $\int_{\mathrm{e}^2}^{\mathrm{e}^8}\frac{1}{kx}\,\mathrm{d}x = \frac{1}{4}$, find the value of k. **(4 marks)**

Challenge

SKILLS INNOVATION

Given $\int_5^{11}\left(\frac{1}{ax+b}\right)\mathrm{d}x = \frac{1}{a}\ln\left(\frac{41}{17}\right)$, and that a and b are integers with $0 < a < 10$, find two different pairs of values for a and b.

7.3 Using trigonometric identities

- Trigonometric identities can be used to integrate expressions. This allows an expression that cannot be integrated to be replaced by an identical expression that can be integrated.

Links Make sure you are familiar with the standard trigonometric identities. ← Pure 2 Section 6.3

Example 5

Find $\int\tan^2 x\,\mathrm{d}x$

Since $\sec^2 x \equiv 1+\tan^2 x$

$\tan^2 x \equiv \sec^2 x - 1$ — You cannot integrate $\tan^2 x$ but you can integrate $\sec^2 x$ directly.

So $\int\tan^2 x\,\mathrm{d}x = \int(\sec^2 x - 1)\,\mathrm{d}x$

$= \int\sec^2 x\,\mathrm{d}x - \int 1\,\mathrm{d}x$ — Using ⑥

$= \tan x - x + c$

Example 6

SKILLS ANALYSIS

Show that $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x \, dx = \frac{\pi}{48} + \frac{1-\sqrt{2}}{8}$

Recall $\cos 2x \equiv 1 - 2\sin^2 x$

So $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$

So $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x \, dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{8}}$

$= \left(\frac{\pi}{16} - \frac{1}{4}\sin\left(\frac{\pi}{4}\right)\right) - \left(\frac{\pi}{24} - \frac{1}{4}\sin\left(\frac{\pi}{6}\right)\right)$

$= \left(\frac{\pi}{16} - \frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)\right) - \left(\frac{\pi}{24} - \frac{1}{4}\left(\frac{1}{2}\right)\right)$

$= \left(\frac{\pi}{16} - \frac{\pi}{24}\right) + \frac{1}{4}\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)$

$= \left(\frac{3\pi}{48} - \frac{2\pi}{48}\right) + \frac{1-\sqrt{2}}{8}$

$= \frac{\pi}{48} + \frac{1-\sqrt{2}}{8}$

You cannot integrate $\sin^2 x$ directly. Use the trigonometric identity to write it in terms of $\cos 2x$

Use the reverse chain rule. If $y = \sin 2x$, $\frac{dy}{dx} = 2\cos 2x$. Adjust for the constant.

Substitute the limits into the integrated expression.

Problem-solving

Being familiar with the exact values for trigonometric functions given in radians will save you lots of time in your exam.

Write $\sin\left(\frac{\pi}{4}\right)$ in its rationalised denominator form, as $\frac{\sqrt{2}}{2}$ rather than $\frac{1}{\sqrt{2}}$. This will make it easier to simplify your fractions.

Watch out This is a 'show that' question so don't use your calculator to simplify the fractions. Show each line of your working carefully.

Example 7

Find:

a $\int \sin 3x \cos 3x \, dx$ **b** $\int (\sec x + \tan x)^2 \, dx$

a $\int \sin 3x \cos 3x \, dx = \int \frac{1}{2}\sin 6x \, dx$

$= -\frac{1}{2} \times \frac{1}{6}\cos 6x + c$

$= -\frac{1}{12}\cos 6x + c$

b $(\sec x + \tan x)^2$

$\equiv \sec^2 x + 2\sec x \tan x + \tan^2 x$

$\equiv \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$

$\equiv 2\sec^2 x + 2\sec x \tan x - 1$

So $\int (\sec x + \tan x)^2 \, dx$

$= \int (2\sec^2 x + 2\sec x \tan x - 1) \, dx$

$= 2\tan x + 2\sec x - x + c$

Remember $\sin 2A \equiv 2\sin A \cos A$, so $\sin 6x \equiv 2\sin 3x \cos 3x$

Use the reverse chain rule.

Simplify $\frac{1}{2} \times \frac{1}{6}$ to $\frac{1}{12}$

Multiply out the bracket.

Write $\tan^2 x$ as $\sec^2 x - 1$. Then all the terms are standard integrals.

Integrate each term using (6) and (9)

Exercise 7C

SKILLS PROBLEM-SOLVING

1 Integrate the following with respect to x:

a $\cot^2 x$ **b** $\cos^2 x$
c $\sin 2x \cos 2x$ **d** $(1 + \sin x)^2$
e $\tan^2 3x$ **f** $(\cot x - \operatorname{cosec} x)^2$
g $(\sin x + \cos x)^2$ **h** $\sin^2 x \cos^2 x$ **i** $\dfrac{1}{\sin^2 x \cos^2 x}$ **j** $(\cos 2x - 1)^2$

Hint For part **a**, use $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$. For part **c**, use $\sin 2A \equiv 2 \sin A \cos A$, making a suitable substitution for A.

2 Find the following integrals:

a $\displaystyle\int \left(\frac{1 - \sin x}{\cos^2 x}\right) dx$ **b** $\displaystyle\int \left(\frac{1 + \cos x}{\sin^2 x}\right) dx$ **c** $\displaystyle\int \left(\frac{\cos 2x}{\cos^2 x}\right) dx$

d $\displaystyle\int \left(\frac{\cos^2 x}{\sin^2 x}\right) dx$ **e** $\displaystyle\int \frac{(1 + \cos x)^2}{\sin^2 x} dx$ **f** $\displaystyle\int (\cot x - \tan x)^2 dx$

g $\displaystyle\int (\cos x - \sin x)^2 dx$ **h** $\displaystyle\int (\cos x - \sec x)^2 dx$ **i** $\displaystyle\int \left(\frac{\cos 2x}{1 - \cos^2 2x}\right) dx$

E/P 3 Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{2 + \pi}{8}$ **(4 marks)**

4 Find the exact value of each of the following:

a $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{\sin^2 x \cos^2 x}\right) dx$ **b** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \operatorname{cosec} x)^2 dx$ **c** $\displaystyle\int_0^{\frac{\pi}{4}} \left(\frac{(1 + \sin x)^2}{\cos^2 x}\right) dx$ **d** $\displaystyle\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \left(\frac{\sin 2x}{1 - \sin^2 2x}\right) dx$

E/P 5 **a** By expanding $\sin(3x + 2x)$ and $\sin(3x - 2x)$ using the double-angle formulae, or otherwise, show that $\sin 5x + \sin x \equiv 2 \sin 3x \cos 2x$ **(4 marks)**

b Hence find $\displaystyle\int \sin 3x \cos 2x \, dx$ **(3 marks)**

E/P 6 $f(x) = 5 \sin^2 x + 7 \cos^2 x$

a Show that $f(x) = \cos 2x + 6$ **(3 marks)**

b Hence, find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} f(x) \, dx$ **(4 marks)**

E/P 7 **a** Show that $\cos^4 x \equiv \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8}$ **(4 marks)**

b Hence, find $\displaystyle\int \cos^4 x \, dx$ **(4 marks)**

7.4 Reverse chain rule

If a function can be written in the form $k\dfrac{f'(x)}{f(x)}$, you can integrate it using the reverse of the chain rule for differentiation.

Example 8

Find

a $\displaystyle\int \frac{2x}{x^2 + 1} dx$ **b** $\displaystyle\int \frac{\cos x}{3 + 2 \sin x} dx$

Problem-solving

If $f(x) = 3 + 2 \sin x$, then $f'(x) = 2 \cos x$

By adjusting for the constant, the numerator is the derivative of the denominator.

a Let $I = \int \frac{2x}{x^2 + 1} dx$

Consider $y = \ln|x^2 + 1|$

Then $\frac{dy}{dx} = \frac{1}{x^2 + 1} \times 2x$

So $I = \ln|x^2 + 1| + c$

b Let $I = \int \frac{\cos x}{3 + 2\sin x} dx$

Consider $y = \ln|3 + 2\sin x|$

Then $\frac{dy}{dx} = \frac{1}{3 + 2\sin x} \times 2\cos x$

So $I = \frac{1}{2}\ln|3 + 2\sin x| + c$

This is equal to the original integrand, so you don't need to adjust it.

Since integration is the reverse of differentiation.

Try differentiating $y = \ln|3 + 2\sin x|$

The derivative of $\ln|3 + 2\sin x|$ is twice the original integrand, so you need to divide it by 2.

To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}dx$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.

Watch out You can't use this method to integrate a function such as $\frac{1}{x^2 + 3}$ because the derivative of $x^2 + 3$ is $2x$, and the numerator does not contain an x term.

You can use a similar method with functions of the form $kf'(x)(f(x))^n$.

Example 9 SKILLS ANALYSIS

Find:

a $\int 3\cos x \sin^2 x\, dx$ **b** $\int x(x^2 + 5)^3\, dx$

a Let $I = \int 3\cos x \sin^2 x\, dx$

Consider $y = \sin^3 x$

$\frac{dy}{dx} = 3\sin^2 x \cos x$

So $I = \sin^3 x + c$

b Let $I = \int x(x^2 + 5)^3\, dx$

Consider $y = (x^2 + 5)^4$

$\frac{dy}{dx} = 4(x^2 + 5)^3 \times 2x$

$= 8x(x^2 + 5)^3$

So $I = \frac{1}{8}(x^2 + 5)^4 + c$

Try differentiating $\sin^3 x$

This is equal to the original integrand, so you don't need to adjust it.

Try differentiating $(x^2 + 5)^4$

The $2x$ comes from differentiating $x^2 + 5$

This is 8 times the required expression so you divide by 8.

- **To integrate an expression of the form $\int k f'(x)(f(x))^n dx$, try $(f(x))^{n+1}$ and differentiate to check, and then adjust any constant.**

Example 10

Use integration to find $\int \frac{\text{cosec}^2 x}{(2 + \cot x)^3}\,dx$

Let $I = \int \frac{\text{cosec}^2 x}{(2 + \cot x)^3}\,dx$

Consider $y = (2 + \cot x)^{-2}$

$\frac{dy}{dx} = -2(2 + \cot x)^{-3} \times (-\text{cosec}^2 x)$

$= 2(2 + \cot x)^{-3}\,\text{cosec}^2 x$

So $I = \frac{1}{2}(2 + \cot x)^{-2} + c$

This is in the form $\int k\text{f}'(x)(\text{f}(x))^n\,dx$ with $\text{f}(x) = 2 + \cot x$ and $n = -3$

Use the chain rule.

This is 2 times the required answer so you need to divide by 2.

Example 11

Given that $\int_0^\theta 5 \tan x \sec^4 x\,dx = \frac{15}{4}$ where $0 < \theta < \frac{\pi}{2}$, find the exact value of θ.

Let $I = \int_0^\theta 5 \tan x \sec^4 x\,dx$

Consider $y = \sec^4 x$

$\frac{dy}{dx} = 4\sec^3 x \times \sec x \tan x$

$= 4\sec^4 x \tan x$

So $I = \left[\frac{5}{4}\sec^4 x\right]_0^\theta = \frac{15}{4}$

$\left(\frac{5}{4}\sec^4 \theta\right) - \left(\frac{5}{4}\sec^4 0\right) = \frac{15}{4}$

$\frac{5}{4}\sec^4 \theta - \frac{5}{4} = \frac{15}{4}$

$\frac{5}{4}\sec^4 \theta = \frac{20}{4}$

$\sec^4 \theta = 4$

$\sec \theta = \pm\sqrt{2}$

$\theta = \frac{\pi}{4}$

This is in the form $\int k\text{f}'(x)(\text{f}(x))^n\,dx$ with $\text{f}(x) = \sec x$ and $n = 4$

This is $\frac{4}{5}$ times the required answer so you need to divide by $\frac{4}{5}$

Substitute the limits into the integrated expression.

$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

Take the 4th root of both sides.

The solutions to $\cos\theta = \pm\frac{1}{\sqrt{2}}$ are $\theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$ The only solution within the given range for θ is $\frac{\pi}{4}$

Online Check your solution by using your calculator.

Exercise 7D

SKILLS ANALYSIS

Hint Decide carefully whether each expression is in the form $k\frac{\text{f}'(x)}{\text{f}(x)}$ or $k\text{f}'(x)(\text{f}(x))^n$

1 Integrate the following functions with respect to x.

a $\frac{x}{x^2 + 4}$ **b** $\frac{e^{2x}}{e^{2x} + 1}$ **c** $\frac{x}{(x^2 + 4)^3}$

d $\frac{e^{2x}}{(e^{2x} + 1)^3}$ **e** $\frac{\cos 2x}{3 + \sin 2x}$ **f** $\frac{\sin 2x}{(3 + \cos 2x)^3}$

g xe^{x^2} **h** $\cos 2x(1 + \sin 2x)^4$ **i** $\sec^2 x \tan^2 x$ **j** $\sec^2 x(1 + \tan^2 x)$

2 Find the following integrals:

a $\int (x + 1)(x^2 + 2x + 3)^4 \, dx$

b $\int \operatorname{cosec}^2 2x \cot 2x \, dx$

c $\int \sin^5 3x \cos 3x \, dx$

d $\int \cos x \, e^{\sin x} \, dx$

e $\int \frac{e^{2x}}{e^{2x} + 3} \, dx$

f $\int x(x^2 + 1)^{\frac{3}{2}} \, dx$

g $\int (2x + 1)\sqrt{x^2 + x + 5} \, dx$

h $\int \frac{2x + 1}{\sqrt{x^2 + x + 5}} \, dx$

i $\int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}} \, dx$

j $\int \frac{\sin x \cos x}{\cos 2x + 3} \, dx$

3 Find the exact value of each of the following:

a $\int_0^3 (3x^2 + 10x)\sqrt{x^3 + 5x^2 + 9} \, dx$

b $\int_{\frac{\pi}{9}}^{\frac{2\pi}{9}} \frac{6 \sin 3x}{1 - \cos 3x} \, dx$

c $\int_4^7 \frac{x}{x^2 - 1} \, dx$

d $\int_0^{\frac{\pi}{4}} \sec^2 x \, e^{4 \tan x} \, dx$

E/P **4** Given that $\int_0^k kx^2 e^{x^3} \, dx = \frac{2}{3}(e^8 - 1)$, find the value of k. **(3 marks)**

P **5** Given that $\int_0^\theta 4 \sin 2x \cos^4 2x \, dx = \frac{4}{5}$, where $0 < \theta < \pi$, find the exact value of θ.

E/P **6** **a** By writing $\cot x = \frac{\cos x}{\sin x}$, find $\int \cot x \, dx$ **(2 marks)**

b Show that $\int \tan x \, dx \equiv \ln|\sec x| + c$ **(3 marks)**

Chapter review 7

1 By choosing a suitable method of integration, find:

a $\int (2x - 3)^7 \, dx$

b $\int x\sqrt{4x - 1} \, dx$

c $\int \sin^2 x \cos x \, dx$

d $\int x \ln x \, dx$

e $\int \frac{4 \sin x \cos x}{4 - 8 \sin^2 x} \, dx$

f $\int \frac{1}{3 - 4x} \, dx$

2 By choosing a suitable method, evaluate the following definite integrals. Write your answers as exact values.

a $\int_{-3}^0 x(x^2 + 3)^5 \, dx$

b $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$

c $\int_1^4 \left(16x^{\frac{3}{2}} - \frac{2}{x}\right) dx$

d $\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} (\cos x + \sin x)(\cos x - \sin x) \, dx$

e $\int_1^4 \left(\frac{4}{16x^2 + 8x - 3}\right) dx$

f $\int_0^{\ln 2} \left(\frac{1}{1 + e^x}\right) dx$

E/P **3** **a** Show that $\int_1^e \frac{1}{x^2} \ln x \, dx = 1 - \frac{2}{e}$ **(5 marks)**

b Given that $p > 1$, show that $\int_1^p \frac{1}{(x + 1)(2x - 1)} \, dx = \frac{1}{3} \ln \frac{4p - 2}{p + 1}$ **(5 marks)**

E/P **4** Given $\int_{\frac{1}{2}}^{b}\left(\frac{2}{x^3}-\frac{1}{x^2}\right)\mathrm{d}x=\frac{9}{4}$, find the value of b. **(4 marks)**

E/P **5** Given $\int_0^{\theta}\cos x\sin^3 x\,\mathrm{d}x=\frac{9}{64}$, where $\theta > 0$, find the smallest possible value of θ. **(4 marks)**

Challenge

SKILLS CREATIVITY

Given $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}(1-\pi\sin kx)\,\mathrm{d}x=\pi(7-6\sqrt{2})$,
find the exact value of k.

Hint Calculate the value of the indefinite integral in terms of k and solve the resulting equation.

Summary of key points

1 $\int x^n\,\mathrm{d}x=\frac{x^{n+1}}{n+1}+c$ $\quad\int e^x\,\mathrm{d}x=e^x+c$ $\quad\int\frac{1}{x}\,\mathrm{d}x=\ln|x|+c$

$\int\cos x\,\mathrm{d}x=\sin x+c$ $\quad\int\sin x\,\mathrm{d}x=-\cos x+c$ $\quad\int\sec^2 x=\tan x+c$

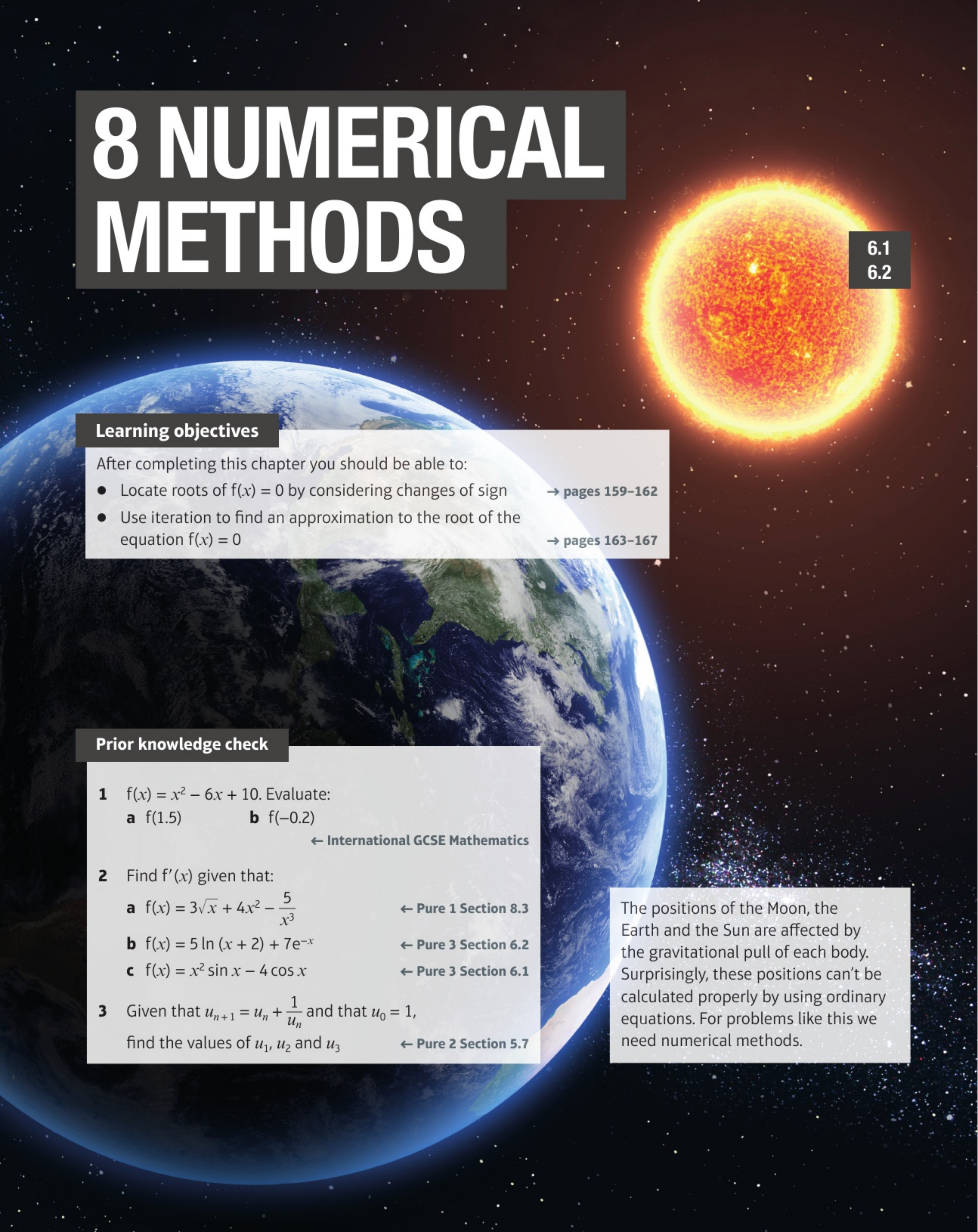
$\int\operatorname{cosec} x\cot x\,\mathrm{d}x=-\operatorname{cosec} x+c$ $\quad\int\operatorname{cosec}^2 x\,\mathrm{d}x=-\cot x+c$ $\quad\int\sec x\tan x\,\mathrm{d}x=\sec x+c$

2 $\int f'(ax+b)\,\mathrm{d}x=\frac{1}{a}f(ax+b)+c$

3 Trigonometric identities can be used to integrate expressions. This allows an expression that cannot be integrated to be replaced by an identical expression that can be integrated.

4 To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}\,\mathrm{d}x$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.

5 To integrate expressions of the form $\int kf'(x)(f(x))^n\,\mathrm{d}x$, try $(f(x))^{n+1}$ and differentiate to check, and then adjust any constant.



8 NUMERICAL METHODS

6.1
6.2

Learning objectives

After completing this chapter you should be able to:

- Locate roots of $f(x) = 0$ by considering changes of sign → pages 159–162
- Use iteration to find an approximation to the root of the equation $f(x) = 0$ → pages 163–167

Prior knowledge check

1 $f(x) = x^2 - 6x + 10$. Evaluate:

a $f(1.5)$ **b** $f(-0.2)$

← International GCSE Mathematics

2 Find $f'(x)$ given that:

a $f(x) = 3\sqrt{x} + 4x^2 - \frac{5}{x^3}$ ← Pure 1 Section 8.3

b $f(x) = 5\ln(x + 2) + 7e^{-x}$ ← Pure 3 Section 6.2

c $f(x) = x^2 \sin x - 4\cos x$ ← Pure 3 Section 6.1

3 Given that $u_{n+1} = u_n + \frac{1}{u_n}$ and that $u_0 = 1$, find the values of u_1, u_2 and u_3 ← Pure 2 Section 5.7

The positions of the Moon, the Earth and the Sun are affected by the gravitational pull of each body. Surprisingly, these positions can't be calculated properly by using ordinary equations. For problems like this we need numerical methods.

8.1 Locating roots

A root of a function is a value of x for which $f(x) = 0$. The graph of $y = f(x)$ will cross the x-axis at points corresponding to the roots of the function.

Notation The following two things are identical:
- the roots of the function $f(x)$
- the roots of the equation $f(x) = 0$

You can sometimes show that a root exists within a given interval by showing that the function changes sign (from positive to negative, or vice versa) within the interval.

- **If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$**

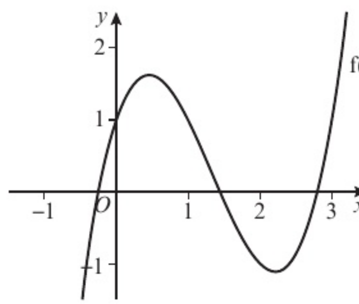
Notation **Continuous** means that the function does not 'jump' from one value to another. If the graph of a function, such as $\tan(x)$, has a vertical asymptote between a and b then the function is not continuous on $[a, b]$.

Example 1 SKILLS REASONING

The diagram shows a sketch of the curve $y = f(x)$, where $f(x) = x^3 - 4x^2 + 3x + 1$

a Explain how the graph shows that $f(x)$ has a root between $x = 2$ and $x = 3$

b Show that $f(x)$ has a root between $x = 1.4$ and $x = 1.5$



a The graph crosses the x-axis between $x = 2$ and $x = 3$. This means that a root of $f(x)$ lies between $x = 2$ and $x = 3$

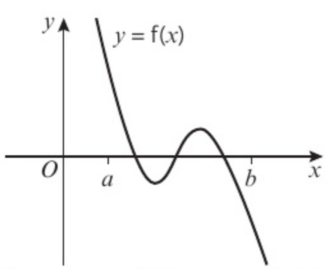
The graph of $y = f(x)$ crosses the x-axis whenever $f(x) = 0$

b $f(1.4) = (1.4)^3 - 4(1.4)^2 + 3(1.4) + 1 = 0.104$
$f(1.5) = (1.5)^3 - 4(1.5)^2 + 3(1.5) + 1 = -0.125$
There is a change of sign between 1.4 and 1.5, so there is at least one root between $x = 1.4$ and $x = 1.5$

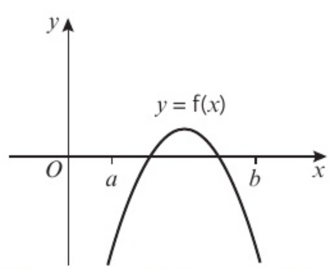
$f(1.4) > 0$ and $f(1.5) < 0$, so there is a change of sign.

$f(x)$ changes sign in the interval $[1.4, 1.5]$, so $f(x)$ must equal zero within this interval.

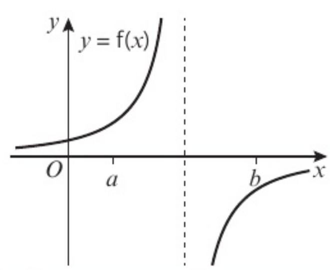
There are three situations you need to watch out for when using the change of sign rule to locate roots. A change of sign does not necessarily mean there is exactly one root. Also, the absence of a sign change does not necessarily mean that a root does not exist in the interval.



There are multiple roots within the interval $[a, b]$. In this case there is an **odd number** of roots.



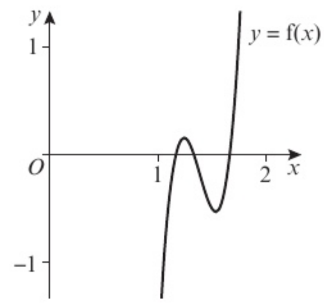
There are multiple roots within the interval $[a, b]$, but a sign change does not occur. In this case there is an **even number** of roots.



There is a vertical asymptote within interval $[a, b]$. A sign change does occur, but there is no root.

Example 2

The graph of the function $f(x) = 54x^3 - 225x^2 + 309x - 140$ is shown in the diagram.



A student observes that f(1.1) and f(1.6) are both negative and states that $f(x)$ has no roots in the interval [1.1, 1.6]

a Explain by reference to the diagram why the student is incorrect.

b Calculate f(1.3) and f(1.5) and use your answer to explain why there are at least 3 roots in the interval $1.1 < x < 1.7$

a The diagram shows that there could be two roots in the interval [1.1, 1.6].

b $f(1.1) = -0.476 < 0$
$f(1.3) = 0.088 > 0$
$f(1.5) = -0.5 < 0$
$f(1.7) = 0.352 > 0$
There is a change of sign between 1.1 and 1.3, between 1.3 and 1.5 and between 1.5 and 1.7, so there are at least three roots in the interval $1.1 < x < 1.7$

Notation The interval [1.1, 1.6] is the set of all real numbers, x, that satisfy $1.1 \leq x \leq 1.6$

Calculate the values of f(1.1), f(1.3), f(1.5) and f(1.7). Comment on the sign of each answer.

$f(x)$ changes sign at least three times in the interval $1.1 < x < 1.7$ so $f(x)$ must equal zero at least three times within this interval.

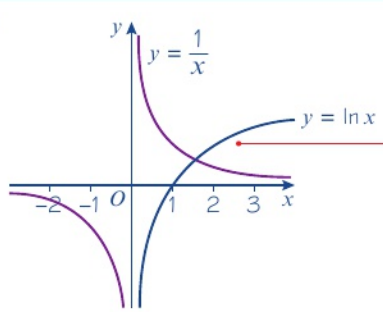
Example 3

a Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagram shows that the function $f(x) = \ln x - \frac{1}{x}$ has only one root.

b Show that this root lies in the interval $1.7 < x < 1.8$

c Given that the root of $f(x)$ is α, show that $\alpha = 1.763$ correct to 3 decimal places.

a



$\ln x - \frac{1}{x} = 0 \Rightarrow \ln x = \frac{1}{x}$

The equation $\ln x = \frac{1}{x}$ has only one solution, so $f(x)$ has only one root.

Sketch $y = \ln x$ and $y = \frac{1}{x}$ on the same axes.

Notice that the curves do intersect.

$f(x)$ has a root where $f(x) = 0$

The curves meet at only one point, so there is only one value of x that satisfies the equation $\ln x = \frac{1}{x}$

b $f(x) = \ln x - \frac{1}{x}$

$f(1.7) = \ln 1.7 - \frac{1}{1.7} = -0.0576...$

$f(1.8) = \ln 1.8 - \frac{1}{1.8} = 0.0322...$

f(1.7) < 0 and f(1.8) > 0, so there is a change of sign.

There is a change of sign between 1.7 and 1.8, so there is at least one root in the interval $1.7 < x < 1.8$

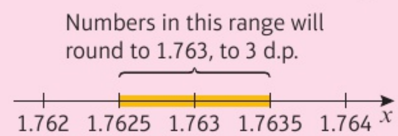
You need to state that there is a change of sign in your conclusion.

c $f(1.7625) = -0.00064... < 0$
$f(1.7635) = 0.00024... > 0$
There is a change of sign in the interval [1.7625, 1.7635] so $1.7625 \leq \alpha \leq 1.7635$, so $\alpha = 1.763$ correct to 3 d.p.

Online Locate the root of $f(x) = \ln x - \frac{1}{x}$ using technology.

Problem-solving

To determine a root to a given degree of accuracy you need to show that it lies within a range of values that will all round to the given value.



Exercise 8A

SKILLS REASONING

1 Show that each of these functions has at least one root in the given interval.

a $f(x) = x^3 - x + 5, -2 < x < -1$

b $f(x) = x^2 - \sqrt{x} - 10, 3 < x < 4$

c $f(x) = x^3 - \frac{1}{x} - 2, -0.5 < x < -0.2$

d $f(x) = e^x - \ln x - 5, 1.65 < x < 1.75$

(E) 2 $f(x) = 3 + x^2 - x^3$

a Show that the equation $f(x) = 0$ has a root, α, in the interval [1.8, 1.9]. **(2 marks)**

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.864$, correct to 3 decimal places. **(3 marks)**

(E) 3 $h(x) = \sqrt[3]{x} - \cos x - 1$, where x is in radians.

a Show that the equation $h(x) = 0$ has a root, α, between $x = 1.4$ and $x = 1.5$ **(2 marks)**

b By choosing a suitable interval, show that $\alpha = 1.441$ is correct to 3 decimal places. **(3 marks)**

(E) 4 $f(x) = \sin x - \ln x, x > 0$, where x is in radians.

a Show that $f(x) = 0$ has a root, α, in the interval [2.2, 2.3]. **(2 marks)**

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 2.219$, correct to 3 decimal places. **(3 marks)**

(P) 5 $f(x) = 2 + \tan x, 0 < x < \pi$, where x is in radians.

a Show that $f(x)$ changes sign in the interval [1.5, 1.6].

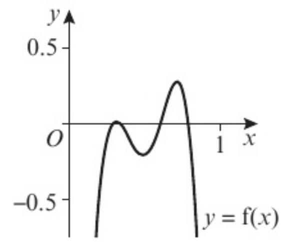
b State with a reason whether or not $f(x)$ has a root in the interval [1.5, 1.6].

P **6** A student observes that the function $f(x) = \frac{1}{x} + 2, x \neq 0$, has a change of sign in the interval $[-1, 1]$. The student writes:

> $y = f(x)$ has a vertical asymptote within this interval so even though there is a change of sign, $f(x)$ has no roots in this interval.

By means of a sketch, or otherwise, explain why the student is incorrect.

7 $f(x) = (105x^3 - 128x^2 + 49x - 6)\cos 2x$, where x is in radians.
The diagram shows a sketch of $y = f(x)$



a Calculate f(0.2) and f(0.8).

b Use your answer to part **a** to make a conclusion about the number of roots of f(x) in the interval $0.2 < x < 0.8$

c Further calculate f(0.3), f(0.4), f(0.5), f(0.6) and f(0.7).

d Use your answers to parts **a** and **c** to make an improved conclusion about the number of roots of f(x) in the interval $0.2 < x < 0.8$

P **8** **a** Using the same axes, sketch the graphs of $y = e^{-x}$ and $y = x^2$

b Explain why the function $f(x) = e^{-x} - x^2$ has only one root.

c Show that the function $f(x) = e^{-x} - x^2$ has a root between $x = 0.70$ and $x = 0.71$

P **9** **a** On the same axes, sketch the graphs of $y = \ln x$ and $y = e^x - 4$

b Write down the number of roots of the equation $\ln x = e^x - 4$

c Show that the equation $\ln x = e^x - 4$ has a root in the interval $[1.4, 1.5]$.

E/P **10** $h(x) = \sin 2x + e^{4x}$

a Show that there is a stationary point, α, of $y = h(x)$ in the interval $-0.9 < x < -0.8$ **(4 marks)**

b By considering the change of sign of $h'(x)$ in a suitable interval, verify that $\alpha = -0.823$ correct to 3 decimal places. **(2 marks)**

E/P **11** **a** On the same axes, sketch the graphs of $y = \sqrt{x}$ and $y = \frac{2}{x}$ **(2 marks)**

b With reference to your sketch, explain why the equation $\sqrt{x} = \frac{2}{x}$ has exactly one real root. **(1 mark)**

c Given that $f(x) = \sqrt{x} - \frac{2}{x}$, show that the equation $f(x) = 0$ has a root r, where $1 < r < 2$ **(2 marks)**

d Show that the equation $\sqrt{x} = \frac{2}{x}$ may be written in the form $x^p = q$, where p and q are integers to be found. **(2 marks)**

e Hence write down the exact value of the root of the equation $\sqrt{x} - \frac{2}{x} = 0$ **(1 mark)**

E/P **12** $f(x) = x^4 - 21x - 18$

a Show that there is a root of the equation $f(x) = 0$ in the interval $[-0.9, -0.8]$. **(3 marks)**

b Find the coordinates of any stationary points on the graph $y = f(x)$ **(3 marks)**

c Given that $f(x) = (x - 3)(x^3 + ax^2 + bx + c)$, find the values of the constants a, b and c. **(3 marks)**

d Sketch the graph of $y = f(x)$ **(3 marks)**

8.2 Fixed point iteration

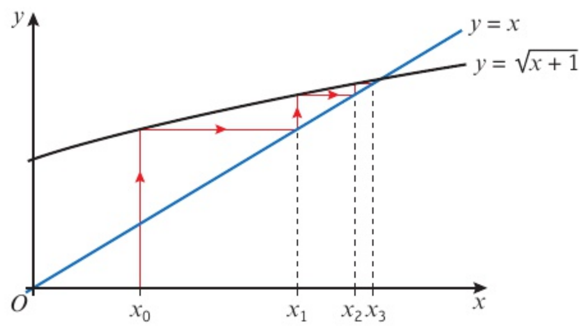
An iterative method can be used to find a value of x for which $f(x) = 0$. To perform an iterative procedure, it is usually necessary to manipulate the algebraic function first.

- **To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$**

Some **iterations** will **converge** to a root. This can happen in two ways. One way is that **successive** iterations get closer and closer to the root from the same direction. Graphically these iterations create a series of steps. The resulting diagram is sometimes referred to as a **staircase diagram**.

$f(x) = x^2 - x - 1$ can produce the iterative formula $x_{n+1} = \sqrt{x_n + 1}$ when $f(x) = 0$. Let $x_0 = 0.5$

Successive iterations produce the following staircase diagram.



Read up from x_0 on the vertical axis to the curve $y = \sqrt{x + 1}$ to find x_1. You can read across to the line $y = x$ to 'map' this value back onto the x-axis. Repeating the process shows the values of x_n converging to the root of the equation $y = \sqrt{x + 1}$, which is also the root of $f(x)$.

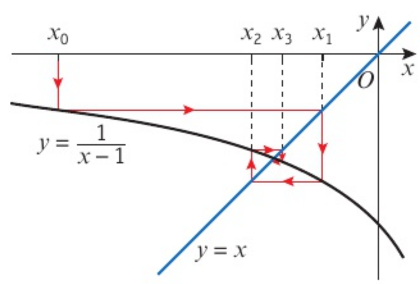
The other way that an iteration converges is that successive iterations alternate being below the root and above the root. These iterations can still converge to the root and the resulting graph is sometimes called a **cobweb diagram**.

Watch out By rearranging the same function in different ways you can find different iterative formulae, which may converge differently.

$f(x) = x^2 - x - 1$ can produce the iterative formula

$x_{n+1} = \dfrac{1}{x_n - 1}$ when $f(x) = 0$. Let $x_0 = -2$

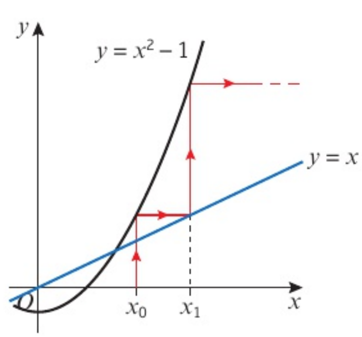
Successive iterations produce the cobweb diagram shown on the right.



Not all iterations or starting values converge to a root. When an iteration moves away from a root, often increasingly quickly, you say that it **diverges**.

$f(x) = x^2 - x - 1$ can produce the iterative formula $x_{n+1} = x_n^2 - 1$ when $f(x) = 0$. Let $x_0 = 2$

Successive iterations diverge from the root, as shown in the diagram below.



Example 4

$f(x) = x^2 - 4x + 1$

a Show that the equation $f(x) = 0$ can be written as $x = 4 - \frac{1}{x}$, $x \neq 0$

$f(x)$ has a root, α, in the interval $3 < x < 4$

b Use the iterative formula $x_{n+1} = 4 - \frac{1}{x_n}$ with $x_0 = 3$ to find the value of x_1, x_2 and x_3

a $f(x) = 0$

$x^2 - 4x + 1 = 0$

$x^2 = 4x - 1$

Add $4x$ to each side and subtract 1 from each side.

$x = 4 - \frac{1}{x}$, $x \neq 0$

Divide each term by x. This step is only valid if $x \neq 0$

b $x_1 = 4 - \frac{1}{x_0} = 3.666666\ldots$

$x_2 = 4 - \frac{1}{x_1} = 3.72727\ldots$

$x_3 = 4 - \frac{1}{x_2} = 3.73170\ldots$

Online Use the iterative formula to work out x_1, x_2 and x_3. You can use your calculator to find each value quickly.

Example 5

$f(x) = x^3 - 3x^2 - 2x + 5$

a Show that the equation $f(x) = 0$ has a root in the interval $3 < x < 4$

b Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$ to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places and taking:

i $x_0 = 1.5$ **ii** $x_0 = 4$

a $f(3) = (3)^3 - 3(3)^2 - 2(3) + 5 = -1$

$f(4) = (4)^3 - 3(4)^2 - 2(4) + 5 = 13$

There is a change of sign in the interval $3 < x < 4$, and f is continuous, so there is a root of $f(x)$ in this interval.

The graph crosses the x-axis between $x = 3$ and $x = 4$

b i $x_1 = \sqrt{\frac{x_0^3 - 2x_0 + 5}{3}} = 1.3385\ldots$

$x_2 = \sqrt{\frac{x_1^3 - 2x_1 + 5}{3}} = 1.2544\ldots$

$x_3 = \sqrt{\frac{x_2^3 - 2x_2 + 5}{3}} = 1.2200\ldots$

Each iteration gets closer to a root, so the sequence $x_0, x_1, x_2, x_3, \ldots$ is **convergent**.

ii $x_1 = \sqrt{\dfrac{x_0^3 - 2x_0 + 5}{3}} = 4.5092...$

$x_2 = \sqrt{\dfrac{x_1^3 - 2x_1 + 5}{3}} = 5.4058...$

$x_3 = \sqrt{\dfrac{x_2^3 - 2x_2 + 5}{3}} = 7.1219...$

Online Explore the iterations graphically using technology.

Each iteration gets further from a root, so the sequence $x_0, x_1, x_2, x_3, \ldots$ is **divergent**.

Exercise 8B

SKILLS REASONING

(P) **1** $f(x) = x^2 - 6x + 2$

a Show that $f(x) = 0$ can be written as:

i $x = \dfrac{x^2 + 2}{6}$ **ii** $x = \sqrt{6x - 2}$ **iii** $x = 6 - \dfrac{2}{x}$

b Starting with $x_0 = 4$, use each iterative formula to find a root of the equation $f(x) = 0$. Round your answers to 3 decimal places.

c Use the quadratic formula to find the roots to the equation $f(x) = 0$, leaving your answer in the form $a \pm \sqrt{b}$, where a and b are constants to be found.

(P) **2** $f(x) = x^2 - 5x - 3$

a Show that $f(x) = 0$ can be written as:

i $x = \sqrt{5x + 3}$ **ii** $x = \dfrac{x^2 - 3}{5}$

b Let $x_0 = 5$. Show that each of the following iterative formulae gives different roots of $f(x) = 0$

i $x_{n+1} = \sqrt{5x_n + 3}$ **ii** $x_{n+1} = \dfrac{x_n^2 - 3}{5}$

(E/P) **3** $f(x) = x^2 - 6x + 1$

a Show that the equation $f(x) = 0$ can be written as $x = \sqrt{6x - 1}$ **(1 mark)**

b Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{6x - 1}$ **(2 marks)**

c Write down the number of roots of $f(x)$. **(1 mark)**

d Use your sketch to explain why the iterative formula $x_{n+1} = \sqrt{6x_n - 1}$ converges to a root of $f(x)$ when $x_0 = 2$ **(1 mark)**

$f(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \dfrac{x_n^2 + 1}{6}$

e By sketching a diagram, explain why the iteration diverges when $x_0 = 10$ **(2 marks)**

(P) **4** $f(x) = xe^{-x} - x + 2$

a Show that the equation $f(x) = 0$ can be written as $x = \ln\left|\dfrac{x}{x-2}\right|, x \neq 2$

$f(x)$ has a root, α, in the interval $-2 < x < -1$

b Use the iterative formula $x_{n+1} = \ln\left|\dfrac{x_n}{x_n - 2}\right|, x \neq 2$, with $x_0 = -1$, to find, to 2 decimal places, the values of x_1, x_2 and x_3

(P) **5** $f(x) = x^3 + 5x^2 - 2$

a Show that $f(x) = 0$ can be written as:

i $x = \sqrt[3]{2 - 5x^2}$ **ii** $x = \frac{2}{x^2} - 5$ **iii** $x = \sqrt{\frac{2 - x^3}{5}}$

b Starting with $x_0 = 10$, use the iterative formula in part **a (ii)** to find a root of the equation $f(x) = 0$. Round your answer to 3 decimal places.

c Starting with $x_0 = 1$, use the iterative formula in part **a (iii)** to find a different root of the equation $f(x) = 0$. Round your answer to 3 decimal places.

d Explain why the iterative formulae in part **a (iii)** cannot be used when $x_0 = 2$

(E/P) **6** $f(x) = x^4 - 3x^3 - 6$

a Show that the equation $f(x) = 0$ can be written as $x = \sqrt[3]{px^4 + q}$, where p and q are constants to be found. **(2 marks)**

b Let $x_0 = 0$. Use the iterative formula $x_{n+1} = \sqrt[3]{px_n^4 + q}$, together with your values of p and q from part **a**, to find, to 3 decimal places, the values of x_1, x_2 and x_3 **(3 marks)**

The root of $f(x) = 0$ is α.

c By choosing a suitable interval, show that $\alpha = -1.132$ to 3 decimal places. **(3 marks)**

(E/P) **7** $f(x) = 3\cos(x^2) + x - 2$

a Show that the equation $f(x) = 0$ can be written as $x = \left(\arccos\left(\frac{2 - x}{3}\right)\right)^{\frac{1}{2}}$ **(2 marks)**

b Use the iterative formula $x_{n+1} = \left(\arccos\left(\frac{2 - x_n}{3}\right)\right)^{\frac{1}{2}}$, $x_0 = 1$, to find, to 3 decimal places, the values of x_1, x_2 and x_3 **(3 marks)**

c Given that $f(x) = 0$ has only one root, α, show that $\alpha = 1.1298$ correct to 4 decimal places. **(3 marks)**

(E/P) **8** $f(x) = 4\cot x - 8x + 3$, $0 < x < \pi$, where x is in radians.

a Show that there is a root α of $f(x) = 0$ in the interval [0.8, 0.9]. **(2 marks)**

b Show that the equation $f(x) = 0$ can be written in the form $x = \frac{\cos x}{2\sin x} + \frac{3}{8}$ **(3 marks)**

c Use the iterative formula $x_{n+1} = \frac{\cos x_n}{2\sin x_n} + \frac{3}{8}$, $x_0 = 0.85$, to calculate the values of x_1, x_2 and x_3 giving your answers to 4 decimal places. **(3 marks)**

d By considering the change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 0.831$ correct to 3 decimal places. **(2 marks)**

(E/P) **9** $g(x) = e^{x-1} + 2x - 15$

a Show that the equation $g(x) = 0$ can be written as $x = \ln(15 - 2x) + 1$, $x < \frac{15}{2}$ **(2 marks)**

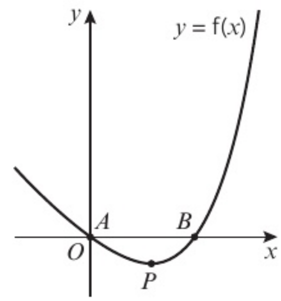
The root of $g(x) = 0$ is α.

The iterative formula $x_{n+1} = \ln(15 - 2x_n) + 1$, $x_0 = 3$, is used to find a value for α.

b Calculate the values of x_1, x_2 and x_3 to 4 decimal places. **(3 marks)**

c By choosing a suitable interval, show that $\alpha = 3.16$ correct to 2 decimal places. **(3 marks)**

E/P **10** The diagram shows a sketch of part of the curve with equation $y = f(x)$, where $f(x) = xe^x - 4x$. The curve cuts the x-axis at the points A and B and has a minimum turning point at P, as shown in the diagram.



a Work out the coordinates of A and the coordinates of B. **(3 marks)**

b Find $f'(x)$. **(3 marks)**

c Show that the x-coordinate of P lies between 0.7 and 0.8. **(2 marks)**

d Show that the x-coordinate of P is the solution to the equation $x = \ln\left(\frac{4}{x+1}\right)$ **(3 marks)**

To find an approximation for the x-coordinate of P, the iterative formula $x_{n+1} = \ln\left(\frac{4}{x_n + 1}\right)$ is used.

e Let $x_0 = 0$. Find the values of x_1, x_2, x_3 and x_4. Give your answers to 3 decimal places. **(3 marks)**

Chapter review 8

E/P **1** $f(x) = x^3 - 6x - 2$

a Show that the equation $f(x) = 0$ can be written in the form $x = \pm\sqrt{a + \frac{b}{x}}$ and state the values of the integers a and b. **(2 marks)**

$f(x) = 0$ has one positive root, α.

The iterative formula $x_{n+1} = \sqrt{a + \frac{b}{x_n}}$, $x_0 = 2$, is used to find an approximate value for α.

b Calculate the values of x_1, x_2, x_3 and x_4 to 4 decimal places. **(3 marks)**

c By choosing a suitable interval, show that $\alpha = 2.602$ is correct to 3 decimal places. **(3 marks)**

E/P **2** $p(x) = 4 - x^2$ and $q(x) = e^x$

a On the same axes, sketch the curves of $y = p(x)$ and $y = q(x)$ **(2 marks)**

b State the number of positive roots and the number of negative roots of the equation $x^2 + e^x - 4 = 0$ **(1 mark)**

c Show that the equation $x^2 + e^x - 4 = 0$ can be written in the form $x = \pm(4 - e^x)^{\frac{1}{2}}$ **(2 marks)**

The iterative formula $x_{n+1} = -(4 - e^{x_n})^{\frac{1}{2}}$, $x_0 = -2$, is used to find an approximate value for the negative root.

d Calculate the values of x_1, x_2, x_3 and x_4 to 4 decimal places. **(3 marks)**

e Explain why the starting value $x_0 = 1.4$ will not produce a valid result with this formula. **(2 marks)**

E/P **3** $g(x) = x^5 - 5x - 6$

a Show that $g(x) = 0$ has a root, α, between $x = 1$ and $x = 2$ **(2 marks)**

b Show that the equation $g(x) = 0$ can be written as $x = (px + q)^{\frac{1}{r}}$, where p, q and r are integers to be found. **(2 marks)**

The iterative formula $x_{n+1} = (px_n + q)^{\frac{1}{r}}$, $x_0 = 1$ is used to find an approximate value for α.

c Calculate the values of x_1, x_2 and x_3 to 4 decimal places. **(3 marks)**

d By choosing a suitable interval, show that $\alpha = 1.708$ is correct to 3 decimal places. **(3 marks)**

E/P **4** $g(x) = x^2 - 3x - 5$

a Show that the equation $g(x) = 0$ can be written as $x = \sqrt{3x + 5}$ **(1 mark)**

b Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{3x + 5}$ **(2 marks)**

c Use your diagram to explain why the iterative formula $x_{n+1} = \sqrt{3x_n + 5}$ converges to a root of $g(x)$ when $x_0 = 1$ **(1 mark)**

$g(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \dfrac{x_n^2 - 5}{3}$

d With reference to a diagram, explain why this iterative formula diverges when $x_0 = 7$ **(3 marks)**

E/P **5** $f(x) = 5x - 4\sin x - 2$, where x is in radians.

a Show that $f(x) = 0$ has a root, α, between $x = 1.1$ and $x = 1.15$ **(2 marks)**

b Show that $f(x) = 0$ can be written as $x = p\sin x + q$, where p and q are rational numbers to be found. **(2 marks)**

c Starting with $x_0 = 1.1$, use the iterative formula $x_{n+1} = p\sin x_n + q$ with your values of p and q to calculate the values of x_1, x_2, x_3 and x_4 to 3 decimal places. **(3 marks)**

E/P **6** **a** On the same axes, sketch the graphs of $y = \dfrac{1}{x}$ and $y = x + 3$ **(2 marks)**

b Write down the number of roots of the equation $\dfrac{1}{x} = x + 3$ **(1 mark)**

c Show that the positive root of the equation $\dfrac{1}{x} = x + 3$ lies in the interval (0.30, 0.31). **(2 marks)**

d Show that the equation $\dfrac{1}{x} = x + 3$ may be written in the form $x^2 + 3x - 1 = 0$ **(2 marks)**

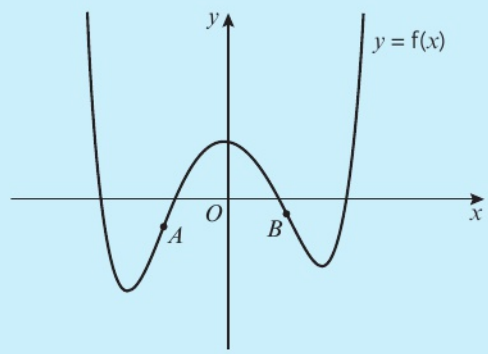
e Use the quadratic formula to find the positive root of the equation $x^2 + 3x - 1 = 0$ to 3 decimal places. **(2 marks)**

Challenge

SKILLS INNOVATION

$f(x) = x^6 + x^3 - 7x^2 - x + 3$

The diagram shows a sketch of $y = f(x)$. Points A and B are the points of inflection on the curve.



a Show that equation $f''(x) = 0$ can be written as:

i $x = \dfrac{7 - 15x^4}{3}$ **ii** $x = \dfrac{7}{15x^3 + 3}$ **iii** $x = \sqrt[4]{\dfrac{7 - 3x}{15}}$

b By choosing a suitable iterative formula and starting value, find an approximation for the x-coordinate of B, correct to 3 decimal places.

c Explain why you cannot use the same iterative formula to find an approximation for the x-coordinate of A.

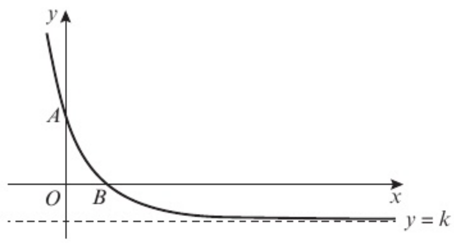
Summary of key points

1 If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$

2 To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$

Review exercise 2

E **1** The graph of the function $f(x) = 3e^{-x} - 1$, $x \in \mathbb{R}$, has an asymptote $y = k$, and crosses the x and y axes at A and B respectively, as shown in the diagram.



a Write down the value of k and the y-coordinate of A. **(2)**

b Find the exact value of the x-coordinate of B, giving your answer as simply as possible. **(2)**

← Pure 3 Section 5.2

E/P **2** A heated metal ball S is dropped into a liquid. As S cools, its temperature, T °C, t minutes after it enters the liquid, is given by $T = 400e^{-0.05t} + 25$, $t \geqslant 0$

a Find the temperature of S as it enters the liquid. **(1)**

b Find how long S is in the liquid before its temperature drops to 300 °C. Give your answer to 3 significant figures. **(3)**

c Find the rate, $\frac{dT}{dt}$, in °C per minute to 3 significant figures, at which the temperature of S is decreasing at the instant $t = 50$ **(3)**

d With reference to the equation given above, explain why the temperature of S can never drop to 20 °C. **(2)**

← Pure 3 Sections 5.2, 5.5

E/P **3** Find the exact solutions to the equations:

a $\ln x + \ln 3 = \ln 6$ **(2)**

b $e^x + 3e^{-x} = 4$ **(4)**

← Pure 3 Section 5.3

E **4** The table below shows the population of Angola between 1970 and 2010.

Year	Population, P (millions)
1970	5.93
1980	7.64
1990	10.33
2000	13.92
2010	19.55

This data can be modelled using an exponential function of the form $P = ab^t$, where t is the time in years since 1970 and a and b are constants.

a Copy and complete the table below, giving your answers to 2 decimal places. **(1)**

Time in years since 1970, t	$\log P$
0	0.77
10	
20	
30	
40	

b Plot a graph of $\log P$ against t using the values from your table and draw in a line of best fit. **(2)**

c By rearranging $P = ab^t$, explain how the graph you have just drawn supports the assumed model. **(3)**

d Use your graph to estimate the values of a and b to 2 significant figures. **(4)**

← Pure 3 Sections 5.4, 5.5

E **5** The function f is defined by

$f : x \to \ln(5x - 2), x \in \mathbb{R}, x > \frac{2}{5}$

a Find an expression for $f^{-1}(x)$. **(2)**

b Write down the domain of $f^{-1}(x)$. **(1)**

c Solve, giving your answer to 3 decimal places,

$\ln(5x - 2) = 2$ **(2)**

← Pure 3 Sections 5.1, 5.3

E **6** The function f is defined by $f : x \to e^x + k, x \in \mathbb{R}$ and k is a positive constant.

a State the range of $f(x)$. **(2)**

b Find $f(\ln k)$, simplifying your answer. **(2)**

c Find f^{-1}, the inverse function of f, in the form $f^{-1} : x \to \ldots$, stating its domain. **(2)**

d On the same axes, sketch the curves with equations $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all points where the graphs cut the axes. **(3)**

← Pure 3 Section 5.1

E **7** The function f is given by

$f : x \to \ln(4 - 2x), x \in \mathbb{R}, x < 2$

a Find an expression for $f^{-1}(x)$. **(3)**

b Sketch the curve with equation $y = f^{-1}(x)$, showing the coordinates of the points where the curve meets the axes. **(3)**

c State the range of $f^{-1}(x)$. **(2)**

The function g is given by

$g : x \to e^x, x \in \mathbb{R}$

d Find the value of gf(0.5) **(2)**

← Pure 3 Sections 5.1, 5.2

E **8** The functions f and g are defined by

$f : x \to 2x + \ln 2, x \in \mathbb{R}$

$g : x \to e^{2x}, x \in \mathbb{R}$

a Prove that the composite function gf is

$gf : x \to 4e^{4x}, x \in \mathbb{R}$ **(4)**

b Sketch the curve with equation $y = gf(x)$, and show the coordinates of the point where the curve crosses the y-axis. **(3)**

c Write down the range of gf. **(2)**

d Find the value of x for which $\frac{d}{dx}[gf(x)] = 3$, giving your answer to 3 significant figures. **(4)**

← Pure 3 Sections 2.3, 5.1, 6.2

E/P **9** **a** By sketching the graphs of $y = -x$ and $y = \ln x, x > 0$, on the same axes, show that the solution to the equation $x + \ln x = 0$ lies between 0 and 1. **(3)**

b Show that $x + \ln x = 0$ may be written in the form

$x = \frac{(2x - \ln x)}{3}$ **(2)**

c Use the iterative formula

$x_{n+1} = \frac{(2x_n - \ln x_n)}{3}, x_0 = 1,$

to find the solution of $x + \ln x = 0$ correct to 5 decimal places. **(3)**

← Pure 3 Sections 5.3, 8.2

E/P **10** A curve has equation $y = \frac{1}{2}x^2 + 4\cos x$

Show that an equation of the normal to the curve at $x = \frac{\pi}{2}$ is

$8y(8 - \pi) - 16x + \pi(\pi^2 - 8\pi + 8) = 0$ **(7)**

← Pure 3 Section 6.1

E/P **11** A curve has equation $y = e^{3x} - \ln(x^2)$.

Show that an equation of the tangent at $x = 2$ is $y - (3e^6 - 1)x - 2 + \ln 4 + 5e^6 = 0$ **(6)**

← Pure 3 Section 6.2

E **12** A curve C has equation $y = (2x - 3)^2 e^{2x}$

a Use the product rule to find $\frac{dy}{dx}$ **(3)**

b Hence find the coordinates of the stationary points of C. **(3)**

← Pure 3 Sections 6.2, 6.4

E **13** The curve C has equation $y = \frac{(x-1)^2}{\sin x}$

a Use the quotient rule to find $\frac{dy}{dx}$ **(3)**

b Show that the equation of the tangent to the curve at $x = \frac{\pi}{2}$ is

$y = (\pi - 2)x + \left(1 - \frac{\pi^2}{4}\right)$ **(4)**

← Pure 3 Sections 6.1, 6.5

E/P **14** **a** Show that if $y = \operatorname{cosec} x$ then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$ **(4)**

b Given $x = \operatorname{cosec} 6y$, find $\frac{dy}{dx}$ in terms of x. **(6)**

← Pure 3 Section 6.6

E/P **15** Assuming standard results for $\sin x$ and $\cos x$, prove that the derivative of $\arcsin x$ is $\frac{1}{\sqrt{1 - x^2}}$ **(5)**

← Pure 3 Section 6.6

E/P **16** Given $\int_a^3 (12 - 3x)^2\, dx = 78$, find the value of a. **(4)**

← Pure 3 Section 7.2

E/P **17** **a** By expanding $\cos(5x + 2x)$ and $\cos(5x - 2x)$ using the double-angle formulae, or otherwise, show that $\cos 7x + \cos 3x \equiv 2 \cos 5x \cos 2x$. **(4)**

b Hence find $\int 6 \cos 5x \cos 2x\, dx$ **(3)**

← Pure 3 Sections 4.3, 7.3

E/P **18** Given that $\int_0^m mx^3 e^{x^4} dx = \frac{3}{4}(e^{81} - 1)$ find the value of m. **(3)**

← Pure 3 Section 7.4

E **19** $f(x) = \frac{5x^2 - 8x + 1}{2x(x-1)^2}$

a Given that $f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ find the values of the constants A, B and C. **(4)**

b Hence find $\int f(x)\, dx$ **(4)**

c Hence show that

$\int_4^9 f(x)\, dx = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$ **(4)**

← Pure 3 Sections 1.2, 6.4, 7.4

E/P **20** **a** Express $\frac{5x + 3}{(2x - 3)(x + 2)}$ in partial fractions. **(3)**

b Hence find the exact value of $\int_2^6 \frac{5x + 3}{(2x - 3)(x + 2)}\, dx$, giving your answer as a single logarithm. **(4)**

← Pure 3 Sections 1.2, 6.4, 7.4

E/P **21** $f(x) = (x^2 + 1) \ln x$

Find the exact value of $\int_1^e f(x)\, dx$ **(7)**

← Pure 3 Section 7.4

E **22** $g(x) = x^3 - x^2 - 1$

a Show that there is a root α of $g(x) = 0$ in the interval [1.4, 1.5]. **(2)**

b By considering a change of sign of $g(x)$ in a suitable interval, verify that $\alpha = 1.466$ correct to 3 decimal places. **(3)**

← Pure 3 Section 8.1

E **23** $p(x) = \cos x + e^{-x}$

a Show that there is a root α of $p(x) = 0$ in the interval [1.7, 1.8]. **(2)**

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.746$ correct to 3 decimal places. **(3)**

← Pure 3 Section 8.1

E **24** $f(x) = e^{x-2} - 3x + 5$

a Show that the equation $f(x) = 0$ can be written as

$$x = \ln(3x - 5) + 2,\ x > \tfrac{5}{3}$$ **(2)**

The root of $f(x) = 0$ is α.

The iterative formula $x_{n+1} = \ln(3x_n - 5) + 2$, $x_0 = 4$ is used to find a value for α.

b Calculate the values of x_1, x_2 and x_3 to 4 decimal places. **(3)**

← Pure 3 Section 8.2

E **25** $f(x) = \dfrac{1}{(x-2)^3} + 4x^2,\ x \neq 2$

a Show that there is a root α of $f(x) = 0$ in the interval [0.2, 0.3]. **(2)**

b Show that the equation $f(x) = 0$ can be written in the form $x = \sqrt[3]{\dfrac{-1}{4x^2}} + 2$ **(3)**

c Use the iterative formula

$x_{n+1} = \sqrt[3]{-1/4x_n^2 + 2}$, $x_0 = 1$ to calculate the

values of x_1, x_2, x_3 and x_4, giving your answers to 4 decimal places. **(3)**

d By considering the change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.524$ correct to 3 decimal places. **(2)**

← Pure 3 Section 8.2

Challenge

1 A curve has equation

$y = -\dfrac{3}{(4-6x)^2},\ x \neq \dfrac{2}{3}$

Find an equation of the normal to the curve at $x = 1$ in the form $ax + by + c = 0$, where a, b and c are integers.

← Pure 3 Section 6.3

2 The functions f and g are defined as $f(x) = x^3 - kx + 1$, where k is a constant, and $g(x) = e^{2x}$, x [R. The graphs of $y = f(x)$ and $y = g(x)$ intersect at the point P, where $x = 0$.

a Confirm that $f(0) = g(0)$ and hence state the coordinates of P.

b Given that the tangents to the graphs at P are perpendicular, find the value of k.

← Pure 3 Section 5.2

3 The volume of a hemisphere V cm³ is related to its radius r cm by the formula $V = \frac{2}{3}\pi r^3$ and the total surface area S cm² is given by the formula $S = \pi r^2 + 2\pi r^2 = 3\pi r^2$. Given that the rate of increase of volume, in cm³ s⁻¹, $\dfrac{dV}{dt} = 6$, find the rate of increase of surface area $\dfrac{dS}{dt}$.

← Pure 3 Section 6.3

Exam practice

Mathematics
International/Advanced Level
Pure Mathematics 3

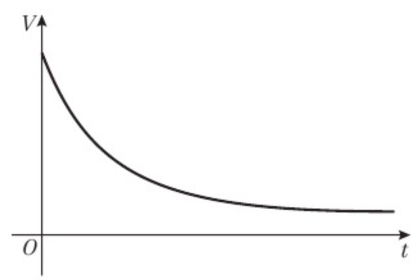
Time: 1 hour 30 minutes
You must have: Mathematical Formulae and Statistical Tables, Calculator
Answer ALL questions

1 Simplify fully $\dfrac{x^2 - 9}{x^2 - 3x} \div \dfrac{2x^2 + 5x - 3}{x^2 + 7x}$ **(4)**

2 Maria wants to predict the value V euros of her new saxophone after t years. She uses the formula

$V = 800e^{-0.2t} + 1000e^{-0.1t} + 200, t \geqslant 0$

The diagram shows a sketch of V against t.



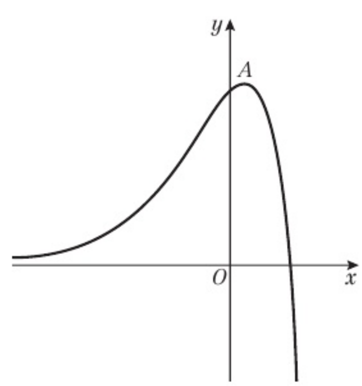
a State the range of V. **(1)**

b Calculate the rate at which the value of Maria's saxophone is decreasing when $t = 15$

Give your answer in euros per year and to the nearest integer. **(3)**

c Calculate the exact value of t when $V = 1400$ **(4)**

3 The diagram shows a sketch of the curve $f(x) = (4 - 3x)e^x, x \in \mathbb{R}$.

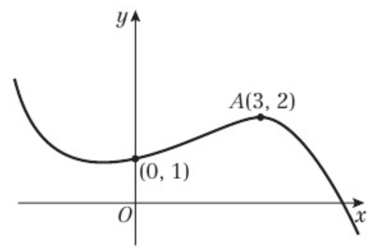


a Using calculus, find the exact coordinates of the turning point at A. **(5)**

b State the range of $f(x)$. **(2)**

c Sketch the curve of $y = |f(x)|$. Show the coordinates where the curve crosses or meets the axes. **(4)**

4 The diagram shows a sketch of the curve $y = f(x)$. The curve passes through the point (0, 1). The point $A(3, 2)$ is a maximum.



On separate axes, sketch the graphs of:

a $y = f(-x) + 1$ **(3)** **b** $y = f(x + 3) + 2$ **(3)**

c $y = 2f(3x)$ **(3)**

On each sketch, show the coordinates where your graph intersects the y-axis and the coordinates of the point to which A is transformed.

5 $f(x) = 3\sin^2 x + 2\cos^2 x$

a Show that $f(x) = \dfrac{5 - \cos 2x}{2}$ **(4)**

b Hence find the exact value of $\int_0^{\frac{\pi}{4}} f(x)\,dx$ **(4)**

6 $y = x^2 + \sin\left(\dfrac{\pi}{2}x\right)$

a Find $\dfrac{dy}{dx}$ **(4)**

b Hence find the equation of the normal to the curve at $x = -1$ **(4)**

7 Given that $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}} (1 - \pi \sin kx)\,dx = \pi(7 - 6\sqrt{2})$, find the exact value of k. **(8)**

8 $f(x) = \dfrac{3x^3 - 10x^2 + 8x + 1}{x^2 - 4x + 4}$

Write $f(x)$ in the form $Ax + B + \dfrac{C}{x - 2} + \dfrac{D}{(x - 2)^2}$ **(7)**

9 $f(x) = \dfrac{1}{4 - x} + 3$

a Calculate f(3.9) and f(4.1). **(2)**

b Explain why the equation $f(x) = 0$ does not have a root in the interval $3.9 < x < 4.1$ **(1)**

The equation $f(x) = 0$ has a single root, α.

c Use algebra to find the exact value of α. **(2)**

10 Integrate the following expressions with respect to x:

a $e^{4x + 3}$ **(2)**

b $\dfrac{\cos 4x}{e^{\sin 4x}}$ **(5)**

TOTAL FOR PAPER: 75 MARKS

GLOSSARY

acute (**angle**) an angle less than 90°

algebraic fraction a fraction where the numerator and denominator are **polynomials**

algebraic long division the process of dividing the **denominator** into the **numerator** of an **algebraic fraction**

appropriate a mathematical function may act as a **model** for a real-life process. If the model describes the process well under all circumstances, it is highly *appropriate*

argument an input to a function

asymptote when a curve approaches but never quite reaches a line, that line is an *asymptote*

cancel (**out**) remove identical values from both the numerator and denominator in order to simplify an expression. For example $\frac{ab}{ac} = \frac{\cancel{a}b}{\cancel{a}c} = \frac{b}{c}$

chord a line segment joining two points on the circumference of a circle

common factor a quantity that will divide without remainder into two or more other quantities

common multiple a multiple of two or more quantities

constant a term that does not include a variable. In the **expression** $3x^2 + 4x + 5$, the term 5 is a *constant*

converge to approach a limit more and more closely

coordinate axes the two perpendicular lines by which the positions of points are measured on a graph

coordinates a set of values, e.g. (3, 2), that show an exact position. The first value represents a point on the x-axis; the second value represents a point on the y-axis

deduce to reach a logical conclusion. If $x + 2 = 3$, we can *deduce* that $x = 1$

denominator the lower part of a fraction. For example, B is the denominator in the fraction $\frac{A}{B}$

derivative the rate of change of a mathematical function; the result of **differentiation**

differentiation calculating the instantaneous rate of change of a function

displacement change of position

expand to write a mathematical **expression** in an extended form. For example, $(x + y)^3$ can be expanded to $x^3 + 3x^2y + 3xy^2 + y^3$

exponential an *exponential* function has the form $f(x) = a^x$

expression a mathematical *expression* contains numbers and/or variables, e.g. $2x^3 + 4\ln x + \sin x$

factor a quantity that divides into another quantity exactly. $x + 1$ is a factor of $x^2 + 3x + 2$

factorise to rewrite an **expression** using brackets. We *factorise* $x^2 + 3x + 2$ to get $(x + 1)(x + 2)$

from first principles proving something without using other proofs such as Pythagoras' theorem

gradient slope

identity an equality between **expressions** that is true for all values of the variables in those expressions

improper algebraic fraction a fraction whose numerator has a degree (power) equal to or greater than the denominator

integrand an **expression** which is to be integrated

intercept (verb) to cross an axis

intercept (noun) the place where a line or curve crosses an axis

intersection the point at which two or more curves cross (intersect)

interval the **limits** of an **expression**, e.g. $-\pi \leqslant \theta \leqslant \pi$

iteration the repeated application of a mathematical process

LHS left-hand side; opposite of **RHS**

limit a value above or below which an **expression** cannot go. The upper limit of $\sin\theta$ is 1

linear where the variables have the power 1. Hence $y = 2x + 3$ is a linear equation but $y = x^2$ and $y = \frac{1}{x}$ ($y = x^{-1}$) are not. A linear equation can be represented by a straight line

logarithm the power to which the base number must be raised in order to get a particular number. For example, $\log_2 32 = 5 \Rightarrow 2^5 = 32$

long term after a long time

midpoint (of a line segment) a **point** on a line segment that divides it into two equal parts

model a mathematical method of describing a real-life process

modulus The positive value of an expression. The *modulus* of –2 is +2. The *modulus* of +2 is also +2

normal a line intersecting a curve at right angles to the tangent at that point

numerator the upper part of a fraction. For example, A is the numerator in the fraction $\frac{A}{B}$

obtuse (**angle**) an angle greater than 90° but less than 180°

origin the point where the y-axis and x-axis intersect

outlier a value that lies well outside the other values in a data set

parallel two lines side by side, the same distance apart at every point

partial fractions when an **algebraic fraction** is converted into a number of simpler fractions, these are called *partial fractions*. For example $\frac{3x^2 - 3x - 2}{(x-1)(x-2)} \equiv 3 + \frac{2}{x-1} + \frac{4}{x-2}$

point marks a location but has no size itself

point of inflection a point where the derivative changes sign

polynomial an expression involving integer powers of a variable, e.g. $x^2 + 5x + 2$

quotient a result obtained by dividing one quantity by another

real a number that can be represented by a (possibly infinite) decimal expansion. Examples include 3, −3, $\sqrt{3}$, $\frac{1}{3}$, log 3, sin 3, π and e

rearrange to put terms in a different order: $3x + x^2 + 2 \rightarrow x^2 + 3x + 2$

reciprocal the reciprocal of a number x is $\frac{1}{x}$. Every number has a reciprocal apart from 0, as $\frac{1}{0}$ is not defined

reflection when an object is mirrored on a line of **symmetry**

RHS right-hand side; opposite of **LHS**

roots (of an equation) the set of all possible solutions

simplify to replace an **expression** with a simpler, usually shorter, one

sketch (noun or verb) a drawing that explains something without necessarily being accurate

stationary point the point on a function where the gradient is zero

stretch to make something longer or (in mathematics only) shorter

substitute to replace something (e.g. a variable) with something else (e.g. a value). If $y = x^3 + 1$ and we substitute $x = 2$, we find that $y = 2^3 + 1 = 9$

successive following one after the other

symmetrical, symmetry two shapes are *symmetrical* if one can be transformed into the other by reflecting, rotating or stretching

translate move (a shape)

translation moves a shape

trend the general direction in which a group of points seems to be going

turning point a point at which $\frac{dy}{dx}$ changes sign It is also known as a maximum, a minimum or a stationary point. However, not all stationary points are turning points. For example, a point of inflection is a stationary point but not a turning point

undefined not having a meaning or a value; for example, the result of division by zero

vertex (plural vertices) where two lines meet at an angle, especially in a shape such as a triangle

ANSWERS

CHAPTER 1

Prior knowledge check

1 a $15x^7$ b $\frac{x}{3y}$

2 a $(x-1)(x-5)$ b $(x+4)(x-4)$ c $(3x-5)(3x+5)$

3 a $\frac{x-3}{x+6}$ b $\frac{x+4}{3x+1}$ c $-\frac{x+5}{x+3}$

Exercise 1A

1 All factors cancel exactly except $\frac{x-8}{8-x} = \frac{x-8}{-(x-8)} = -1$

2 $a = 5, b = 12$

3 $\frac{x-4}{2x+10}$

4 a $\frac{2x^2-3x-2}{6x-8} \div \frac{x-2}{3x^2+14x-24}$
$= \frac{2x^2-3x-2}{6x-8} \times \frac{3x^2+14x-24}{x-2}$
$= \frac{(2x+1)(x-2)}{2(3x-4)} \times \frac{(3x-4)(x+6)}{x-2}$
$= \frac{(2x+1)(x+6)}{2} = \frac{2x^2+13x+6}{2}$

b $f'(x) = 2x + \frac{13}{2}$; $f'(4) = \frac{29}{2}$

Exercise 1B

1 a $\frac{7}{12}$ b $\frac{7}{20}$ c $\frac{p+q}{pq}$ d $\frac{7}{8x}$ e $\frac{3-x}{x^2}$ f $\frac{2a-15}{10b}$

2 a $\frac{x+3}{x(x+1)}$ b $\frac{-x+7}{(x-1)(x+2)}$ c $\frac{8x-2}{(2x+1)(x-1)}$
d $\frac{-x-5}{6}$ e $\frac{2x-4}{(x+4)^2}$ f $\frac{23x+9}{6(x+3)(x-1)}$

3 a $\frac{x+3}{(x+1)^2}$ b $\frac{3x+1}{(x-2)(x+2)}$ c $\frac{-x-7}{(x+1)(x+3)^2}$
d $\frac{3x+3y+2}{(y-x)(y+x)}$ e $\frac{2x+5}{(x+2)^2(x+1)}$ f $\frac{7x+8}{(x+2)(x+3)(x-4)}$

4 $\frac{2x-19}{(x+5)(x-3)}$

5 a $\frac{6x^2+14x+6}{x(x+1)(x+2)}$ b $\frac{-x^2-24x-8}{3x(x-2)(2x+1)}$
c $\frac{9x^2-14x-7}{(x-1)(x+1)(x-3)}$

6 $\frac{50x+3}{(6x+1)(6x-1)}$

7 a $g(x) = x + \frac{6}{x+2} + \frac{36}{x^2-2x-8}$
$= \frac{x(x+2)(x-4)}{(x+2)(x-4)} + \frac{6(x-4)}{(x+2)(x-4)} + \frac{36}{(x+2)(x-4)}$
$= \frac{x^3-2x^2-2x+12}{(x+2)(x-4)}$

b Divide $x^3-2x^2-2x+12$ by $(x+2)$ to give x^2-4x+6

Exercise 1C

1 $A = 1, B = 1, C = 2, D = -6$
2 $a = 2, b = -3, c = 5, d = -10$
3 $p = 1, q = 2, r = 4$
4 $m = 2, n = 4, p = 7$
5 $A = 4, B = 1, C = -8$ and $D = 3$
6 $A = 4, B = -13, C = 33$ and $D = -27$
7 $p = 1, q = 0, r = 2, s = 0$ and $t = -6$
8 $a = 2, b = 1, c = 1, d = 5$ and $e = -4$
9 $A = 3, B = -4, C = 1, D = 4, E = 1$
10 a $(x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$
b $(x-1)(x^2+1)$, $a = 1, b = -1, c = 1, d = 0$ and $e = 1$

Chapter review 1

1 a x^3-7 b $\frac{x+4}{x-1}$ c $\frac{2x-1}{2x+1}$

2 $3x^2+5$

3 $2x^2-2x+5$

4 a $\frac{1}{3}$ b $\frac{2(x^2+4)(x-5)}{(x^2-7)(x+4)}$ c $\frac{2x+3}{x}$

5 a $\frac{2x-4}{x-4}$ b $\frac{4(e^6-1)}{e^6-2}$

6 a $a = \frac{3}{4}, b = -\frac{13}{8}, c = -\frac{5}{8}$
b $g'(x) = \frac{3}{2}x - \frac{13}{8}$, $g'(-2) = -\frac{37}{8}$

7 $\frac{6x^2+18x+5}{x^2-3x-10}$

8 $x + \frac{3}{x-1} - \frac{12}{x^2+2x-3}$
$= \frac{x(x+3)(x-1)}{(x+3)(x-1)} + \frac{3(x+3)}{(x+3)(x-1)} - \frac{12}{(x+3)(x-1)}$
$= \frac{(x^2+3x+3)(x-1)}{(x+3)(x-1)} = \frac{x^2+3x+3}{x+3}$

9 $A = 1, B = -4, C = 3, D = 8$
10 $A = 2, B = -4, C = 6, D = -11$
11 $A = 1, B = 0, C = 1, D = 3$

Challenge

1 $A = 2, B = -3, C = \frac{34}{11}, D = \frac{73}{11}$

2 $(ax^3+bx^2+cx+d) \div (x-p)$
$= (ax^2+(b+ap)x+d)+(c+bp+ap^2)$
with a remainder of $d+cp+bp^2+ap^3$
$f(p) = ap^3+bp^2+cp+d = 0$, which matches the remainder, so $(x-p)$ is a factor of $f(x)$.

3 a $f(-3) = 0$ or $f(x) = (x+3)(2x^2+3x+1)$
b $\frac{1}{(x+3)} + \frac{8}{(2x+1)} - \frac{5}{(x+1)}$

CHAPTER 2

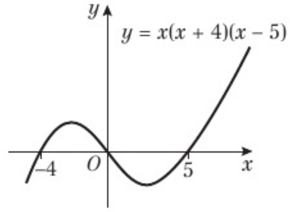
Prior knowledge check

1 a $y = \frac{9-5x}{7}$ b $y = \frac{5p-8x}{2}$ c $y = \frac{5x-4}{8+9x}$

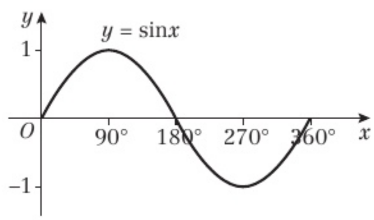
2 **a** $25x^2 - 30x + 5$ or $5(5x^2 - 6x + 1)$

b $\frac{1}{6x - 14}$ **c** $\frac{3x + 7}{-x - 1}$

3 **a**



b



4 **a** 28 **b** 0 **c** 18

Exercise 2A

1 **a** $\frac{3}{4}$ **b** 0.28 **c** 8 **d** $\frac{19}{56}$ **e** 4 **f** 11

2 **a** 5 **b** 46 **c** 40

3 **a** 16 **b** 65 **c** 0

4 **a** Positive $|x|$ graph with vertex at (1, 0), y-intercept at (0, 1)

b Positive $|x|$ graph with vertex at $(-1\frac{1}{2}, 0)$, y-intercept at (0, 3)

c Positive $|x|$ graph with vertex at $(\frac{7}{4}, 0)$, y-intercept at (0, 7)

d Positive $|x|$ graph with vertex at (10, 0), y-intercept at (0, 5)

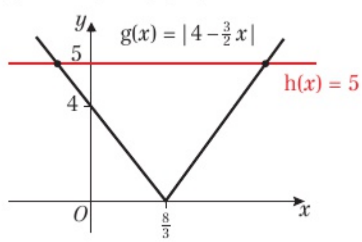
e Positive $|x|$ graph with vertex at (7, 0), y-intercept at (0, 7)

f Positive $|x|$ graph with vertex at $(\frac{3}{2}, 0)$, y-intercept at (0, 6)

g Negative $|x|$ graph with vertex and y-intercept at (0, 0)

h Negative $|x|$ graph with vertex at $(\frac{1}{3}, 0)$, y-intercept at (0, –1)

5 **a**



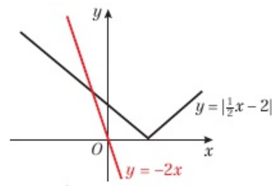
b $x = -\frac{2}{3}$ and $x = 6$

6 **a** $x = 2$ or $x = -\frac{4}{3}$ **b** $x = 7$ or $x = 3$

c No solution **d** $x = 1$ or $x = -\frac{1}{7}$

e $x = -\frac{2}{5}$ or $x = 2$ **f** $x = 24$ or $x = -12$

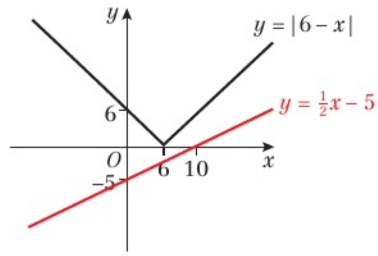
7 **a**



b $x = -\frac{4}{3}$

8 $x = -3, x = 4$

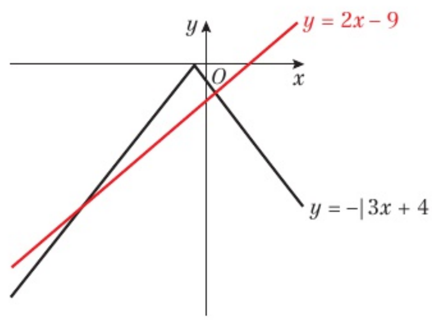
9 **a**



b The two graphs do not intersect, therefore there are no solutions to the equation $|6 - x| = \frac{1}{2}x - 5$

10 Value for x cannot be negative as it equals a modulus.

11 **a**



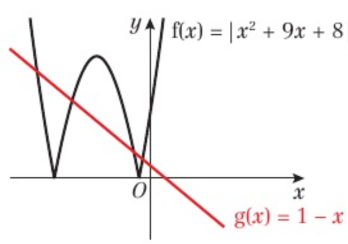
b $x < -13$ or $x > 1$

12 $-23 < x < \frac{5}{3}$

13 **a** $k = -3$ **b** Solution is $x = 6$

Challenge

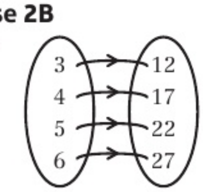
a



b There are 4 solutions: $x = -5 \pm 3\sqrt{2}$ and $x = -4 \pm \sqrt{7}$

Exercise 2B

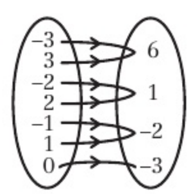
1 **a** **i**



ii one-to-one

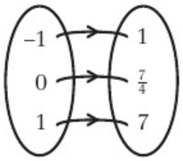
iii {f(x) = 12, 17, 22, 27}

b **i**



ii many-to-one

iii {f(x) = –3, –2, 1, 6}

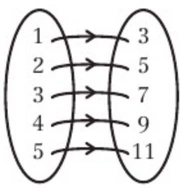
c i 

ii one-to-one

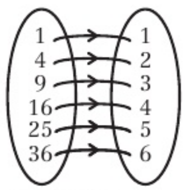
iii $\{f(x) = 1, \frac{7}{4}, 7\}$

2 a i one-to-one ii function
b i one-to-one ii function
c i one-to-many ii not a function
d i one to many ii not a function
e i one to one
ii not valid at the asymptote, so not a function
f i many to one ii function

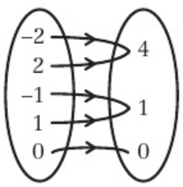
3 a 6 b $\pm 2\sqrt{5}$ c 4 d 2, –3

4 a i 

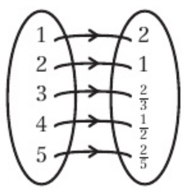
ii one-to-one

b i 

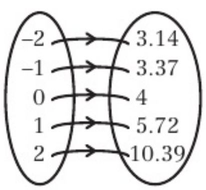
ii one-to-one

c i 

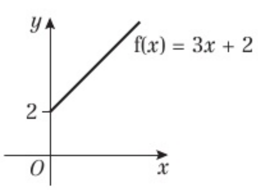
ii many-to-one

d i 

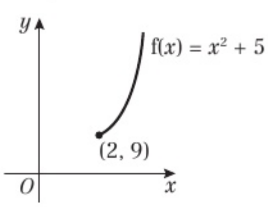
ii one-to-one

e i 

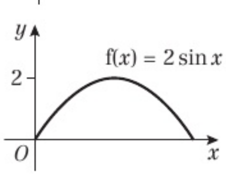
ii one-to-one

5 a i 

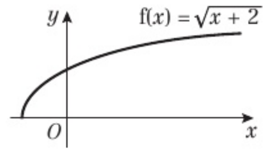
ii $f(x) \geqslant 2$
iii one-to-one

b i 

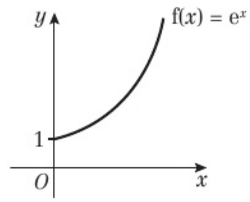
ii $f(x) \geqslant 9$
iii one-to-one

c i 

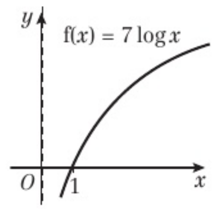
ii $0 \leqslant f(x) \leqslant 2$
iii many-to-one

d i 

ii $f(x) \geqslant 0$
iii one-to-one

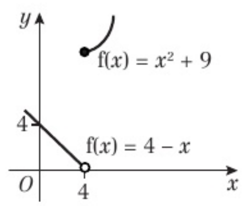
e i 

ii $f(x) \geqslant 1$
iii one-to-one

f i 

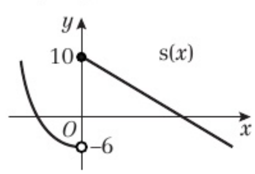
ii $f(x) \in \mathbb{R}$
iii one-to-one

6 a g(x) is not a function because it is not defined for $x = 4$

b 

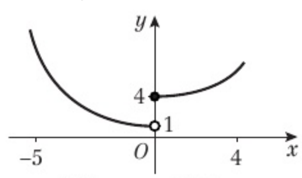
c i 1 ii 109

d $a = -86$ or $a = 9$

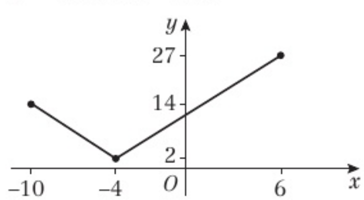
7 a 

b –7

c –2 and 5

8 a 

b $a = -3.91$ or $a = 3.58$

9 a 

b Range $\{2 \leqslant h(x) \leqslant 27\}$

c $a = -9, a = 0$

10 $c = \frac{2}{5}, d = \frac{44}{5}$

11 $a = 2, b = -1$

12 $a = 3$

Exercise 2C

1 a 7 b $\frac{9}{4}$ or 2.25 c 0.25 d –47 e –26

2 a $4x^2 - 15$ b $16x^2 + 8x - 3$ c $\frac{1}{x^2} - 4$
d $\frac{4}{x} + 1$ e $16x + 5$

3 **a** $fg(x) = 3x^2 - 2$ **b** $x = 1$

4 **a** $qp(x) = \dfrac{4x - 5}{x - 2}$ **b** $x = \frac{9}{4}$

5 **a** 23 **b** $x = \frac{13}{7}$ or $x = \frac{13}{5}$

6 **a** $f^2(x) = f\left(\dfrac{1}{x+1}\right) = \dfrac{1}{\left(\dfrac{1}{x+1}\right) + 1} = \dfrac{x+1}{x+2}$

b $f^3(x) = \dfrac{x+2}{2x+3}$

7 **a** 2^{x+3} **b** $2^x + 3$

8 **a** $20x$ **b** x^{20}

9 **a** $(x + 3)^3 - 1$, $qp(x) > -1$ **b** 999 **c** $x = 2$

10 $3 \pm \dfrac{\sqrt{6}}{2}$

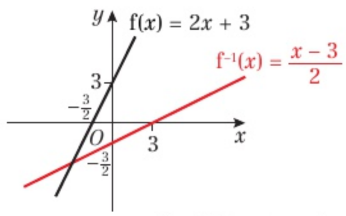
11 **a** $-8 \leqslant g(x) \leqslant 12$ **b** 6 **c** 10.5

Exercise 2D

1 **a** **i** $y \in \mathbb{R}$ **ii** $f^{-1}(x) = \dfrac{x-3}{2}$

iii Domain: $x \in \mathbb{R}$, Range: $y \in \mathbb{R}$

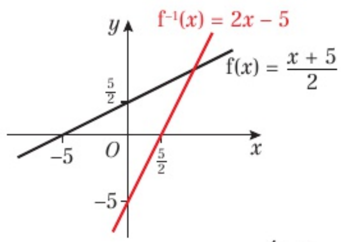
iv



b **i** $y \in \mathbb{R}$ **ii** $f^{-1}(x) = 2x - 5$

iii Domain: $x \in \mathbb{R}$, Range: $y \in \mathbb{R}$

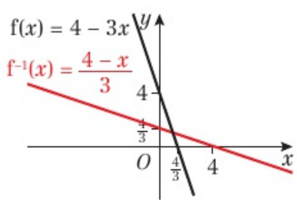
iv



c **i** $y \in \mathbb{R}$ **ii** $f^{-1}(x) = \dfrac{4-x}{3}$

iii Domain: $x \in \mathbb{R}$, Range: $y \in \mathbb{R}$

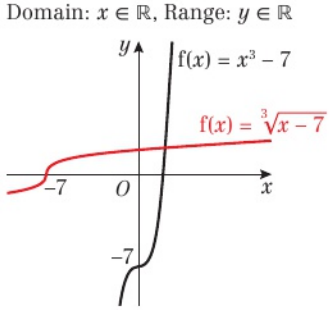
iv



d **i** $y \in \mathbb{R}$ **ii** $f^{-1}(x) = \sqrt[3]{x+7}$

iii Domain: $x \in \mathbb{R}$, Range: $y \in \mathbb{R}$

iv



2 **a** $f^{-1}(x) = 10 - x$, $x \in \mathbb{R}$ **b** $g^{-1}(x) = 5x$, $x \in \mathbb{R}$

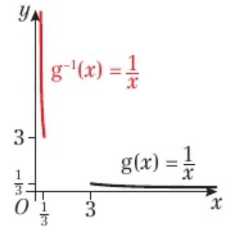
c $h^{-1}(x) = \dfrac{3}{x}$, $x \neq 0$ **d** $k^{-1}(x) = x + 8$, $x \in \mathbb{R}$

3 Domain becomes $x < 4$

4 **a** **i** $0 < g(x) \leqslant \frac{1}{3}$ **ii** $g^{-1}(x) = \dfrac{1}{x}$

iii $x \in \mathbb{R}$, $0 < x \leqslant \frac{1}{3}$, $g^{-1}(x) \geqslant 3$

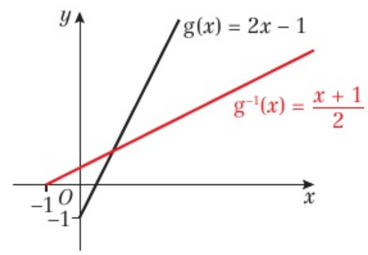
iv



b **i** $g(x) \geqslant -1$ **ii** $g^{-1}(x) = \dfrac{x+1}{2}$

iii $x \in \mathbb{R}$, $x \geqslant -1$, $g^{-1}(x) \geqslant 0$

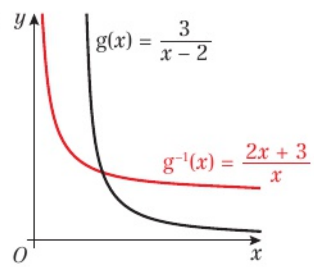
iv



c **i** $g(x) > 0$ **ii** $g^{-1}(x) = \dfrac{2x+3}{x}$

iii $x \in \mathbb{R}$, $x > 0$, $g^{-1}(x) > 2$

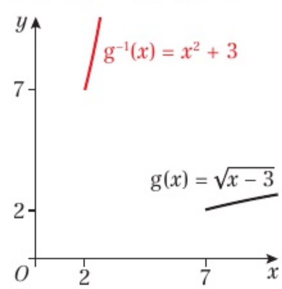
iv



d **i** $g(x) \geqslant 2$ **ii** $g^{-1}(x) = x^2 + 3$

iii $x \in \mathbb{R}$, $x \geqslant 2$, $g^{-1}(x) \geqslant 7$

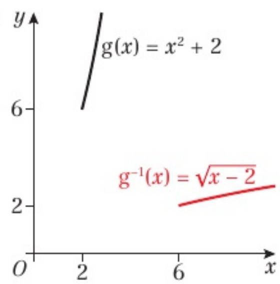
iv



e **i** $g(x) > 6$ **ii** $g^{-1}(x) = \sqrt{x-2}$

iii $x \in \mathbb{R}, x > 6, g^{-1}(x) > 2$

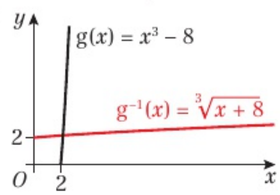
iv



f **i** $g(x) \geqslant 0$ **ii** $g^{-1}(x) = \sqrt[3]{x+8}$

iii $x \in \mathbb{R}, x \geqslant 0, g^{-1}(x) \geqslant 2$

iv



5 $t^{-1}(x) = \sqrt{x+4} + 3, x \in \mathbb{R}, x \geqslant 0$

6 **a** -2 **b** $m^{-1}(x) = \sqrt{x-5} - 2$ **c** $x > 5$

7 **a** h(x) tends to infinity

b 7

c $h^{-1}(x) = \dfrac{2x+1}{x-2} \quad x \in \mathbb{R}, x \neq 2$

d $2 + \sqrt{5}, 2 - \sqrt{5}$

8 **a** $nm(x) = x$

b The functions m and n are inverse of one another as $mn(x) = nm(x) = x$

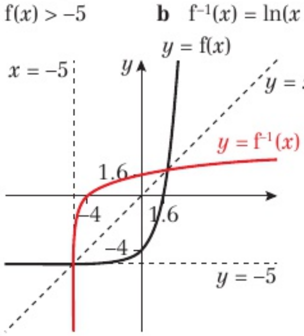
9 $st(x) = \dfrac{3}{\dfrac{3-x}{x} + 1} = x$, $ts(x) = \dfrac{3 - \dfrac{3}{x-1}}{\dfrac{3}{x+1}} = x$

10 **a** $f^{-1}(x) = -\sqrt{\dfrac{x+3}{2}} \quad x \in \mathbb{R}, x > -3$

b $a = -1$

11 **a** $f(x) > -5$ **b** $f^{-1}(x) = \ln(x+5) \quad x \in \mathbb{R}, x > -5$

c



d $g^{-1}(x) = e^x + 4, x \in \mathbb{R}$ **e** $x = 1.95$

12 **a** $f(x) = \dfrac{3(x+2)}{x^2 + x - 20} - \dfrac{2}{x-4}$

$= \dfrac{3(x+2)}{(x+5)(x-4)} - \dfrac{2(x+5)}{(x+5)(x-4)} = \dfrac{x-4}{(x+5)(x-4)}$

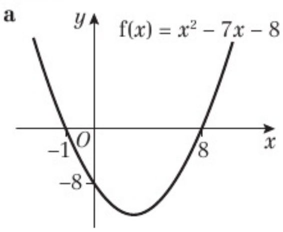
$= \dfrac{1}{x+5}$

b $y \in \mathbb{R}, y < \frac{1}{9}$

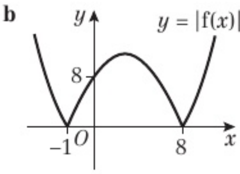
c $f^{-1}: x \rightarrow \dfrac{1}{x} - 5$. Domain is $x \in \mathbb{R}, x < \frac{1}{9}$ and $x \neq 0$

Exercise 2E

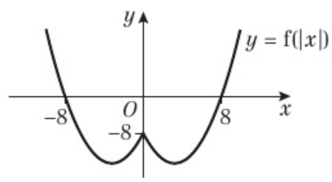
1 **a**



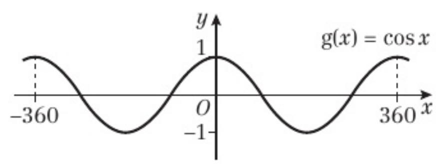
b



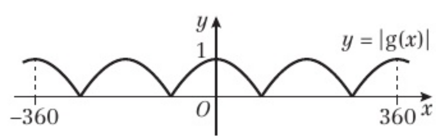
c



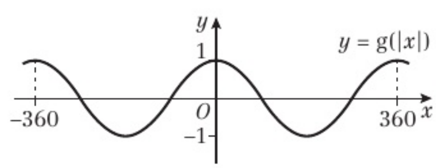
2 **a**



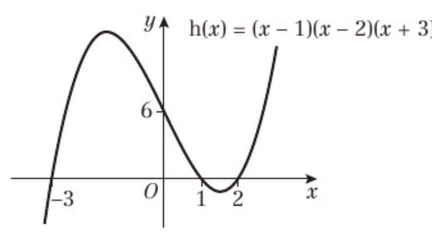
b



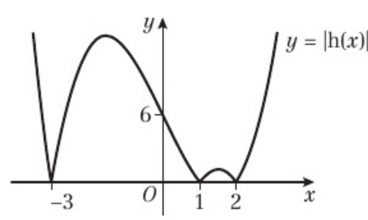
c



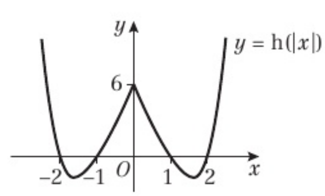
3 **a**



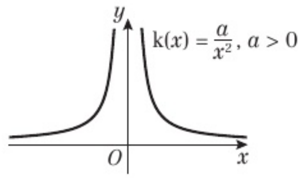
b



c

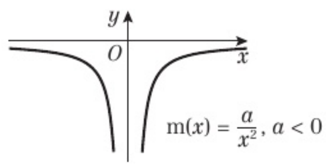


4 a



b Both these graphs would match the original graph.

c

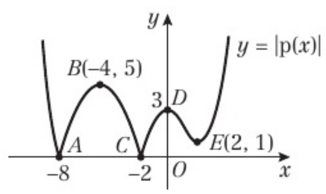


d i True, $|k(x)| = \left|\frac{a}{x^2}\right| = \left|\frac{-a}{x^2}\right| = |m(x)|$

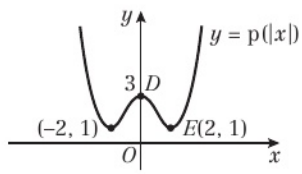
ii False, $k(|x|) = \frac{a}{|x|^2} \neq \frac{-a}{|x|^2} = m(|x|)$

iii True, $m(|x|) = \frac{-a}{|x|^2} = \frac{-a}{x^2} = m(x)$

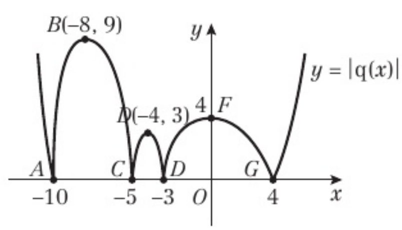
5 a



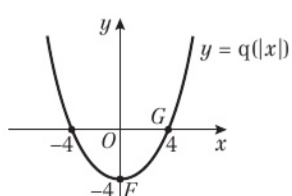
b



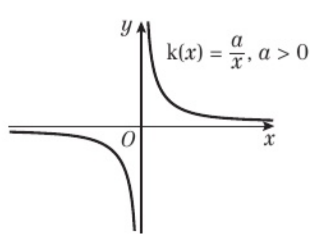
6 a



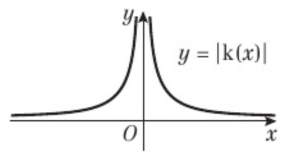
b



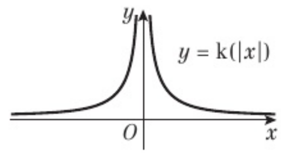
7 a



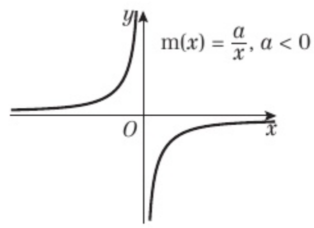
b



c

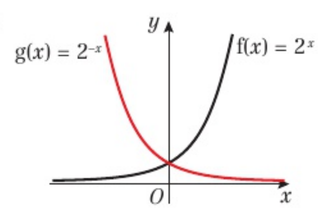


8 a



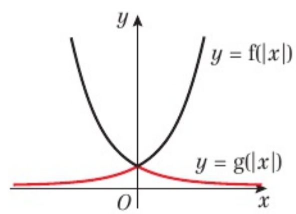
b They are reflections of each other in the x-axis. $|m(x)| = -m(|x|)$

9 a

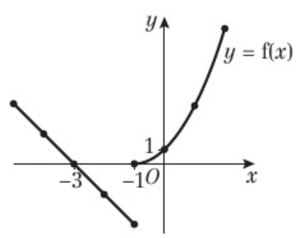


b They would be the same as the original graph.

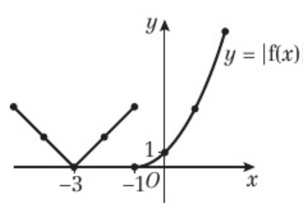
c



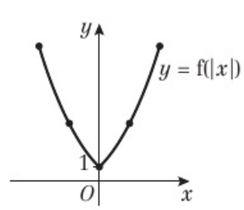
10 a $-4 < f(x) \leqslant 9$



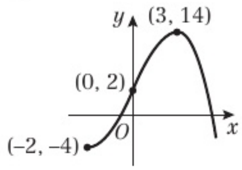
b

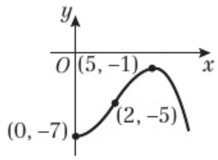


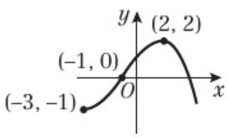
c

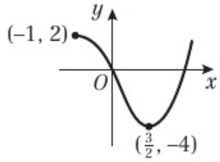


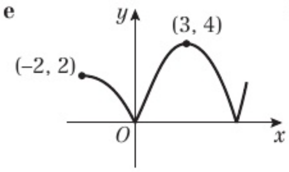
Exercise 2F

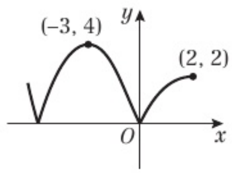
1 a


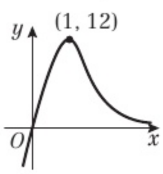
b


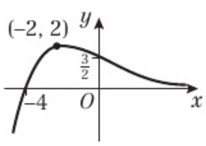
c


d


e


f


2 a


b


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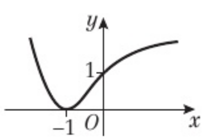
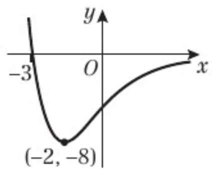
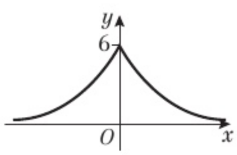
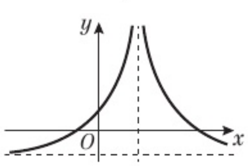
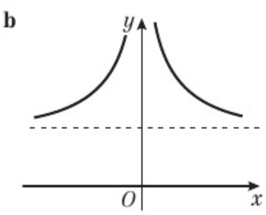


d


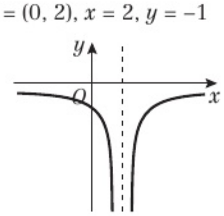
e


3 a


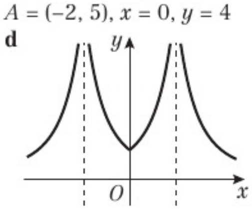
$A = (0, 2), x = 2, y = -1$

b


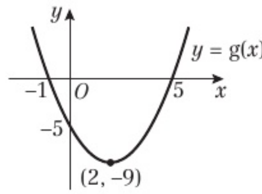
$A = (-2, 5), x = 0, y = 4$

c


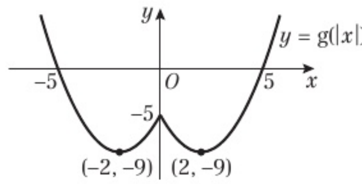
$A = (0, -1), x = 1, y = 0$

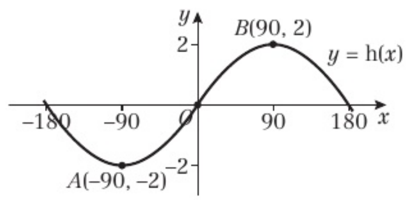
d


$A = (0, 1), x = 2, x = -2, y = 0$

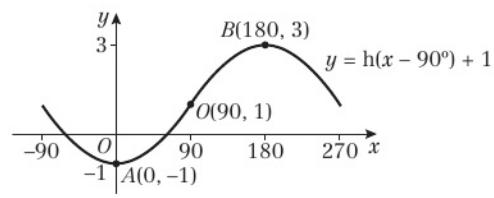
4 a


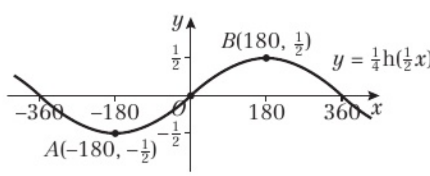
b i (6, −18) ii (1, −9) iii (2, 9)

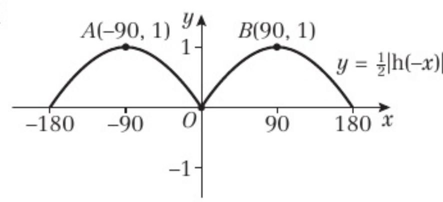
c


5 a


b $A(-90, -2)$ and $B(90, 2)$

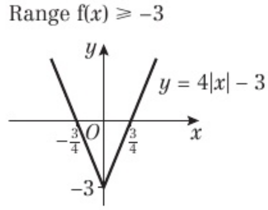
c i


ii


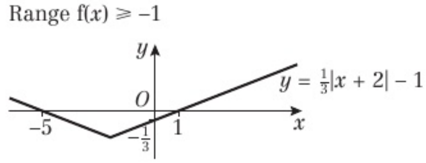
iii


Exercise 2G

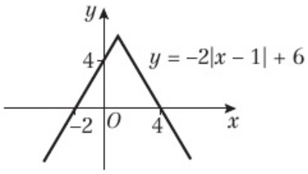
1 a Range $f(x) \geqslant -3$



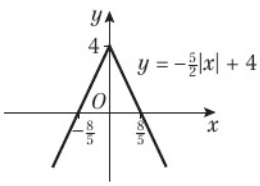
b Range $f(x) \geqslant -1$



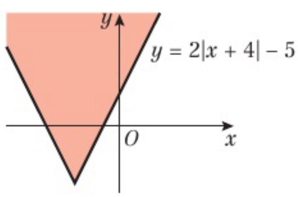
c Range f(x) ⩽ 6



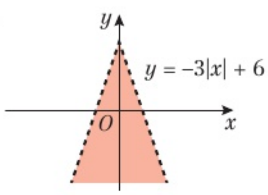
d Range f(x) ⩽ 4



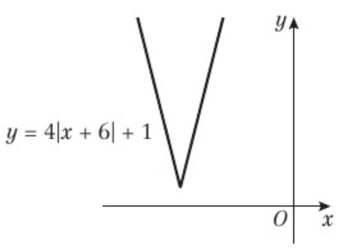
2 a, b



3 a, b

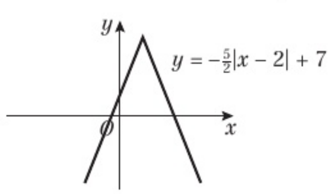


4 a



b $f(x) \geqslant 1$ **c** $x = -\frac{16}{3}$ and $x = -\frac{48}{7}$

5 a



b $g(x) \leqslant 7$ **c** $x = -\frac{2}{3}$ or $x = \frac{22}{7}$

6 $k < 14$

7 $b = 2$

8 a $h(x) \geqslant -7$

b Original function is many-to-one, therefore the inverse is one-to-many, which is not a function.

c $-\frac{1}{2} < x < \frac{5}{2}$ **d** $k < -\frac{23}{3}$

9 a $a = 10$ **b** $P(-3, 10)$ and $Q(2, 0)$

c $x = -\frac{6}{7}$ or $x = -6$

10 a $m(x) \leqslant 7$ **b** $x = -\frac{35}{23}$ or $x = -5$

c $k < 7$

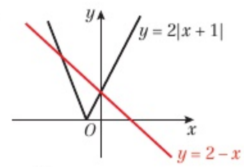
Challenge

1 a $A(3, -6)$ and $B(7, -2)$ **b** 6 units²

2 Graphs intersect at $x = \frac{1}{3}$ and $x = \frac{17}{3}$
Maximum point of f(x) is (3, 10). Minimum point of g(x) is (3, 2). Using area of a kite, area = $\frac{64}{3}$

Chapter review 2

1 a

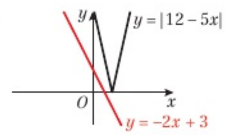


b $x = 0, x = -4$

2 $k > -\frac{11}{4}$

3 $x = -\frac{24}{19}$ or $x = \frac{40}{21}$

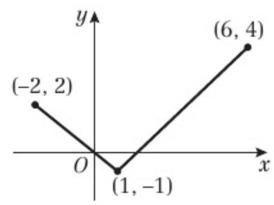
4 a



b The graphs do not intersect, so there are no solutions.

5 a i one-to-many **ii** not a function
b i one-to-one **ii** function
c i many-to-one **ii** function
d i many-to-one **ii** function
e i one-to-one **ii** not a function
f i one-to-one **ii** not a function

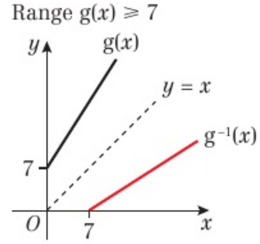
6 a



b $\frac{1}{2}$ and $1\frac{1}{2}$

7 a $pq(x) = 4x^2 + 10x$ **b** $x = \dfrac{-3 \pm \sqrt{21}}{4}$

8 a Range $g(x) \geqslant 7$



b $g^{-1}(x) = \dfrac{x - 7}{2}, x \in \mathbb{R}, x \geqslant 7$

c $g^{-1}(x)$ is a reflection of g(x) in the line $y = x$

9 a $f^{-1}(x) = \dfrac{x + 3}{x - 2}, x \in \mathbb{R}, x > 2$

b i Range $f^{-1}(x) > 1$ **ii** $x \in \mathbb{R}, x > 2$

10 a $f(x) = \dfrac{x}{x^2 - 1} - \dfrac{1}{x + 1} = \dfrac{x}{(x - 1)(x + 1)} - \dfrac{1}{x + 1}$

$= \dfrac{x}{(x - 1)(x + 1)} - \dfrac{x - 1}{(x - 1)(x + 1)} = \dfrac{1}{(x - 1)(x + 1)}$

b $f(x) > 0$ **c** $x = 6$

11 a $20, 28, \frac{1}{9}$ **b** $f(x) \geqslant -8, g(x) \in \mathbb{R}$

c $g^{-1}(x) = \sqrt[3]{x-1}, x \in \mathbb{R}$

d $4(x^3 - 1)$ **e** $a = \frac{5}{3}$

12 a $a = -3$ **b** $f^{-1}: x \mapsto \sqrt{x+13} - 3, x > -4$

13 a $f^{-1}(x) = \dfrac{x+1}{4}, x \in \mathbb{R}$

b $gf(x) = \dfrac{3}{8x-3}, x \in \mathbb{R}, x \neq \frac{3}{8}$

c -0.076 and 0.826 (3 d.p.)

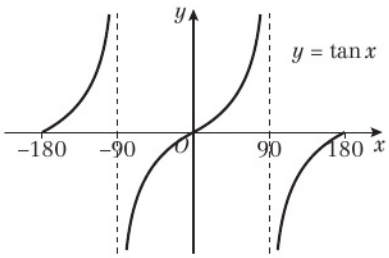
14 a $f^{-1}(x) = \dfrac{2x}{x-1}, x \in \mathbb{R}, x \neq 1$

b Range $f^{-1}(x) \in \mathbb{R}, f^{-1}(x) \neq 2$

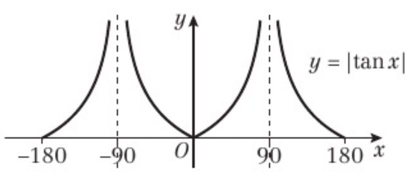
c -1 **d** $1, \frac{6}{5}$

15 a $8, 9$ **b** -45 and $5\sqrt{2}$

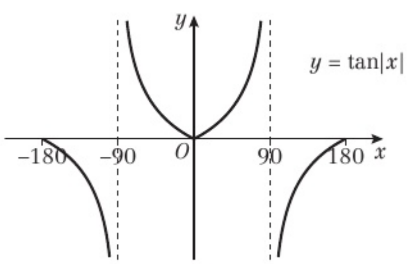
16 a



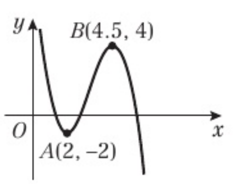
b



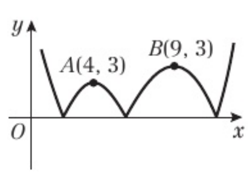
c



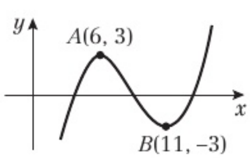
17 a



b

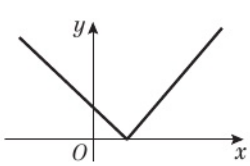


c



18 a $g(x) \geqslant 0$ **b** $x = 0, x = 8$

c



$x = 2$ and $x = 6$

19 a Positive $|x|$ graph with vertex at $\left(\frac{a}{2}, 0\right)$ and y-intercept at $(0, a)$.

b Positive $|x|$ graph with vertex at $\left(\frac{a}{4}, 0\right)$ and y-intercept at $(0, a)$.

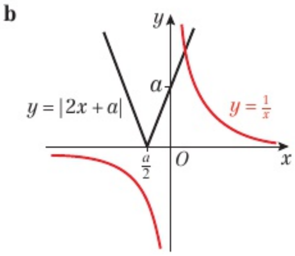
c $a = 6, a = 10$

20 a Positive $|x|$ graph with vertex at $(2a, 0)$ and y-intercept at $(0, a)$.

b $x = \dfrac{3a}{2}, x = 3a$

c Negative $|x|$ graph with x-intercepts at $(a, 0)$ and $(3a, 0)$ and y-intercept at $(0, -a)$.

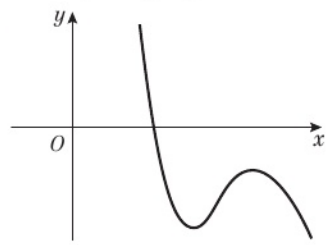
21 a, b



c One intersection point **d** $x = \dfrac{-a + \sqrt{(a^2 + 8)}}{4}$

22 a $(1, 2), \left(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4}\right)$

b

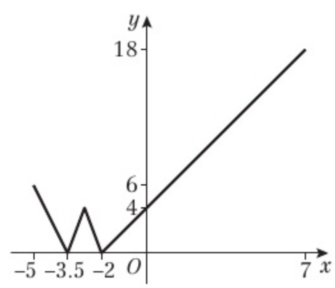


c $(3, -6)$, Minimum

$\left(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2}\right)$, Maximum

23 a $-2 \leqslant f(x) \leqslant 18$ **b** 0

c



d $x = 2$ or $x = 5$

24 a $p(x) \leqslant 10$

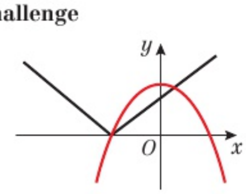
b Original function is many-to-one, therefore the inverse is one-to-many, which is not a function.

c $-11 < x < 3$

d $k > 8$

Challenge

a



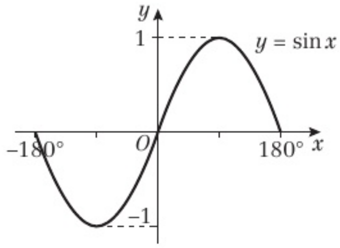
b $(-a, 0), (a, 0), (0, a^2)$ **c** $a = 5$

Online Worked solutions are available in SolutionBank.

CHAPTER 3

Prior knowledge check

1



a 53.1°, 126.9° (1 d.p.)   **b** −23.6°, −156.4° (1 d.p.)

2 $\frac{1}{\sin x \cos x} - \frac{1}{\tan x} = \frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} = \frac{1 - \cos^2 x}{\sin x \cos x}$

$= \frac{\sin^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$

3 0.308, 1.26, 1.88, 2.83, 3.45, 4.40, 5.02, 5.98 (3 s.f.)

Exercise 3A

1 **a** +ve   **b** −ve   **c** −ve   **d** +ve
e −ve

2 **a** −5.76   **b** −1.02   **c** −1.02   **d** 5.67
e 0.577   **f** −1.36   **g** −3.24   **h** 1.04

3 **a** 1   **b** −1   **c** −1   **d** −2
e $-\frac{2\sqrt{3}}{3}$ **f** −1   **g** 2   **h** 2
i $-\sqrt{2}$ **j** $\frac{\sqrt{3}}{3}$ **k** $\frac{2\sqrt{3}}{3}$ **l** $-\sqrt{2}$

4 $\text{cosec}(\pi - x) = \frac{1}{\sin(\pi - x)} = \frac{1}{\sin x} = \text{cosec}\,x$

5 $\cot 30° \sec 30° = \frac{1}{\tan 30°} \times \frac{1}{\cos 30°} = \frac{\sqrt{3}}{1} \times \frac{2}{\sqrt{3}} = 2$

6 $\text{cosec}\left(\frac{2\pi}{3}\right) + \sec\left(\frac{2\pi}{3}\right) = \frac{1}{\sin\left(\frac{2\pi}{3}\right)} + \frac{1}{\cos\left(\frac{2\pi}{3}\right)}$

$= \frac{1}{\frac{\sqrt{3}}{2}} + \frac{1}{-\frac{1}{2}}$

$= -2 + \frac{2}{\sqrt{3}} = -2 + \frac{2}{3}\sqrt{3}$

Challenge

a Using triangle OBP, $OB\cos\theta = 1$
$\Rightarrow OB = \frac{1}{\cos\theta} = \sec\theta$

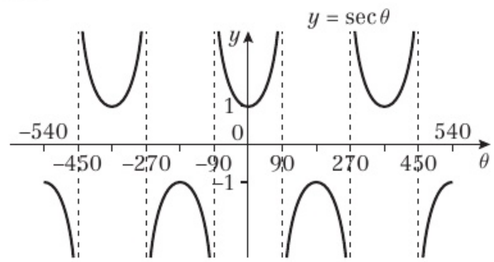
b Using triangle OAP, $OA\sin\theta = 1$
$\Rightarrow OA = \frac{1}{\sin\theta} = \text{cosec}\,\theta$

c Using Pythagoras' theorem, $AP^2 = OA^2 - OP^2$
So $AP^2 = \text{cosec}^2\theta - 1 = \frac{1}{\sin^2\theta} - 1$
$= \frac{1 - \sin^2\theta}{\sin^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$

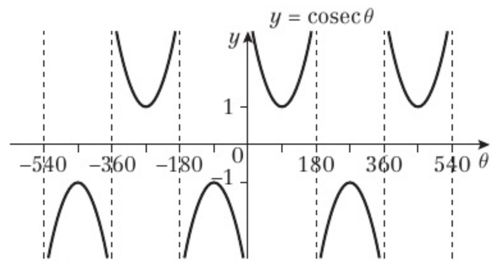
Therefore $AP = \cot\theta$

Exercise 3B

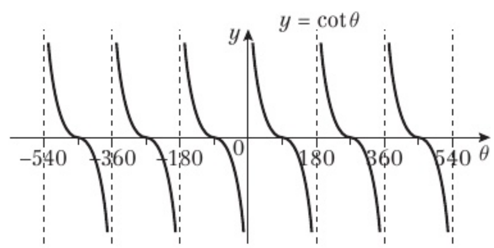
1 **a**



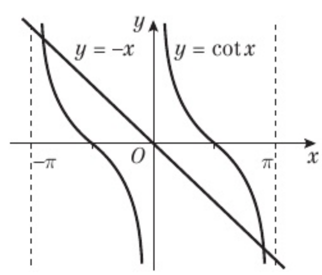
b



c

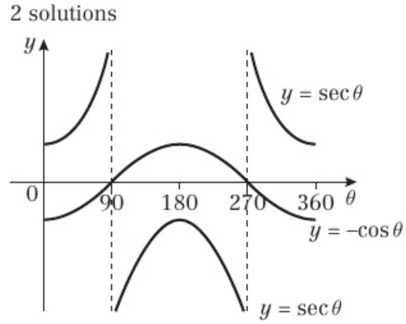


2 **a**



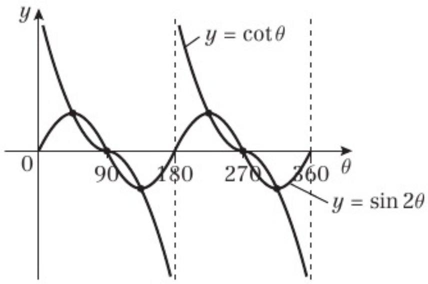
b 2 solutions

3 **a**



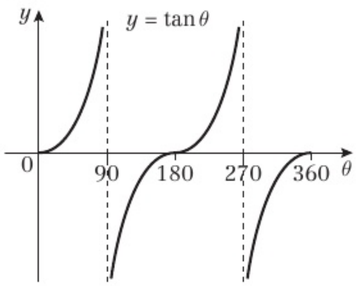
b The solutions of $\sec\theta = -\cos\theta$ are the θ values of the points of intersection of $y = \sec\theta$ and $y = -\cos\theta$. As they do not meet, there are no solutions.

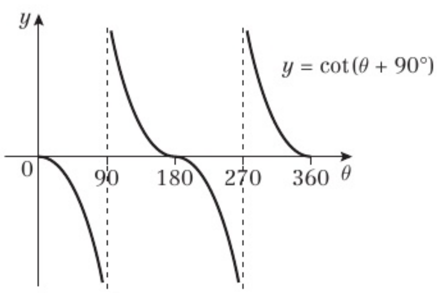
4 **a**



b 6

5 **a**





b $\cot(\theta + 90°) = -\tan\theta$

6 **a** **i** The graph of $y = \tan\left(\theta + \frac{\pi}{2}\right)$ is the same as that of $y = \tan\theta$ translated by $\frac{\pi}{2}$ to the left.

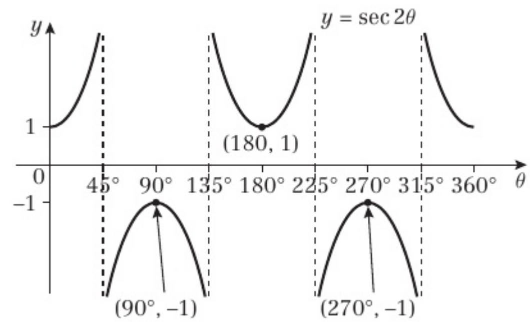
ii The graph of $y = \cot(-\theta)$ is the same as that of $y = \cot\theta$ reflected in the y-axis.

iii The graph of $y = \text{cosec}\left(\theta + \frac{\pi}{4}\right)$ is the same as that of $y = \text{cosec}\,\theta$ translated by $\frac{\pi}{4}$ to the left.

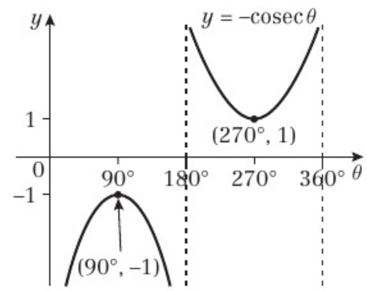
iv The graph of $y = \sec\left(\theta - \frac{\pi}{4}\right)$ is the same as that of $y = \sec\theta$ translated by $\frac{\pi}{4}$ to the right.

b $\tan\left(\theta + \frac{\pi}{2}\right) = \cot(-\theta)$; $\text{cosec}\left(\theta + \frac{\pi}{4}\right) = \sec\left(\theta - \frac{\pi}{4}\right)$

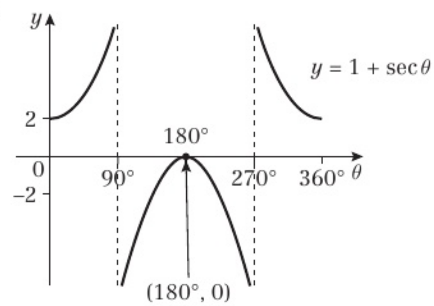
7 **a**



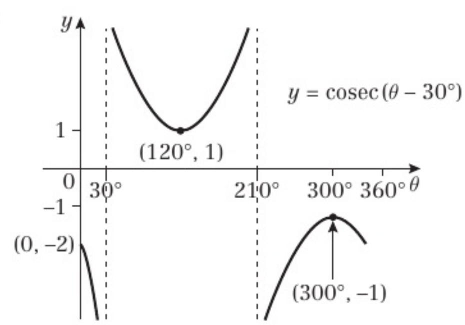
b



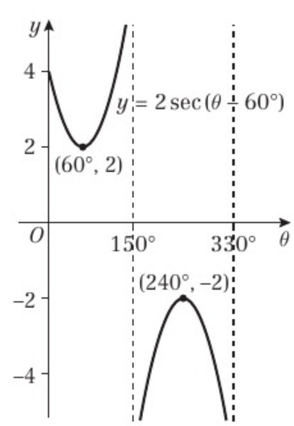
c



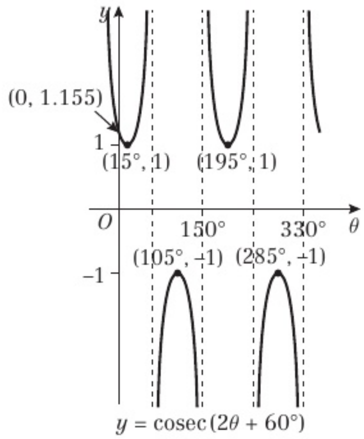
d



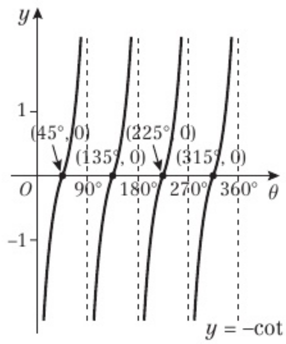
e



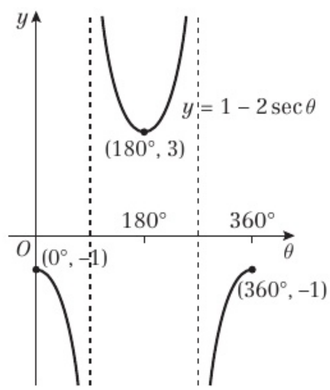
f



g

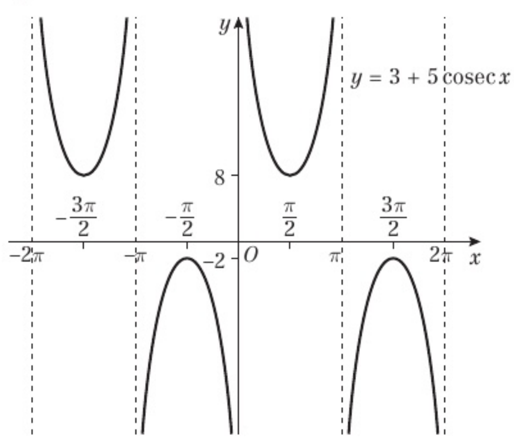


h



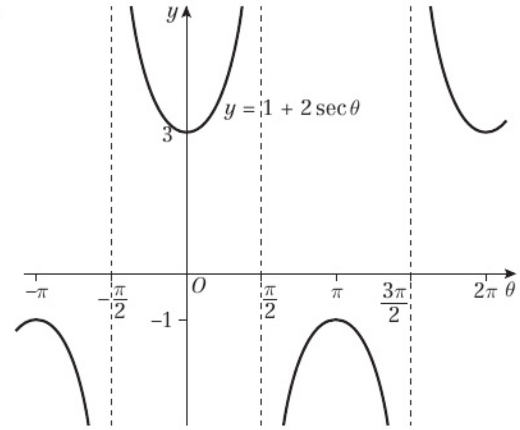
8 **a** $\frac{2\pi}{3}$ **b** 4π **c** π **d** 2π

9 **a**



b $-2 < k < 8$

10 **a**



b $\theta = -\pi, 0, \pi, 2\pi$

c Max $= \frac{1}{3}$, first occurs at $\theta = 2\pi$

Min $= -1$, first occurs at $\theta = \pi$

Exercise 3C

1 **a** $\operatorname{cosec}^3\theta$ **b** $4\cot^6\theta$ **c** $\frac{1}{2}\sec^2\theta$

d $\cot^2\theta$ **e** $\sec^5\theta$ **f** $\operatorname{cosec}^2\theta$

g $2\sqrt{\cot\theta}$ **h** $\sec^3\theta$

2 **a** $\frac{5}{4}$ **b** $-\frac{1}{2}$ **c** $\pm\sqrt{3}$

3 **a** $\cos\theta$ **b** 1 **c** $\sec 2\theta$

d 1 **e** 1 **f** $\cos A$

g $\cos x$

4 **a** $\text{LHS} = \cos\theta + \sin\theta\frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$

$= \frac{1}{\cos\theta} = \sec\theta = \text{RHS}$

b $\text{LHS} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta} = \frac{1}{\sin\theta} \times \frac{1}{\cos\theta}$

$= \operatorname{cosec}\theta\sec\theta = \text{RHS}$

c $\text{LHS} = \frac{1}{\sin\theta} - \sin\theta = \frac{1 - \sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta}$

$= \cos\theta \times \frac{\cos\theta}{\sin\theta} = \cos\theta\cot\theta = \text{RHS}$

d $\text{LHS} = (1 - \cos x)\left(1 + \frac{1}{\cos x}\right) = 1 - \cos x + \frac{1}{\cos x} - 1$

$= \frac{1}{\cos x} - \cos x = \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$

$= \sin x \times \frac{\sin x}{\cos x} = \sin x\tan x = \text{RHS}$

e $\text{LHS} = \frac{\cos^2 x + (1 - \sin x)^2}{(1 - \sin x)\cos x}$

$= \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{(1 - \sin x)\cos x}$

$= \frac{2 - 2\sin x}{(1 - \sin x)\cos x} = \frac{2(1 - \sin x)}{(1 - \sin x)\cos x}$

$= 2\sec x = \text{RHS}$

f $\text{LHS} = \frac{\cos\theta}{1 + \frac{1}{\tan\theta}} = \frac{\cos\theta}{\left(\frac{\tan\theta + 1}{\tan\theta}\right)}$

$= \frac{\cos\theta\tan\theta}{\tan\theta + 1} = \frac{\sin\theta}{1 + \tan\theta} = \text{RHS}$

5 a $45°, 315°$ b $199°, 341°$
c $112°, 292°$ d $30°, 150°$
e $30°, 150°, 210°, 330°$ f $36.9°, 90°, 143°, 270°$
g $26.6°, 207°$ h $45°, 135°, 225°, 315°$

6 a $90°$ b $\pm 109°$
c $-164°, 16.2°$ d $41.8°, 138°$
e $\pm 45°, \pm 135°$ f $\pm 60°$
g $-173°, -97.2°, 7.24°, 82.8°$
h $-152°, -36.5°, 28.4°, 143°$

7 a π b $\frac{5\pi}{6}, \frac{11\pi}{6}$ c $\frac{2\pi}{3}, \frac{4\pi}{3}$ d $\frac{\pi}{4}, \frac{3\pi}{4}$

8 a $\frac{AB}{AD} = \cos\theta \Rightarrow AD = 6\sec\theta$

$\frac{AC}{AB} = \cos\theta \Rightarrow AC = 6\cos\theta$

$CD = AD - AC \Rightarrow CD = 6\sec\theta - 6\cos\theta$
$= 6(\sec\theta - \cos\theta)$

b 2 cm

9 a $\frac{\operatorname{cosec} x - \cot x}{1 - \cos x} = \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{1 - \cos x} = \frac{1}{\sin x} \times \frac{1 - \cos x}{1 - \cos x}$
$= \operatorname{cosec} x$

b $x = \frac{\pi}{6}, \frac{5\pi}{6}$

10 a $\frac{\sin x \tan x}{1 - \cos x} - 1 = \frac{\sin^2 x}{\cos x(1 - \cos x)} - 1$
$= \frac{\sin^2 x - \cos x + \cos^2 x}{\cos x(1 - \cos x)} = \frac{1 - \cos x}{\cos x(1 - \cos x)}$
$= \frac{1}{\cos x} = \sec x$

b Would need to solve $\sec x = -\frac{1}{2}$, which is equivalent to $\cos x = -2$, which has no solutions.

11 $x = 11.3°, 191.3°$ (1 d.p.)

Exercise 3D

1 a $\sec^2\left(\frac{1}{2}\theta\right)$ b $\tan^2\theta$ c 1
d $\tan\theta$ e 1 f 3
g $\sin\theta$ h 1 i $\cos\theta$
j 1 k $4\operatorname{cosec}^4(2\theta)$

2 $\pm\sqrt{k - 1}$

3 a $\frac{1}{2}$ b $-\frac{\sqrt{3}}{2}$

4 a $-\frac{5}{4}$ b $-\frac{4}{5}$ c $-\frac{3}{5}$

5 a $-\frac{7}{24}$ b $-\frac{25}{7}$

6 a LHS $= (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$
$= 1(\sec^2\theta + \tan^2\theta) =$ RHS

b LHS $= (1 + \cot^2 x) - (1 - \cos^2 x)$
$= \cot^2 x + \cos^2 x =$ RHS

c LHS $= \frac{1}{\cos^2 A}\left(\frac{\cos^2 A}{\sin^2 A} - \cos^2 A\right) = \frac{1}{\sin^2 A} - 1$
$= \operatorname{cosec}^2 A - 1 = \cot^2 A =$ RHS

d RHS $= \tan^2\theta \times \cos^2\theta = \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta = \sin^2\theta$
$= 1 - \cos^2\theta =$ LHS

e LHS $= \frac{1 - \tan^2 A}{\sec^2 A} = \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$
$= \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A$
$= 1 - 2\sin^2 A =$ RHS

f LHS $= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$
$= \frac{1}{\cos^2\theta\sin^2\theta} = \sec^2\theta\operatorname{cosec}^2\theta =$ RHS

g LHS $= \operatorname{cosec} A(1 + \tan^2 A) = \operatorname{cosec} A\left(1 + \frac{\sin^2 A}{\cos^2 A}\right)$
$= \operatorname{cosec} A + \frac{1}{\sin A} \cdot \frac{\sin^2 A}{\cos^2 A} = \operatorname{cosec} A + \frac{\sin A}{\cos A} \cdot \frac{1}{\cos A}$
$= \operatorname{cosec} A + \tan A \sec A =$ RHS

h LHS $= \sec^2\theta - \sin^2\theta = (1 + \tan^2\theta) - (1 - \cos^2\theta)$
$= \tan^2\theta + \cos^2\theta =$ RHS

7 $\frac{\sqrt{2}}{4}$

8 a $20.9°, 69.1°, 201°, 249°$ b $\pm\frac{\pi}{3}$
c $-153°, -135°, 26.6°, 45°$ d $\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$
e $120°$ f $0, \frac{\pi}{4}, \pi$
g $0°, 180°$ h $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{4}, \frac{4\pi}{3}$

9 a $1 + \sqrt{2}$

b $\cos k = \frac{1}{1 + \sqrt{2}} = \frac{\sqrt{2} - 1}{(\sqrt{2} - 1)(\sqrt{2} + 1)} = \sqrt{2} - 1$

c $65.5°, 294.5°$

10 a $b = \frac{4}{a}$

b $c^2 = \cot^2 x = \frac{\cos^2 x}{\sin^2 x} = \frac{b^2}{1 - b^2} = \frac{\left(\frac{4}{a}\right)^2}{1 - \left(\frac{4}{a}\right)^2}$
$= \frac{16}{a^2} \times \frac{a^2}{(a^2 - 16)} = \frac{16}{a^2 - 16}$

11 a $\frac{1}{x} = \frac{1}{\sec\theta + \tan\theta} = \frac{\sec\theta - \tan\theta}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}$
$= \frac{\sec\theta - \tan\theta}{(\sec^2\theta - \tan^2\theta)} = \frac{\sec\theta - \tan\theta}{1}$

b $x^2 + \frac{1}{x^2} + 2 = \left(x + \frac{1}{x}\right)^2 = (2\sec\theta)^2 = 4\sec^2\theta$

12 $p = 2(1 + \tan^2\theta) - \tan^2\theta = 2 + \tan^2\theta$
$\Rightarrow \tan^2\theta = p - 2 \Rightarrow \cot^2\theta = \frac{1}{p - 2}$
$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \frac{1}{p - 2} = \frac{(p - 2) + 1}{p - 2} = \frac{p - 1}{p - 2}$

Exercise 3E

1 a $\frac{\pi}{2}$ b $\frac{\pi}{2}$ c $-\frac{\pi}{4}$ d $-\frac{\pi}{6}$
e $\frac{3\pi}{4}$ f $-\frac{\pi}{6}$ g $\frac{\pi}{3}$ h $\frac{\pi}{3}$

2 a 0 b $-\frac{\pi}{3}$ c $\frac{\pi}{2}$

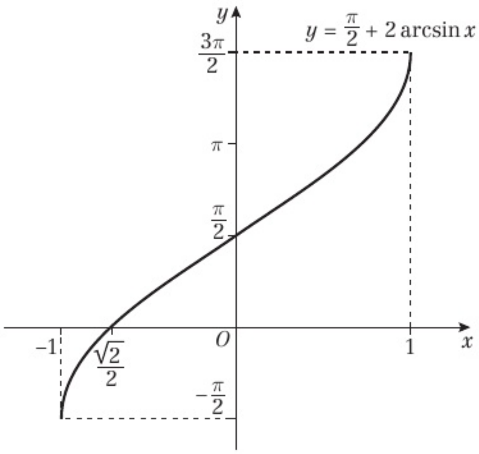
3 a $\frac{1}{2}$ b $-\frac{1}{2}$ c -1 d 0

4 a $\frac{\sqrt{3}}{2}$ b $\frac{\sqrt{3}}{2}$ c -1 d 2
e -1 f 1

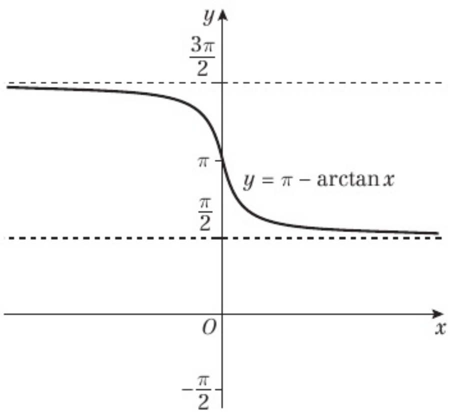
5 $\alpha, \pi - \alpha$

6 a $0 < x < 1$
b i $\sqrt{1 - x^2}$ ii $\frac{x}{\sqrt{1 - x^2}}$
c i no change ii no change

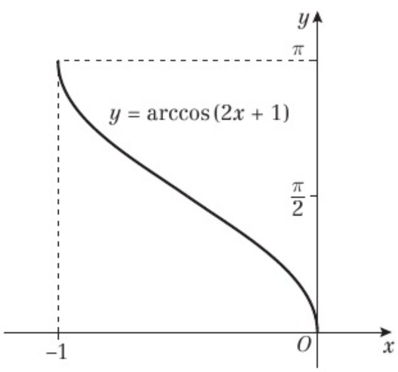
7 **a**



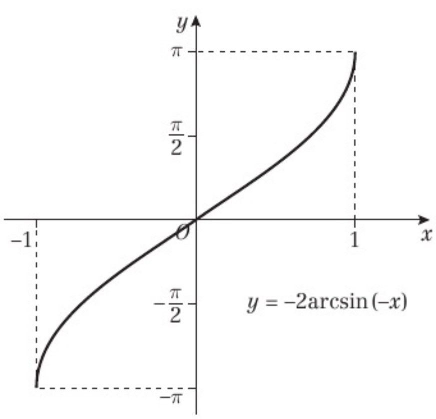
b



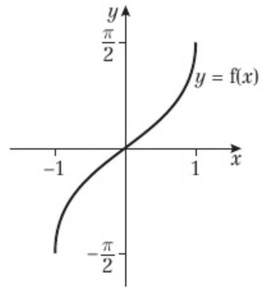
c



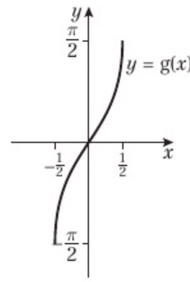
d



8 **a**



b



Range: $-\frac{\pi}{2} \leqslant f(x) \leqslant \frac{\pi}{2}$

c g: $x \to \arcsin 2x$, $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$

d g^{-1}: $x \to \frac{1}{2}\sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

9 **a** Let $y = \arccos x$. $x \in [0,1] \Rightarrow y \in \left[0, \frac{\pi}{2}\right]$

$\cos y = x$, so $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$

(Note, $\sin y \neq -\sqrt{1 - x^2}$ since $y \in \left[0, \frac{\pi}{2}\right]$, so $\sin y \geqslant 0$)

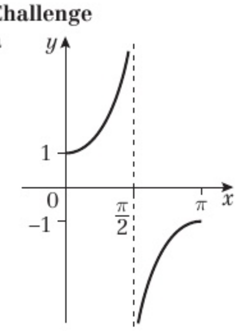
$y = \arcsin\sqrt{1 - x^2}$

Therefore, $\arccos x = \arcsin\sqrt{1 - x^2}$ for $x \in [0,1]$

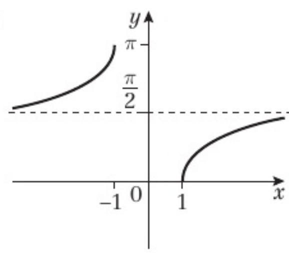
b For $x \in (-1,0)$, $\arccos x \in \left(\frac{\pi}{2}, \pi\right)$, but arcsin only has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Challenge

a



b



Range: $0 \leqslant \operatorname{arcsec} x \leqslant \pi$, $\operatorname{arcsec} x \neq \frac{\pi}{2}$

Chapter review 3

1 $-125.3°$, $\pm 54.7°$

2 $p = \frac{8}{q}$

3 $p^2q^2 = \sin^2\theta \times 4^2\cot^2\theta = 16\sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta}$

$= 16\cos^2\theta = 16(1 - \sin^2\theta) = 16(1 - p^2)$

4 **a** **i** $60°$

ii $30°, 41.8°, 138.2°, 150°$

b **i** $30°, 165°, 210°, 345°$

ii $45°, 116.6°, 225°, 296.6°$

c **i** $\frac{71\pi}{60}, \frac{101\pi}{60}$

ii $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

5 $-\frac{8}{5}$

6 **a** LHS $= \left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)(\sin\theta + \cos\theta)$

$= \dfrac{(\sin^2\theta + \cos^2\theta)}{\cos\theta\sin\theta}(\sin\theta + \cos\theta)$

$= \dfrac{\sin\theta}{\sin\theta\cos\theta} + \dfrac{\cos\theta}{\cos\theta\sin\theta}$

$= \sec\theta + \text{cosec}\,\theta =$ RHS

b LHS $= \dfrac{\frac{1}{\sin x}}{\frac{1}{\sin x} - \sin x}$

$= \dfrac{\frac{1}{\sin x}}{\frac{1-\sin^2 x}{\sin x}} = \dfrac{1}{\sin x} \times \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x =$ RHS

c LHS $= 1 - \sin x + \text{cosec}\,x - 1 = \dfrac{1}{\sin x} - \sin x$

$= \dfrac{1 - \sin^2 x}{\sin x} = \dfrac{\cos^2 x}{\sin x} = \cos x\dfrac{\cos x}{\sin x} = \cos x \cot x$

$=$ RHS

d LHS $= \dfrac{\cot x(1 + \sin x) - \cos x(\text{cosec}\,x - 1)}{(\text{cosec}\,x - 1)(1 + \sin x)}$

$= \dfrac{\cot x + \cos x - \cot x + \cos x}{\text{cosec}\,x - 1 + 1 - \sin x} = \dfrac{2\cos x}{\text{cosec}\,x - \sin x}$

$= \dfrac{2\cos x}{\frac{1}{\sin x} - \sin x} = \dfrac{2\cos x}{\left(\frac{1-\sin^2 x}{\sin x}\right)} = \dfrac{2\cos x\sin x}{\cos^2 x} = 2\tan x$

e LHS $= \dfrac{\text{cosec}\,\theta + 1 + \text{cosec}\,\theta - 1}{(\text{cosec}^2\theta - 1)} = \dfrac{2\,\text{cosec}\,\theta}{\cot^2\theta}$

$= \dfrac{2}{\sin\theta}\cdot\dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{2\sin\theta}{\cos^2\theta} = \dfrac{2}{\cos\theta}\cdot\dfrac{\sin\theta}{\cos\theta}$

$= 2\sec\theta\tan\theta =$ RHS

f LHS $= \dfrac{\sec^2\theta - \tan^2\theta}{\sec^2\theta} = \dfrac{1}{\sec^2\theta} = \cos^2\theta =$ RHS

7 **a** LHS $= \dfrac{\sin^2 x + (1 + \cos x)^2}{(1 + \cos x)\sin x}$

$= \dfrac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1 + \cos x)\sin x} = \dfrac{2 + 2\cos x}{(1 + \cos x)\sin x}$

$= \dfrac{2(1 + \cos x)}{(1 + \cos x)\sin x} = \dfrac{2}{\sin x} = 2\,\text{cosec}\,x$

b $-\dfrac{\pi}{3}, -\dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

8 RHS $= \left(\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2 = \dfrac{(1 + \cos\theta)^2}{\sin^2\theta} = \dfrac{(1 + \cos\theta)^2}{1 - \cos^2\theta}$

$= \dfrac{(1 + \cos\theta)^2}{(1 - \cos\theta)(1 + \cos\theta)} = \dfrac{1 + \cos\theta}{1 - \cos\theta} =$ LHS

9 **a** $-2\sqrt{2}$

b $\text{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{1}{8} = \frac{9}{8}$

$\Rightarrow \text{cosec}\,A = \pm\dfrac{3}{2\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{4}$

As A is obtuse, cosec A is +ve, $\Rightarrow \text{cosec}\,A = \dfrac{3\sqrt{2}}{4}$

10 **a** $\dfrac{1}{k}$ **b** $k^2 - 1$ **c** $-\dfrac{1}{\sqrt{k^2 - 1}}$ **d** $-\dfrac{k}{\sqrt{k^2 - 1}}$

11 $\dfrac{\pi}{12}, \dfrac{17\pi}{12}$

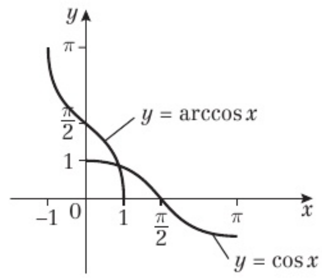
12 $\dfrac{\pi}{3}$

13 $\dfrac{\pi}{3}, \dfrac{5\pi}{6}, \dfrac{4\pi}{3}, \dfrac{11\pi}{6}$

14 **a** $(\sec x - 1)(\text{cosec}\,x - 2)$ **b** 30°, 150°

15 $2 - \sqrt{3}$

16



17 **a** $-\frac{1}{3}$ **b i** $-\frac{5}{3}$, **ii** $-\frac{4}{3}$ **c** 126.9°

18 $pq = (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = \sec^2\theta - \tan^2\theta$

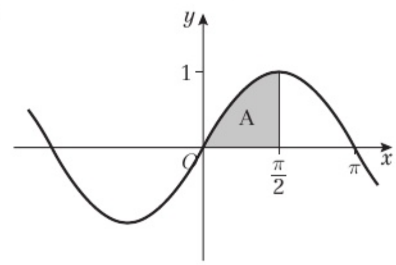
$= 1 \Rightarrow p = \dfrac{1}{q}$

19 **a** LHS $= (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

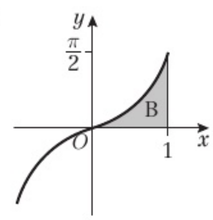
$= 1 \times (\sec^2\theta + \tan^2\theta) = \sec^2\theta + \tan^2\theta =$ RHS

b −153.4°, −135°, 26.6°, 45°

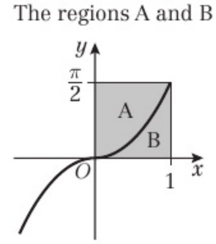
20 **a**



b



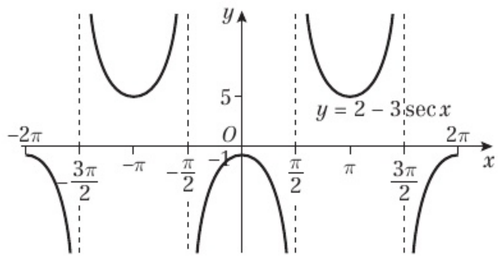
c The regions A and B fit together to make a rectangle.



Area $= 1 \times \dfrac{\pi}{2} = \dfrac{\pi}{2}$

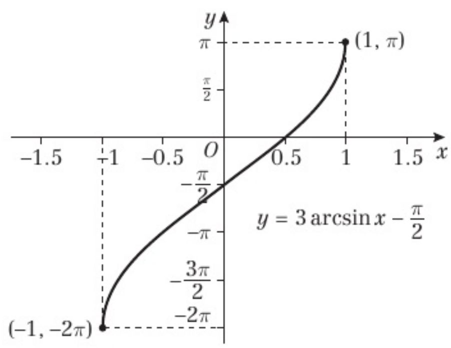
21 $\cot 60° \sec 60° = \dfrac{1}{\tan 60°} \times \dfrac{1}{\cos 60°} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

22 a



b $-1 < k < 5$

23 a



b $\left(\frac{1}{2}, 0\right)$

24 a Let $y = \arccos x$. So $\cos y = x$, $\sin y = \sqrt{1 - x^2}$

Thus $\tan y = \frac{\sqrt{1 - x^2}}{x}$, which is valid for $x \in (0, 1]$

Therefore $\arccos x = \arctan \frac{\sqrt{1 - x^2}}{x}$ for $0 < x \leq 1$

b Letting $y = \arccos x$, $x \in (-1, 0) \Rightarrow y \in \left(\frac{\pi}{2}, \pi\right)$

$\tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{1 - x^2}}{x}$

$\arctan \frac{\sqrt{1 - x^2}}{x}$ gives values in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

so for $y \in \left(\frac{\pi}{2}, \pi\right)$ you need to add π:

$y = \pi + \arctan \frac{\sqrt{1 - x^2}}{x}$

Therefore $\arccos x = \pi + \arctan \frac{\sqrt{1 - x^2}}{x}$

CHAPTER 4

Prior knowledge check

1 **a** $\frac{1}{\sqrt{2}}$ **b** $\frac{\sqrt{3}}{2}$ **c** $\sqrt{3}$

2 **a** 194.2°, 245.8° **b** 45°, 165°, 225°, 345° **c** 270°

3 **a** $\text{LHS} \equiv \cos x + \sin x \tan x \equiv \cos x + \sin x\left(\frac{\sin x}{\cos x}\right)$

$\equiv \frac{\cos^2 x + \sin^2 x}{\cos x} \equiv \frac{1}{\cos x} \equiv \sec x \equiv \text{RHS}$

b $\text{LHS} \equiv \cot x \sec x \sin x \equiv \left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{1}\right)$

$\equiv 1 \equiv \text{RHS}$

c $\text{LHS} \equiv \frac{\cos^2 x + \sin^2 x}{1 + \cot^2 x} \equiv \frac{1}{\text{cosec}^2 x} \equiv \sin^2 x \equiv \text{RHS}$

Exercise 4A

1 **a** **i** $(\alpha - \beta) + \beta = \alpha$, so $\angle FAB = \alpha$

ii $\angle FAB = \angle ABD$ (alternate angles)

$\angle CBE = 90 - \alpha$, so $\angle BCE = 90 - (90 - \alpha) = \alpha$

iii $\cos \beta = \frac{AB}{1} \Rightarrow AB = \cos \beta$

iv $\sin \beta = \frac{BC}{1} \Rightarrow BC = \sin \beta$

b **i** $\sin \alpha = \frac{AD}{\cos \beta} \Rightarrow AD = \sin \alpha \cos \beta$

ii $\cos \alpha = \frac{BD}{\cos \beta} \Rightarrow BD = \cos \alpha \cos \beta$

c **i** $\cos \alpha = \frac{CE}{\sin \beta} \Rightarrow CE = \cos \alpha \sin \beta$

ii $\sin \alpha = \frac{BE}{\sin \beta} \Rightarrow BE = \sin \alpha \sin \beta$

d **i** $\sin(\alpha - \beta) = \frac{FC}{1} \Rightarrow FC = \sin(\alpha - \beta)$

ii $\cos(\alpha - \beta) = \frac{FA}{1} \Rightarrow FA = \cos(\alpha - \beta)$

e **i** $FC + CE = AD$, so $FC = AD - CE$

$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

ii $AF = DB + BE$

$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

2 $\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

$= \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

3 $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(P + (-Q)) = \sin P \cos(-Q) + \cos P \sin(-Q)$

$\sin(P - Q) = \sin P \cos Q - \cos P \sin Q$

4 Example: with $A = 60°$, $B = 30°$

$\sin(A + B) = \sin 90° = 1$; $\sin A + \sin B = \frac{\sqrt{3}}{2} + \frac{1}{2} \neq 1$

[You can find examples of A and B for which the statement is true, e.g. $A = 30°$, $B = -30°$, but one counter-example shows that it is not an identity.]

5 $\cos(\theta - \theta) \equiv \cos\theta \cos\theta + \sin\theta \sin\theta$

$\Rightarrow \sin^2\theta + \cos^2\theta \equiv 1$ as $\cos 0 = 1$

6 **a** $\sin\left(\frac{\pi}{2} - \theta\right) \equiv \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$

$\equiv (1)\cos\theta - (0)\sin\theta = \cos\theta$

b $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta$

$\equiv (0)\cos\theta - (1)\sin\theta = \sin\theta$

7 $\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$

8 $\cos\left(x + \frac{\pi}{3}\right) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

9 **a** sin 35° **b** sin 35° **c** cos 210° **d** tan 31°
e $\cos\theta$ **f** $\cos 7\theta$ **g** $\sin 3\theta$ **h** $\tan 5\theta$
i $\sin A$ **j** $\cos 3x$

10 **a** $\sin\left(x + \frac{\pi}{4}\right)$ or $\cos\left(x - \frac{\pi}{4}\right)$ **b** $\cos\left(x + \frac{\pi}{4}\right)$

c $\sin\left(x + \frac{\pi}{3}\right)$ or $\cos\left(x - \frac{\pi}{6}\right)$ **d** $\sin\left(x - \frac{\pi}{4}\right)$

11 $\cos y = \sin x \cos y + \sin y \cos x$
Divide by $\cos x \cos y \Rightarrow \sec x = \tan x + \tan y$
so $\tan y = \sec x - \tan x$

12 $\dfrac{\tan x - 3}{3\tan x + 1}$ 13 2

14 a $\frac{5}{3}$ b $\sqrt{3}$ c $-\left(\dfrac{8+5\sqrt{3}}{11}\right)$

15 $\dfrac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x} = \dfrac{1}{2} \Rightarrow (2+\sqrt{3})\tan x = 1 - 2\sqrt{3}$, so

$\tan x = \dfrac{1-2\sqrt{3}}{2+\sqrt{3}} = \dfrac{(1-2\sqrt{3})(2-\sqrt{3})}{1} = 8 - 5\sqrt{3}$

16 Write θ as $\left(\theta + \dfrac{2\pi}{3}\right) - \dfrac{2\pi}{3}$ and $\theta + \dfrac{4\pi}{3}$ as $\left(\theta + \dfrac{2\pi}{3}\right) + \dfrac{2\pi}{3}$
Use the addition formulae for cos and simplify.

Challenge

a i Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}x(y\cos B)(\sin A) = \frac{1}{2}xy\sin A\cos B$
ii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}y(x\cos A)(\sin B) = \frac{1}{2}xy\cos A\sin B$
iii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}xy\sin(A+B)$

b Area of large triangle = area T_1 + area T_2
$\frac{1}{2}xy\sin(A+B) = \frac{1}{2}xy\sin A\cos B + \frac{1}{2}xy\cos A\sin B$
$\sin(A+B) = \sin A\cos B + \cos A\sin B$

Exercise 4B

1 a $\dfrac{\sqrt{2}(\sqrt{3}+1)}{4}$ b $\dfrac{\sqrt{2}(\sqrt{3}+1)}{4}$ c $\dfrac{\sqrt{2}(\sqrt{3}-1)}{4}$ d $\sqrt{3}-2$

2 a 1 b 0 c $\dfrac{\sqrt{3}}{2}$ d $\dfrac{\sqrt{2}}{2}$ e $\dfrac{\sqrt{2}}{2}$
f $-\frac{1}{2}$ g $\sqrt{3}$ h $\dfrac{\sqrt{3}}{3}$ i 1 j $\sqrt{2}$

3 a $\tan(45° + 30°) = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45°\tan 30°}$

b $\tan 75° = \dfrac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \dfrac{3+\sqrt{3}}{3-\sqrt{3}} = \dfrac{(3+\sqrt{3})(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}$
$= \dfrac{12+6\sqrt{3}}{9-3} = 2+\sqrt{3}$

4 $-\frac{6}{7}$

5 a $\cos 105° = \cos(45° + 60°)$
$= \cos 45°\cos 60° - \sin 45°\sin 60°$
$= \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2} - \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} = \dfrac{1-\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$
b $a = 2, b = 3$

6 a $\dfrac{3+4\sqrt{3}}{10}$ b $\dfrac{4+3\sqrt{3}}{10}$ c $\dfrac{10(3\sqrt{3}-4)}{11}$

7 a $\frac{3}{5}$ b $\frac{4}{5}$ c $\dfrac{3-4\sqrt{3}}{10}$ d $\frac{1}{7}$

8 a $-\frac{77}{85}$ b $-\frac{36}{85}$ c $\frac{36}{77}$

9 a $-\frac{36}{325}$ b $\frac{204}{253}$ c $-\frac{325}{36}$

10 a 45° b 225°

Exercise 4C

1 $\sin 2A = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$

2 a $\cos 2A = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$
b i $\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$
$= 2\cos^2 A - 1$
ii $\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A$

3 $\tan 2A = \dfrac{\tan A + \tan A}{1 - \tan A\tan A} = \dfrac{2\tan A}{1 - \tan^2 A}$

4 a $\sin 20°$ b $\cos 50°$ c $\cos 80°$
d $\tan 10°$ e $\operatorname{cosec} 49°$ f $3\cos 60°$
g $\frac{1}{2}\sin 16°$ h $\cos\left(\frac{\pi}{8}\right)$

5 a $\dfrac{\sqrt{2}}{2}$ b $\dfrac{\sqrt{3}}{2}$ c $\frac{1}{2}$ d 1

6 a $(\sin A + \cos A)^2 = \sin^2 A + 2\sin A\cos A + \cos^2 A$
$= 1 + \sin 2A$
b $\left(\sin\dfrac{\pi}{8} + \cos\dfrac{\pi}{8}\right)^2 = 1 + \sin\dfrac{\pi}{4} = 1 + \dfrac{\sqrt{2}}{2} = \dfrac{2+\sqrt{2}}{2}$

7 a $\cos 6\theta$ b $3\sin 4\theta$ c $\tan\theta$
d $2\cos\theta$ e $\sqrt{2}\cos\theta$ f $\frac{1}{4}\sin^2 2\theta$
g $\sin 4\theta$ h $-\frac{1}{2}\tan 2\theta$ i $\cos^2 2\theta$

8 $q = \dfrac{p^2}{2} - 1$

9 a $y = 2(1 - x)$ b $2xy = 1 - x^2$
c $y^2 = 4x^2(1 - x^2)$ d $y^2 = \dfrac{2(4-x)}{3}$

10 $-\frac{7}{8}$

11 $\pm\frac{1}{5}$

12 a i $\frac{24}{7}$ ii $\frac{24}{25}$ iii $\frac{7}{25}$ b $\frac{336}{625}$

13 a i $-\frac{7}{9}$ ii $\dfrac{2\sqrt{2}}{3}$ iii $-\dfrac{9\sqrt{2}}{8}$
b $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = -\dfrac{4\sqrt{2}}{9} \times -\dfrac{9}{7} = \dfrac{4\sqrt{2}}{7}$

14 −3

15 mn

16 a $\cos 2\theta = \dfrac{3^2 + 6^2 - 5^2}{2 \times 3 \times 6} = \dfrac{20}{36} = \dfrac{5}{9}$ b $\dfrac{\sqrt{2}}{3}$

17 a $\frac{3}{4}$ b $m = \tan 2\theta = \dfrac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2} = \dfrac{3}{2} \times \dfrac{16}{7} = \dfrac{24}{7}$

18 a $\cos 2A = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$
$= \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$
b $4\cos 2x = 6\cos^2 x - 3\sin 2x$
$\cos 2x + 3\cos 2x - 6\cos^2 x + 3\sin 2x = 0$
$\cos 2x + 3(2\cos^2 x - 1) - 6\cos^2 x + 3\sin 2x = 0$
$\cos 2x - 3 + 3\sin 2x = 0$
$\cos 2x + 3\sin 2x - 3 = 0$

19 $\tan 2A \equiv \dfrac{\sin 2A}{\cos 2A} \equiv \dfrac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$
$\equiv \dfrac{\frac{2\sin A\cos A}{\cos^2 A}}{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}} \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

Exercise 4D

1 a 51.7°, 231.7° b 170.1°, 350.1°
c 56.5°, 303.5° d 150°, 330°

2 a $\sin\left(\theta + \dfrac{\pi}{4}\right) \equiv \sin\theta\cos\dfrac{\pi}{4} + \cos\theta\sin\dfrac{\pi}{4}$
$\equiv \dfrac{1}{\sqrt{2}}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta \equiv \dfrac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$

b $0, \frac{\pi}{2}, 2\pi$ **c** $0, \frac{\pi}{2}, 2\pi$

3 **a** 30°, 270° **b** 30°, 270°

4 **a** $3(\sin x \cos y - \cos x \sin y)$
$- (\sin x \cos y + \cos x \sin y) = 0$
$\Rightarrow 2\sin x \cos y - 4\cos x \sin y = 0$
Divide throughout by $2\cos x \cos y$
$\Rightarrow \tan x - 2\tan y = 0$, so $\tan x = 2\tan y$

b Using **a** $\tan x = 2\tan y = 2\tan 45° = 2$
so $x = 63.4°, 243.4°$

5 **a** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ **b** ±38.7°
c 30°, 150°, 210°, 330° **d** $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$
e 60°, 300°, 443.6°, 636.4° **f** $\frac{\pi}{8}, \frac{5\pi}{8}$
g $\frac{\pi}{4}, \frac{5\pi}{4}$
h 0°, 30°, 150°, 180°, 210°, 330° **i** $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$
j −104.0°, 0°, 76.0°
k 0°, 35.3°, 144.7°, 180°, 215.3°, 324.7°, 360°

6 51.3°

7 **a** $5\sin 2\theta = 10\sin\theta\cos\theta$, so equation becomes
$10\sin\theta\cos\theta + 4\sin\theta = 0$, or $2\sin\theta(5\cos\theta + 2) = 0$
b 0°, 180°, 113.6°, 246.4°

8 **a** $2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta = 1$
$\Rightarrow 2\sin\theta\cos\theta - 2\sin^2\theta = 0$
$\Rightarrow 2\sin\theta(\cos\theta - \sin\theta) = 0$
b 0°, 180°, 45°, 225°

9 **a** LHS $= \cos^2 2\theta + \sin^2 2\theta - 2\sin 2\theta\cos 2\theta$
$= 1 - \sin 4\theta =$ RHS
b $\frac{\pi}{24}, \frac{17\pi}{24}$

10 **a** **i** $\text{RHS} = \dfrac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} = 2\dfrac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \times \dfrac{\cos^2\left(\frac{\theta}{2}\right)}{1}$

$= 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \sin\theta$

ii $\text{RHS} = \dfrac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1 - \tan^2\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)}$

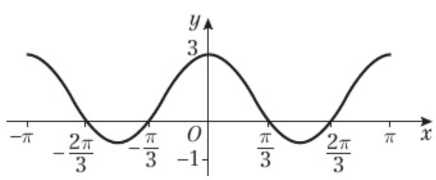
$= \cos^2\left(\frac{\theta}{2}\right)\left\{1 - \tan^2\left(\frac{\theta}{2}\right)\right\} = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$

$= \cos\theta =$ LHS

b **i** 90°, 323.1° **ii** 13.3°, 240.4°

11 **a** LHS $\equiv \dfrac{3(1 + \cos 2x)}{2} - \dfrac{(1 - \cos 2x)}{2}$
$\equiv 1 + 2\cos 2x$

b



Crosses y-axis at (0, 3)

Crosses x-axis at $\left(-\frac{2\pi}{3}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right)$

12 **a** $2\cos^2\left(\frac{\theta}{2}\right) - 4\sin^2\left(\frac{\theta}{2}\right) = 2\left(\frac{1 + \cos\theta}{2}\right) - 4\left(\frac{1 - \cos\theta}{2}\right)$
$= 1 + \cos\theta - 2 + 2\cos\theta = 3\cos\theta - 1$
b 131.8°, 228.2°

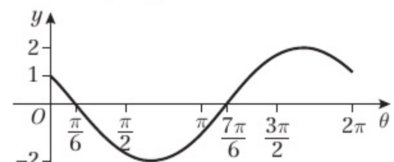
13 **a** $(\sin^2 A + \cos^2 A)^2 \equiv \sin^4 A + \cos^4 A + 2\sin^2 A\cos^2 A$
So $1 \equiv \sin^4 A + \cos^4 A + \dfrac{(2\sin A\cos A)^2}{2}$
$\Rightarrow$ $2 \equiv 2(\sin^4 A + \cos^4 A) + \sin^2 2A$
$\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$
b Using **a**: $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$
$\equiv \frac{1}{2}\left\{2 - \dfrac{(1 - \cos 4A)}{2}\right\} \equiv \dfrac{(4 - 1 + \cos 4A)}{4} \equiv \dfrac{3 + \cos 4A}{4}$
c $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

14 **a** $\cos 3\theta \equiv \cos(2\theta + \theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$
$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta$
$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$
$\equiv 4\cos^3\theta - 3(\sin^2\theta + \cos^2\theta)\cos\theta$
$\equiv 4\cos^3\theta - 3\cos\theta$
b $\frac{\pi}{9}, \frac{5\pi}{9}$ and $\frac{7\pi}{9}$

Exercise 4E

1 $R = 13$; $\tan\alpha = \frac{12}{5}$ **2** 35.3° **3** 41.8°

4 **a** $\cos\theta - \sqrt{3}\sin\theta \equiv R\cos(\theta + \alpha)$ gives $R = 2$, $\alpha = \frac{\pi}{3}$
b $y = 2\cos\left(\theta + \frac{\pi}{3}\right)$



5 **a** $25\cos(\theta + 73.7°)$ **b** (0, 7)
c 25, −25 **d** **i** 2 **ii** 0 **iii** 1

6 **a** $R = \sqrt{10}$, $\alpha = 71.6°$ **b** $\theta = 69.2°, 327.7°$

7 **a** $\sqrt{5}\cos(2\theta + 1.107)$ **b** $\theta = 0.60, 1.44$

8 **a** 6.9°, 66.9° **b** 16.6°, 65.9°
c 8.0°, 115.9° **d** −165.2°, 74.8°

9 **a** $5\sin(3\theta - 53.1°)$
b Minimum value is −5,
when $3\theta - 53.1° = 270° \Rightarrow \theta = 107.7°$
c 21.6°, 73.9°, 141.6°

10 **a** $5\left(\frac{1 - \cos 2\theta}{2}\right) - 3\left(\frac{1 + \cos 2\theta}{2}\right) + 3\sin 2\theta$
$\equiv 1 + 3\sin 2\theta - 4\cos 2\theta$, so $a = 3$, $b = -4$, $c = 1$
b Maximum = 6, minimum = −4 **c** 14.8°, 128.4°

11 **a** $R = \sqrt{10}$, $\alpha = 18.4°$, $\theta = 69.2°, 327.7°$
b $9\cos^2\theta = 4 - 4\sin\theta + \sin^2\theta$
$\Rightarrow 9(1 - \sin^2\theta) = 4 - 4\sin\theta + \sin^2\theta$
So $10\sin^2\theta - 4\sin\theta - 5 = 0$
c 69.2°, 110.8°, 212.3°, 327.7°
d When you square you are also solving
$3\cos\theta = -(2 - \sin\theta)$. The other two solutions are for this equation.

12 **a** $\dfrac{\cos\theta}{\sin\theta} \times \sin\theta + 2\sin\theta = \dfrac{1}{\sin\theta} \times \sin\theta \Rightarrow$
$\cos\theta + 2\sin\theta = 1$
b $\theta = 126.9°$ (1 d.p.)

13 a $\sqrt{2}\cos\theta\cos\frac{\pi}{4} + \sqrt{2}\sin\theta\sin\frac{\pi}{4} + \sqrt{3}\sin\theta - \sin\theta = 2$
$\Rightarrow \cos\theta + \sin\theta - \sin\theta + \sqrt{3}\sin\theta = 2$
$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = 2$

b $\frac{\pi}{3}$

14 a $R = 41, \alpha = 77.320°$ **b i** $\frac{18}{91}$ **ii** $77.320°$

15 a $R = 13, \alpha = 22.6°$ **b** $\theta = 48.7°, 108.7°$
c $a = 12, b = -5, c = 12$ **d** minimum value $= -1$

Exercise 4F

1 a $\text{LHS} = \frac{\cos^2 A - \sin^2 A}{\cos A + \sin A} = \frac{(\cancel{\cos A + \sin A})(\cos A - \sin A)}{\cancel{\cos A + \sin A}}$
$= \cos A - \sin A = \text{RHS}$

b $\text{RHS} = \frac{\cancel{2}}{\cancel{2}\sin A\cos A}\{\sin B\cos A - \cos B\sin A\}$
$= \frac{\sin B}{\sin A} - \frac{\cos B}{\cos A} = \text{LHS}$

c $\text{LHS} = \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \tan\theta = \text{RHS}$

d $\text{LHS} = \frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{1 + \frac{\sin^2\theta}{\cos^2\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}}$
$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta} = \frac{1}{\cos 2\theta} = \sec 2\theta = \text{RHS}$

e $\text{LHS} = 2\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$
$= 2\sin\theta\cos\theta = \sin 2\theta = \text{RHS}$

f $\text{LHS} = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta} = \frac{\sin(3\theta - \theta)}{\sin\theta\cos\theta}$
$= \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}$

g $\text{LHS} = \frac{1}{\sin\theta} - \frac{2\cos 2\theta\cos\theta}{\sin 2\theta} = \frac{1}{\sin\theta} - \frac{\cancel{2}\cos 2\theta\cancel{\cos\theta}}{\cancel{2}\sin\theta\cancel{\cos\theta}}$
$= \frac{1 - \cos 2\theta}{\sin\theta} = \frac{1 - (1 - 2\sin^2\theta)}{\sin\theta} = 2\sin\theta = \text{RHS}$

h $\text{LHS} = \frac{\frac{1}{\cos\theta} - 1}{\frac{1}{\cos\theta} + 1} = \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)}{1 + \left(2\cos^2\frac{\theta}{2} - 1\right)}$
$= \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2} = \text{RHS}$

i $\text{LHS} = \frac{1 - \tan x}{1 + \tan x} = \frac{\cos x - \sin x}{\cos x + \sin x}$
$= \frac{(\cos x - \sin x)(\cos x - \sin x)}{\cos^2 x - \sin^2 x}$
$= \frac{\cos^2 x + \sin^2 x - 2\sin x\cos x}{\cos^2 x - \sin^2 x} = \frac{1 - \sin 2x}{\cos 2x} = \text{RHS}$

2 a $\text{LHS} = \sin(A + 60°) + \sin(A - 60°) = \sin A\cos 60°$
$+ \cos A\sin 60° + \sin A\cos 60° - \cos A\sin 60°$
$= 2\sin A\cos 60° \equiv \sin A = \text{RHS}$

b $\text{LHS} = \frac{\cos A}{\sin B} - \frac{\sin A}{\cos B} = \frac{\cos A\cos B - \sin A\sin B}{\sin B\cos B}$
$\equiv \frac{\cos(A + B)}{\sin B\cos B} = \text{RHS}$

c $\text{LHS} = \frac{\sin(x + y)}{\cos x\cos y} = \frac{\sin x\cos y + \cos x\sin y}{\cos\ \cos y}$
$= \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} \equiv \tan x + \tan y = \text{RHS}$

d $\text{LHS} = \frac{\cos(x + y)}{\sin x\sin y} + 1 = \frac{\cos x\cos y - \sin x\sin y}{\sin x\sin y} + 1$
$= \frac{\cos x\cos y}{\sin x\sin y} - \frac{\sin x\sin y}{\sin x\sin y} + 1 = \frac{\cos x\cos y}{\sin x\sin y}$
$\equiv \cot x\cot y = \text{RHS}$

e $\text{LHS} = \cos\left(\theta + \frac{\pi}{3}\right) + \sqrt{3}\sin\theta = \cos\theta\cos\frac{\pi}{3}$
$- \sin\theta\sin\frac{\pi}{3} + \sqrt{3}\sin\theta = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta + \sqrt{3}\sin\theta$
$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \equiv \sin\left(\theta + \frac{\pi}{6}\right) = \text{RHS}$

f $\text{LHS} = \cot(A + B) = \frac{\cos(A + B)}{\sin(A + B)}$
$= \frac{\cos A\cos B - \sin A\sin B}{\sin A\cos B + \cos A\sin B}$
$= \frac{\frac{\cos A\cos B}{\sin A\sin B} - \frac{\sin A\sin B}{\sin A\sin B}}{\frac{\sin A\cos B}{\sin A\sin B} + \frac{\cos A\sin B}{\sin A\sin B}} \equiv \frac{\cot A\cot B - 1}{\cot A + \cot B} = \text{RHS}$

g $\text{LHS} = \sin^2(45° + \theta) + \sin^2(45° - \theta) = (\sin(45° + \theta))^2$
$+ (\sin(45° - \theta))^2 = (\sin 45°\cos\theta + \cos 45°\sin\theta)^2$
$+ (\sin 45°\cos\theta - \cos 45°\sin\theta)^2$
$= \left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)^2 + \left(\frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta\right)^2$
$= \frac{1}{2}\cos^2\theta + \cos\theta\sin\theta + \frac{1}{2}\sin^2\theta + \frac{1}{2}\cos^2\theta$
$- \cos\theta\sin\theta + \frac{1}{2}\sin^2\theta = \cos^2\theta + \sin^2\theta \equiv 1 = \text{RHS}$

h $\text{LHS} = \cos(A + B)\cos(A - B)$
$= (\cos A\cos B - \sin A\sin B) \times (\cos A\cos B + \sin A\sin B)$
$= (\cos^2 A\cos^2 B) - (\sin^2 A\sin^2 B) = (\cos^2 A(1 - \sin^2 B))$
$- ((1 - \cos^2 A)\sin^2 B) = \cos^2 A - \cos^2 A\sin^2 B$
$- \sin^2 B + \cos^2 A\sin^2 B \equiv \cos^2 A - \sin^2 B = \text{RHS}$

3 a $\text{LHS} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$
$= \frac{1}{\left(\frac{1}{2}\right)\sin 2\theta} = 2\operatorname{cosec} 2\theta = \text{RHS}$

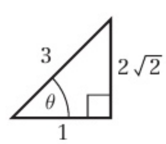
b 4

4 a Use $\sin 3\theta \equiv \sin(2\theta + \theta)$ and substitute $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

b Use $\cos 3\theta \equiv \cos(2\theta + \theta)$ and substitute $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

c $\tan 3\theta \equiv \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}$
$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

d



$\tan\theta = 2\sqrt{2}$

so $\tan 3\theta = \frac{6\sqrt{2} - 16\sqrt{2}}{1 - 24} = \frac{-10\sqrt{2}}{-23} = \frac{10\sqrt{2}}{23}$

5 **a** **i** $\cos x \equiv 2\cos^2 \frac{x}{2} - 1$

$\Rightarrow 2\cos^2 \frac{x}{2} \equiv 1 + \cos x \Rightarrow \cos^2 \frac{x}{2} \equiv \frac{1 + \cos x}{2}$

ii $\cos x \equiv 1 - 2\sin^2 \frac{x}{2}$

$\Rightarrow 2\sin^2 \frac{x}{2} \equiv 1 - \cos x \Rightarrow \sin^2 \frac{x}{2} \equiv \frac{1 - \cos x}{2}$

b **i** $\frac{2\sqrt{5}}{5}$ **ii** $\frac{\sqrt{5}}{5}$ **iii** $\frac{1}{2}$

c $\cos^4 \frac{A}{2} \equiv \left(\frac{1 + \cos A}{2}\right)^2 \equiv \frac{1 + 2\cos A + \cos^2 A}{4}$

$\equiv \frac{1 + 2\cos A + \left(\frac{1 + \cos 2A}{2}\right)}{4}$

$\equiv \frac{2 + 4\cos A + 1 + \cos 2A}{8} \equiv \frac{3 + 4\cos A + \cos 2A}{8}$

6 LHS $\equiv \cos^4 \theta \equiv (\cos^2 \theta)^2 \equiv \left(\frac{1 + \cos 2\theta}{2}\right)^2$

$\equiv \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta) \equiv \frac{1}{4} + \frac{1}{2}\cos 2\theta$

$+ \frac{1}{4}\left(\frac{1 + \cos 4\theta}{2}\right) \equiv \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta$

$\equiv \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \equiv$ RHS

7 $[\sin(x + y) + \sin(x - y)][\sin(x + y) - \sin(x - y)]$
$\equiv [2\sin x \cos y][2\cos x \sin y]$
$\equiv [2\sin x \cos x][2\cos y \sin y]$
$\equiv \sin 2x \sin 2y$

8 $2\cos\left(2\theta + \frac{\pi}{3}\right) \equiv 2\left(\cos 2\theta \cos \frac{\pi}{3} - \sin 2\theta \sin \frac{\pi}{3}\right)$

$\equiv 2\left(\cos 2\theta \frac{1}{2} - \sin 2\theta \frac{\sqrt{3}}{2}\right) \equiv \cos 2\theta - \sqrt{3}\sin 2\theta$

9 $4\cos\left(2\theta - \frac{\pi}{6}\right) \equiv 4\cos 2\theta \cos \frac{\pi}{6} + 4\sin 2\theta \sin \frac{\pi}{6}$

$\equiv 2\sqrt{3}\cos 2\theta + 2\sin 2\theta \equiv 2\sqrt{3}(1 - 2\sin^2\theta) + 4\sin\theta\cos\theta$
$\equiv 2\sqrt{3} - 4\sqrt{3}\sin^2\theta + 4\sin\theta\cos\theta$

10 **a** RHS $= \sqrt{2}\left\{\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right\}$

$= \sqrt{2}\left\{\sin\theta\frac{1}{\sqrt{2}} + \cos\theta\frac{1}{\sqrt{2}}\right\} = \sin\theta + \cos\theta =$ LHS

b RHS $= 2\left\{\sin 2\theta \cos\frac{\pi}{6} - \cos 2\theta \sin\frac{\pi}{6}\right\}$

$= 2\left\{\sin 2\theta \frac{\sqrt{3}}{2} - \cos 2\theta \frac{1}{2}\right\} = \sqrt{3}\sin 2\theta - \cos 2\theta =$ LHS

Challenge

1 **a** $\cos(A + B) - \cos(A - B)$
$\equiv \cos A \cos B - \sin A \sin B - (\cos A \cos B + \sin A \sin B)$
$\equiv -2\sin A \sin B$

b Let $A + B = P$ and $A - B = Q$. Solve to get $A = \frac{P + Q}{2}$

and $B = \frac{P - Q}{2}$. Then use result from part **a** to get

$\cos P - \cos Q = -2\sin\left(\frac{P + Q}{2}\right)\sin\left(\frac{P - Q}{2}\right)$

c $-\frac{3}{2}(\cos 8x - \cos 6x)$

2 **a** $\sin(A + B) + \sin(A - B)$
$= \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$
$= 2\sin A \cos B$
Let $A + B = P$ and $A - B = Q$

$\therefore A = \frac{P + Q}{2}$ and $B = \frac{P - Q}{2}$

$\therefore \sin P + \sin Q = 2\sin\left(\frac{P + Q}{2}\right)\cos\left(\frac{P - Q}{2}\right)$

b $\frac{11\pi}{24} = \frac{P + Q}{2}, \frac{5\pi}{24} = \frac{P - Q}{2}$

$\frac{22\pi}{24} = P + Q, \frac{10\pi}{24} = P - Q$

$\frac{32\pi}{24} = 2P \Rightarrow P = \frac{2\pi}{3}, Q = \frac{\pi}{4},$

$\sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{3} + \sqrt{2}}{2}$

Chapter review 4

1 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{\sqrt{3}}{3}$

2 $\sin x = \frac{1}{\sqrt{5}}$, so $\cos x = \frac{2}{\sqrt{5}}$

$\cos(x - y) = \sin y \Rightarrow \frac{2}{\sqrt{5}}\cos y + \frac{1}{\sqrt{5}}\sin y = \sin y$

$\Rightarrow (\sqrt{5} - 1)\sin y = 2\cos y \Rightarrow \tan y = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2}$

3 **a** $\tan A = 2, \tan B = \frac{1}{3}$ **b** 45°

4 Use the sine rule and addition formulae to get

$\frac{1}{20}\sin\theta \times \frac{\sqrt{3}}{2} = \frac{9}{20}\cos\theta \times \frac{1}{2}$

Then rearrange to get $\tan\theta = 3\sqrt{3}$

5 75°

6 **a** **i** $\frac{56}{65}$ **ii** $\frac{120}{119}$

b Use $\cos\{180° - (A + B)\} \equiv -\cos(A + B)$ and expand. You can work out all the required trig. ratios (A and B are acute).

7 **a** Use $\cos 2x \equiv 1 - 2\sin^2 x$ **b** $\frac{4}{5}$

c **i** Use $\tan x = 2, \tan y = \frac{1}{3}$ in the expansion of $\tan(x + y)$

ii Find $\tan(x - y) = 1$ and note that $x - y$ has to be acute.

8 **a** Show that both sides are equal to $\frac{5}{6}$

b $\frac{3k}{2}$ **c** $\frac{12k}{4 - 9k^2}$

9 **a** $\sqrt{3}\sin 2\theta = 1 - 2\sin^2\theta = \cos 2\theta$

$\Rightarrow \sqrt{3}\tan 2\theta = 1 \Rightarrow \tan 2\theta = \frac{1}{\sqrt{3}}$

b $\frac{\pi}{12}, \frac{7\pi}{12}$

10 **a** $a = 2, b = 5, c = -1$ **b** 0.187, 2.95

11 **a** $\cos(x - 60°) = \cos x \cos 60° + \sin x \sin 60°$

$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$

So $\left(2 - \frac{\sqrt{3}}{2}\right)\sin x = \frac{1}{2}\cos x \Rightarrow \tan x = \frac{\frac{1}{2}}{2 - \frac{\sqrt{3}}{2}} = \frac{1}{4 - \sqrt{3}}$

b 23.8°, 203.8°

12 a $\cos(x + 20°) = \sin(90° - 20° - x) = \sin(70° - x)$
Using addition formulae:
$\cos x \cos 20° - \sin x \sin 20°$
$= \sin 70° \cos x - \cos 70° \sin x$
Rearrange to get: $\sin x(5\cos 70°) + \cos x(3\sin 70°) = 0$
$\Rightarrow \tan x = \dfrac{\sin x}{\cos x} = -\dfrac{3\sin 70°}{5\cos 70°} = -\dfrac{3}{5}\tan 70°$

b 121.2°

13 a Find $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$ and insert in expansions on LHS. Result follows.

b 0.6, 0.8

14 a Example: $A = 60°, B = 0°$; $\sec(A + B) = 2$
$\sec A + \sec B = 2 + 1 = 3$

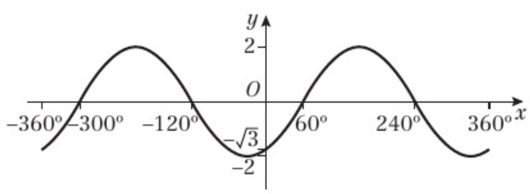
b LHS $= \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} \equiv \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$
$\equiv \dfrac{1}{\frac{1}{2}\sin 2\theta} \equiv 2\operatorname{cosec} 2\theta =$ RHS

15 a Setting $\theta = \frac{\pi}{8}$ gives resulting quadratic equation in t,
$t^2 + 2t - 1 = 0$, where $t = \tan\left(\frac{\pi}{8}\right)$
Solving this and taking +ve value for t gives result.

b Expanding $\tan\left(\frac{\pi}{4} + \frac{\pi}{8}\right)$ gives answer: $\sqrt{2} + 1$

16 a $2\sin(x - 60)°$

b



Graph crosses y-axis at $(0°, -\sqrt{3})$
Graph crosses x-axis at $(-300°, -0)$, $(-120°, 0)$, $(60°, 0)$, $(240°, 0)$

17 a $R = 25, \alpha = 1.29$ **b** 32 **c** $\theta = 0.12, 1.17$

18 a $2.5\sin(2x + 0.927)$ **b** $\frac{3}{2}\sin 2x + 2\cos 2x + 2$ **c** 4.5

19 a $\alpha = 14.0°$ **b** 0°, 151.9°, 360°

20 a $R = \sqrt{13}, \alpha = 56.3°$ **b** $\theta = 17.6°, 229.8°$

21 a LHS $= \dfrac{1}{\cos\theta}\cdot\dfrac{1}{\sin\theta} \equiv \dfrac{1}{\frac{1}{2}\sin 2\theta} \equiv 2\operatorname{cosec} 2\theta =$ RHS

b LHS $= \dfrac{1 + \tan x}{1 - \tan x} - \dfrac{1 - \tan x}{1 + \tan x}$
$\equiv \dfrac{(1 + \tan x)^2 - (1 - \tan x)^2}{(1 + \tan x)(1 - \tan x)}$
$\equiv \dfrac{(1 + 2\tan x + \tan^2 x) - (1 - 2\tan x + \tan^2 x)}{1 - \tan^2 x}$
$\equiv \dfrac{4\tan x}{1 - \tan^2 x} = \dfrac{2(2\tan x)}{1 - \tan^2 x} = 2\tan 2x =$ RHS

c LHS $= -\frac{1}{2}[\cos 2x - \cos 2y] \equiv \frac{1}{2}[\cos 2y - \cos 2x]$
$\equiv \frac{1}{2}[2\cos^2 y - 1 - (2\cos^2 x - 1)]$
$\equiv \frac{1}{2}[2\cos^2 y - 2\cos^2 x] \equiv \cos^2 y - \cos^2 x =$ RHS

d LHS $= 2\cos 2\theta + 1 + (2\cos^2 2\theta - 1)$
$\equiv 2\cos 2\theta(1 + \cos 2\theta) \equiv 2\cos 2\theta(2\cos^2\theta)$
$\equiv 4\cos^2\theta\cos 2\theta \equiv$ RHS

22 a $\dfrac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)} \equiv \dfrac{2\sin^2 x}{2\cos^2 x}$
$\equiv \tan^2 x = \sec^2 x - 1$

b $\dfrac{\pi}{3}, -\dfrac{\pi}{3}$

23 a LHS $= \cos^4 2\theta - \sin^4 2\theta$
$\equiv (\cos^2 2\theta - \sin^2 2\theta)(\cos^2 2\theta + \sin^2 2\theta)$
$\equiv (\cos^2 2\theta - \sin^2 2\theta)(1)$
$\equiv \cos 4\theta =$ RHS

b 15°, 75°, 105°, 165°

24 a Use $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$

b $\sin 360° = 0$, $2 - 2\cos(360°) = 2 - 2 = 0$

c 26.6°, 206.6°

Challenge

1 a $\dfrac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv \dfrac{2\cos 3\theta\cos\theta}{-2\cos 3\theta\sin\theta} \equiv -\cot\theta$

b $\cos 5x + \cos x + 2\cos 3x$
$\equiv 2\cos 3x\cos 2x + 2\cos 3x$
$\equiv 2\cos 3x(\cos 2x + 1)$
$\equiv 2\cos 3x(2\cos^2 x)$
$\equiv 4\cos^2 x\cos 3x$

2 a As $\angle OAB = \angle OBA \Rightarrow \angle AOB = \pi - 2\theta$, so $\angle BOD = 2\theta$
$OB = 1$, $OD = \cos 2\theta$
$BD = \sin 2\theta$, $AB = 2\cos\theta$
$\sin\theta = \dfrac{BD}{AB} = \dfrac{BD}{2\cos\theta}$
So $BD = 2\sin\theta\cos\theta$
But $BD = \sin 2\theta$
So $\sin 2\theta \equiv 2\sin\theta\cos\theta$

b $AB = 2\cos\theta$
$AD = (2\cos\theta)\cos\theta = 2\cos^2\theta$
$OD = 2\cos^2\theta - 1$
From part **a**, $OD = \cos 2\theta$, so $\cos 2\theta = 2\cos^2\theta - 1$

Review exercise 1

1 $\dfrac{4x - 3}{x(x - 3)}$

2 a $f(x) = \dfrac{(x + 2)^2 - 3(x + 2) + 3}{(x + 2)^2} = \dfrac{x^2 + x + 1}{(x + 2)^2}$

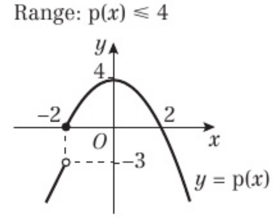
b $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$

c $x^2 + x + 1 > 0$ from **b** and $(x + 2)^2 > 0$ as $x \neq -2$

3 $d = 3, e = 6, f = -14$

4 $x > \frac{2}{3}$ or $x < -5$

5 a Range: $p(x) \leqslant 4$



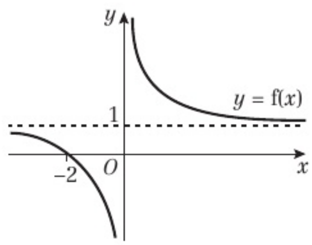
b $a = -\frac{25}{4}$ or $a = 2\sqrt{6}$

6 a $qp(x) = \dfrac{-5x - 18}{x + 4}$
$a = -5, b = -18, c = 1, d = 4$

b $x = -\frac{39}{10}$

c $r^{-1}(x) = \dfrac{-4x - 18}{x + 5}, x \in \mathbb{R}, x \neq -5$

7 a



b $\dfrac{\left(\dfrac{x+2}{x}\right)+2}{\left(\dfrac{x+2}{x}\right)} = \dfrac{x+2+2x}{x+2} = \dfrac{3x+2}{x+2}$

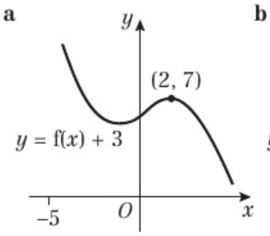
c ln 13 **d** $g^{-1}(x) = \dfrac{e^x+5}{2}, x \in \mathbb{R}$

8 a $3(1-2x) = 1-2(3x+b), b = -\frac{2}{3}$

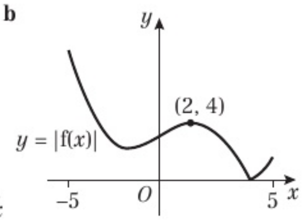
b $p^{-1}(x) = \dfrac{3x+2}{9}$, $q^{-1}(x) = \dfrac{1-x}{2}$

c $p^{-1}(x)q^{-1}(x) = q^{-1}(x)p^{-1}(x) = \dfrac{-3x+7}{18}$,
$a = -3, b = 7, c = 18$

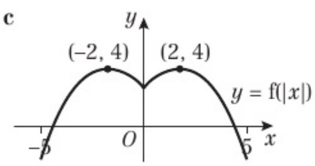
9 a



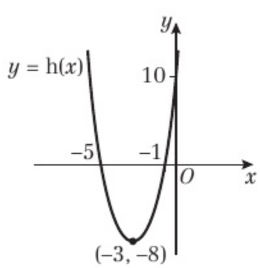
b



c

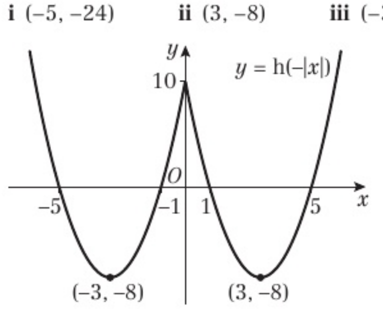


10 a

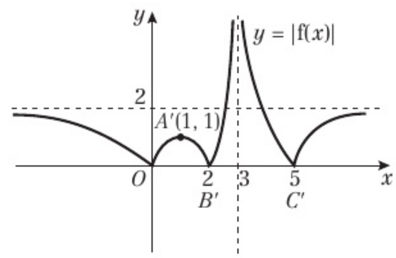


b **i** (−5, −24) **ii** (3, −8) **iii** (−3, 8)

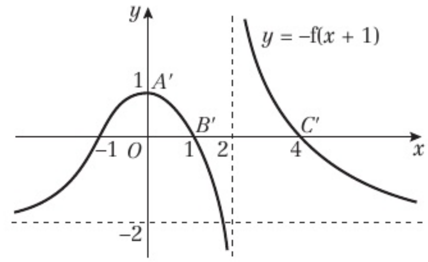
c



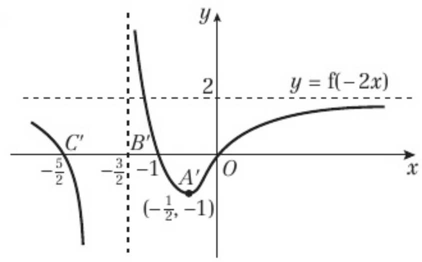
11 a i



ii



iii

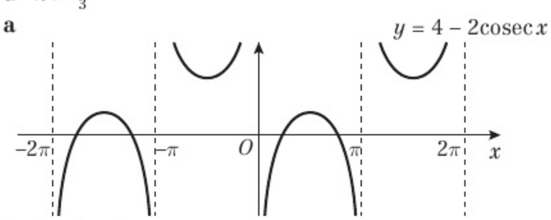


b **i** 6 **ii** 4

12 a $b = -9$ **b** $A(9, -3), B(15, 0)$
c $x = 15, x = -21$

13 a $f(x) \leqslant 8$
b The function is not one-to-one.
c $-\frac{32}{3} < x < -\frac{8}{7}$
d $k > \frac{44}{3}$

14 a



b $2 < k < 6$

15 a $\dfrac{\pi}{3}$ **b** $k = 2$ **c** $-\dfrac{11\pi}{2}, -\dfrac{5\pi}{12}$

16 a $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} = \dfrac{\cos^2 x + (1-\sin x)^2}{\cos x(1-\sin x)}$

$= \dfrac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{\cos x(1-\sin x)} = \dfrac{2-2\sin x}{\cos x(1-\sin x)}$

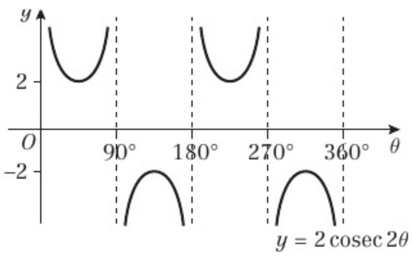
$= \dfrac{2}{\cos x} = 2\sec x$

b $x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{11\pi}{4}, \dfrac{13\pi}{4}$

17 a $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \dfrac{1}{\frac{1}{2}\sin 2\theta} = \dfrac{2}{\sin 2\theta} = 2\,\text{cosec}\,2\theta$

b



c 20.9°, 69.1°, 200.9°, 249.1°

18 a Note the angle $BDC = \theta$

$\cos\theta = \frac{BC}{10} \Rightarrow BC = 10\cos\theta$

$\sin\theta = \frac{BC}{BD} \Rightarrow BD = 10\cot\theta$

b $10\cot\theta = \frac{10}{\sqrt{3}} \Rightarrow \cot\theta = \frac{1}{\sqrt{3}}, \theta = \frac{\pi}{3}$

$DC = 10\cos\theta\cot\theta = 10\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{5}{\sqrt{3}}$

19 a $\sin^2\theta + \cos^2\theta = 1$

$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \sec^2\theta$

b 0.0°, 131.8°, 228.2°

20 a $ab = 2, a = \frac{2}{b}$

b $\frac{4 - b^2}{a^2 - 1} = \frac{4 - b^2}{\frac{4}{b^2} - 1} = \frac{4 - b^2}{\frac{4 - b^2}{b^2}} = b^2$

21 a $\frac{\pi}{2} - y = \arccos x$ **b** $\frac{\pi}{2}$

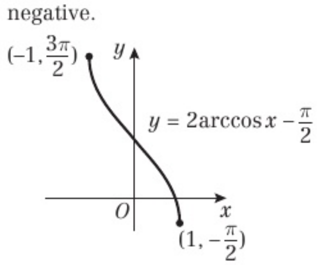
22 a $\arccos\frac{1}{x} = p \Rightarrow \cos p = \frac{1}{x}$

Use Pythagorean theorem to show that opposite side of right-angled triangle is $\sqrt{x^2 - 1}$

$\sin p = \frac{\sqrt{x^2 - 1}}{x} \Rightarrow p = \arcsin\frac{\sqrt{x^2 - 1}}{x}$

b Possible answer: cannot take the square root of a negative number and for $0 \leqslant x \leqslant 1$, $x^2 - 1$ is negative.

23 a



b $\left(\frac{1}{\sqrt{2}}, 0\right)$

24 $\tan\left(x + \frac{\pi}{6}\right) = \frac{1}{6} \Rightarrow \frac{\tan x + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}\tan x} = \frac{1}{6}$

$6\tan x + 2\sqrt{3} = 1 - \frac{\sqrt{3}}{3}\tan x$

$\left(\frac{18 + \sqrt{3}}{3}\right)\tan x = 1 - 2\sqrt{3}$

$\tan x = \frac{3 - 6\sqrt{3}}{18 + \sqrt{3}} \times \frac{18 - \sqrt{3}}{18 - \sqrt{3}} = \frac{72 - 111\sqrt{3}}{321}$

25 a $\sin(x + 30°) = 2\sin(x - 60°)$

$\sin x\cos 30° + \cos x\sin 30°$
$= 2(\sin x\cos 60° - \cos x\sin 60°)$

$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right)$

$\sqrt{3}\sin x + \cos x = 2\sin x - 2\sqrt{3}\cos x$

$(-2 + \sqrt{3})\sin x = (-1 - 2\sqrt{3})\cos x$

$\frac{\sin x}{\cos x} = \frac{-1 - 2\sqrt{3}}{-2 + \sqrt{3}} = \frac{-1 - 2\sqrt{3}}{-2 + \sqrt{3}} \times \frac{-2 - \sqrt{3}}{-2 - \sqrt{3}}$

$= \frac{2 + 4\sqrt{3} + \sqrt{3} + 6}{4 + 2\sqrt{3} - 2\sqrt{3} - 3} = 8 + 5\sqrt{3}$

b $8 - 5\sqrt{3}$

26 a $\sin 165° = \sin(120° + 45°)$
$= \sin 120°\cos 45° + \cos 120°\sin 45°$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{-1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$

b $\text{cosec } 165° = \frac{1}{\sin 165°}$

$= \frac{4}{(\sqrt{6} - \sqrt{2})} \times \frac{(\sqrt{6} + \sqrt{2})}{(\sqrt{6} + \sqrt{2})} = \frac{4(\sqrt{6} + \sqrt{2})}{6 - 2} = \sqrt{6} + \sqrt{2}$

27 a $\cos A = \frac{3}{4} \Rightarrow \sin A = \frac{-\sqrt{7}}{4}$

$\sin 2A = 2\sin A\cos A = 2\left(\frac{-\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = \frac{-3\sqrt{7}}{8}$

b $\cos 2A = 2\cos^2 A - 1 = \frac{1}{8}$

$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{\left(\frac{-3\sqrt{7}}{8}\right)}{\left(\frac{1}{8}\right)} = -3\sqrt{7}$

28 a −180°, 0°, 30°, 150°, 180°

b −148.3°, −58.3°, 31.7°, 121.7° (1 d.p.)

29 a $3\sin x + 2\cos x = \sqrt{13}\sin(x + 0.588\ldots)$

b 169

c $\Rightarrow x = 2.273, 5.976$ (3 d.p.)

30 a $\cot\theta - \tan\theta = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$

$= \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = \frac{2\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta$

b $\theta = -2.95, -1.38, 0.190, 1.76$ (3 s.f.)

31 a $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$
$= (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$
$= \cos^3\theta - 3\sin^2\theta\cos\theta$
$= \cos^3\theta - 3(1 - \cos^2\theta)\cos\theta$
$= 4\cos^3\theta - 3\cos\theta$

b $\sec 3\theta = \frac{-27}{19\sqrt{2}} = \frac{-27\sqrt{2}}{38}$

32 $\sin^4\theta = (\sin^2\theta)(\sin^2\theta)$

$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\sin^4\theta = \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)$$
$$= \frac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)$$
$$= \frac{1}{4}\left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$
$$= \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

Challenge

1 **a** $(x+2)^2 + (y-3)^2 = 25$ **b** 15 units^2

2 A: $x = \dfrac{19-\sqrt{41}}{4}$, B: $x = \dfrac{16}{3}$, C: $x = \dfrac{19+\sqrt{41}}{4}$

3 **a** $\sin x$ **b** $\cos x$ **c** $\operatorname{cosec} x$
d $\cot x$ **e** $\tan x$ **f** $\sec x$

CHAPTER 5

Prior knowledge check

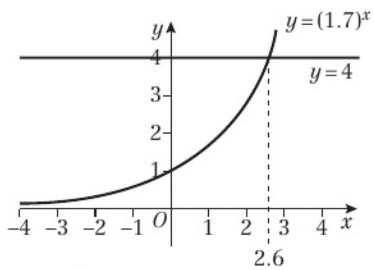
1 **a** 125 **b** $\frac{1}{3}$ **c** 32 **d** 49 **e** 1

2 **a** 6^6 **b** y^{21} **c** 2^6 **d** x^4

3 gradient 1.5, y-intercept 4.1

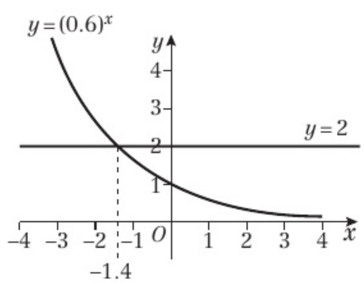
Exercise 5A

1 **a**



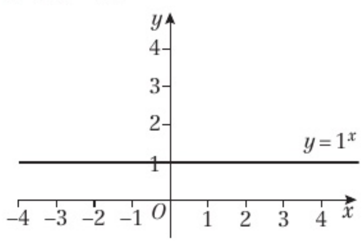
b $x \approx 2.6$

2 **a**



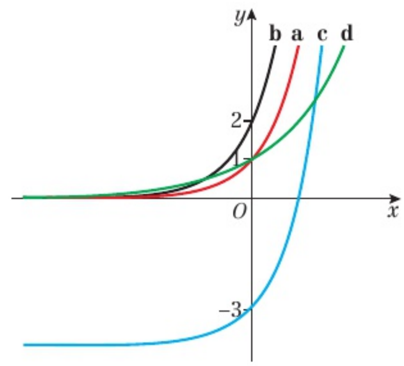
b $x \approx -1.4$

3



4 **a** True, because $a^0 = 1$ whenever a is positive
b False, for example when $a = \frac{1}{2}$
c True, because when a is positive, $a^x > 0$ for all values of x

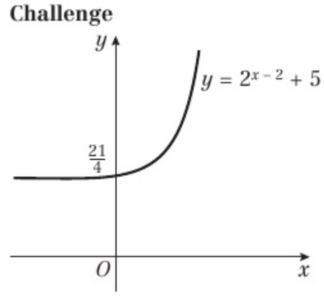
5



6 $k = 3$, $a = 2$

7 **a** As x increases, y decreases
b $p = 1.2$, $q = 0.2$

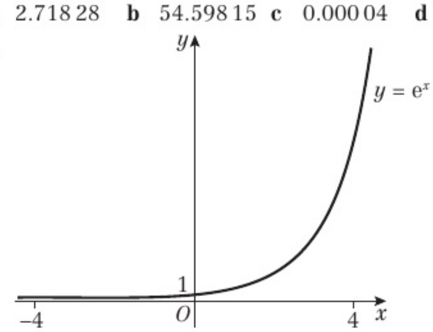
Challenge



Exercise 5B

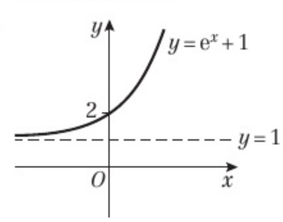
1 **a** 2.718 28 **b** 54.598 15 **c** 0.000 04 **d** 1.221 40

2 **a**

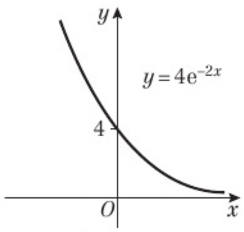


b Student's own answers
c e = 2.71828…
e^3 = 20.08553…

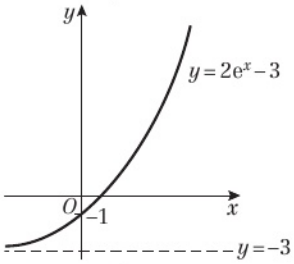
3 **a**



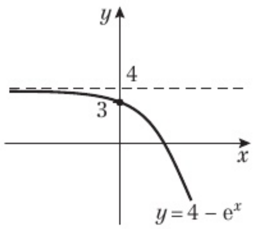
b



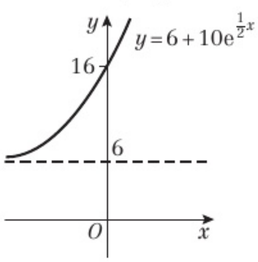
c



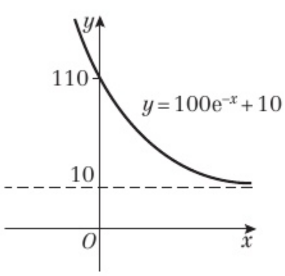
d



e

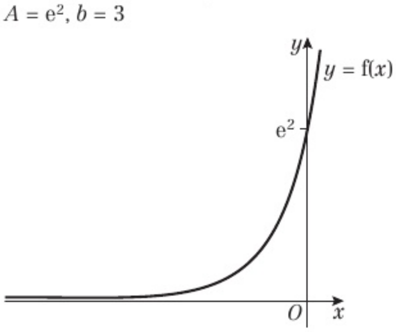


f



4 **a** $A = 1$, $C = 5$, b is positive
b $A = 4$, $C = 0$, b is negative
c $A = 6$, $C = 2$, b is positive

5 $A = e^2$, $b = 3$



6 **a** $6e^{6x}$ **b** $-\frac{1}{3}e^{-\frac{1}{3}x}$ **c** $14e^{2x}$
d $2e^{0.4x}$ **e** $3e^{3x} + 2e^{x}$ **f** $2e^{2x} + e^{x}$

7 **a** $3e^6$ **b** 3 **c** $3e^{-1.5}$

8 $f'(x) = 0.2e^{0.2x}$
The gradient of the tangent when $x = 5$ is
$f'(5) = 0.2e^1 = 0.2e$
The equation of the tangent is therefore $y = (0.2e)x + c$
At (5, e), $e = 0.2e \times 5 + c$, so $c = 0$ and when $x = 0$, $y = 0$

Exercise 5C

1 **a** $\ln 6$ **b** $\frac{1}{2}\ln 11$ **c** $3 - \ln 20$
d $\frac{1}{4}\ln\left(\frac{1}{3}\right)$ **e** $\frac{1}{2}\ln 3 - 3$ **f** $5 - \ln 19$

2 **a** e^2 **b** $\frac{e}{4}$ **c** $\frac{1}{2}e^4 - \frac{3}{2}$
d $\frac{1}{6}(e^{\frac{3}{2}} + 2)$ **e** $18 - e^{\frac{1}{2}}$ **f** 2, 5

3 **a** $\ln 2, \ln 6$ **b** $\frac{1}{2}\ln 2, 0$ **c** e^3, e^{-5}
d $\ln 4, 0$ **e** $\ln 5, \ln\left(\frac{1}{3}\right)$ **f** e^6, e^{-2}

4 $\ln 3, 2\ln 2$

5 **a** $\frac{1}{8}(e^2 + 3)$ **b** $\frac{1}{5}(\ln 3 + 40)$ **c** $\frac{1}{5}\ln 7, 0$
d e^3, e^{-1}

6 $\dfrac{1 + \ln 5}{4 + \ln 3}$

7 **a** The constant 6 represents the initial concentration of the banned medicine in mg/l
b 4.91 mg/l
c $3 = 6e^{-\frac{t}{10}}$
$\frac{1}{2} = e^{-\frac{t}{10}}$
$\ln\left(\frac{1}{2}\right) = -\dfrac{t}{10}$
$t = -10\ln\left(\frac{1}{2}\right) = 6.931... = 6$ hours 56 minutes

8 **a** $(0, 3 + \ln 4)$ **b** $(4 - e^{-3})$

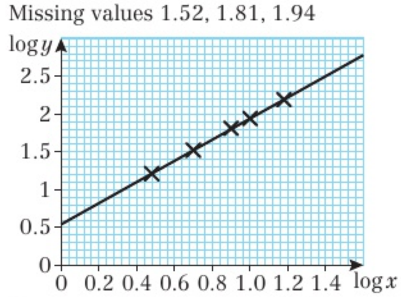
Challenge
As $y = 2$ is an asymptote, $C = 2$
Substituting (0, 5) gives $5 = Ae^0 + 2$, so A is 3.
Substituting (6, 10) gives $10 = 3e^{6B} + 2$
Rearranging this gives $B = \frac{1}{6}\ln\left(\frac{8}{3}\right)$

Exercise 5D

1 **a** $\log S = \log(4 \times 7^x)$
$= \log 4 + \log 7^x \log S$
$= \log 4 + x\log 7$
b gradient $\log 7$, intercept $\log 4$

2 **a** $\log A = \log(6x^4)$
$= \log 6 + \log x^4$
$= \log 6 + 4\log x$
b gradient 4, intercept $\log 6$

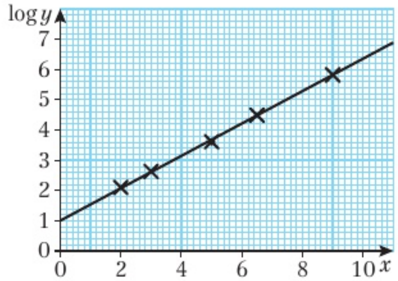
3 **a** Missing values 1.52, 1.81, 1.94
b



c Approximately $a = 3.5$, $n = 1.4$

4 **a** Missing values 2.63, 3.61, 4.49, 5.82

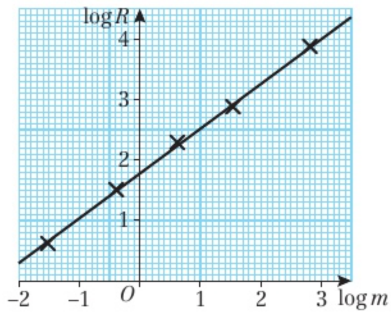
b



c Approximately $b = 3.4$, $a = 10$

5 **a** Missing values −0.39, 0.62, 1.54, 2.81

b

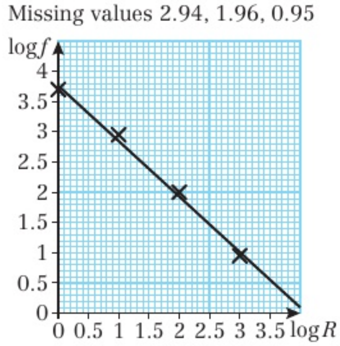


c Approximately $a = 60$, $b = 0.75$

d Approximately 1,600 kcal per day (2 s.f.)

6 **a** Missing values 2.94, 1.96, 0.95

b



c Approximately $A = 5800$, $b = -0.9$

d Approximately 690 times

7 **a** Missing values 0.98, 1.08, 1.13, 1.26, 1.37

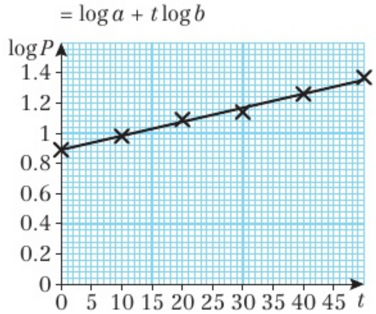
b $P = ab^t$

$\log P = \log(ab^t)$

$= \log a + \log b^t$

$= \log a + t \log b$

c



d Approximately $a = 7.6$, $b = 1.0$

e The rate of growth is often proportional to the size of the population

8 **a** $\log N = 0.095t + 1.6$

b $a = 40$, $b = 1.2$

c The constant a represents the initial number of sick people.

d 9500 people. After 30 days people may start to recover, or the disease may stop spreading as quickly.

9 **a** $\log A = 2 \log w - 0.1049$

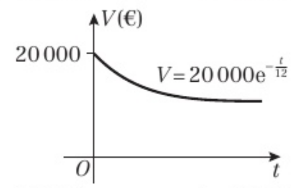
b $q = 2$, $p = 0.7854$

c Circles: p is approximately one quarter π, and the width is twice the radius, so $A = \frac{\pi}{4}w^2 = \frac{\pi}{4}(2r)^2 = \pi r^2$.

Exercise 5E

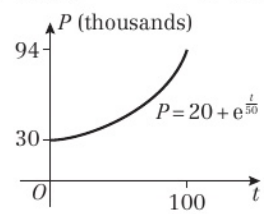
1 **a** €20 000 **b** €14 331

c



2 **a** 30 000 **b** 38 221

c

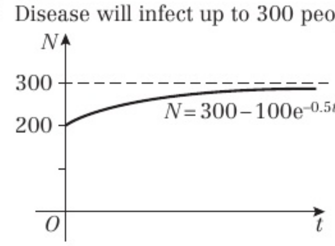


d Model predicts population of the country to be over 200 million, this is highly unlikely and by 2500 new factors are likely to affect population growth. Model not valid for predictions that far into the future.

3 **a** 200

b Disease will infect up to 300 people.

c



4 **a** **i** 15 rabbits **ii** 132 rabbits

b The initial number of rabbits

c $\frac{dR}{dm} = 2.4e^{0.2m}$

When $m = 6$, $\frac{dR}{dm} = 7.97 \approx 8$

d The rabbits may begin to run out of food or space

5 **a** 0.565 bars

b $\frac{dp}{dh} = -0.13e^{-0.13h} = -0.13p$, $k = -0.13$

c The atmospheric pressure decreases exponentially as the altitude increases

d 12%

6 **a** Model 1: 15 733 Dirhams

Model 2: 15 723 Dirhams

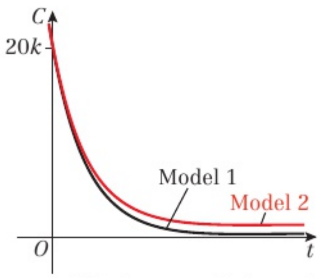
Similar results

b Model 1: 1814 Dirhams

Model 2: 2484 Dirhams

Model 2 predicts a larger value

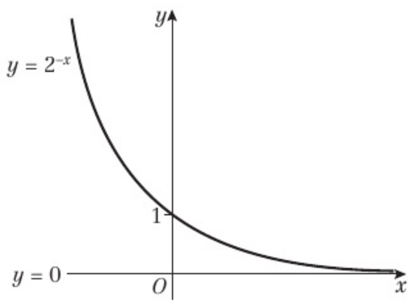
c



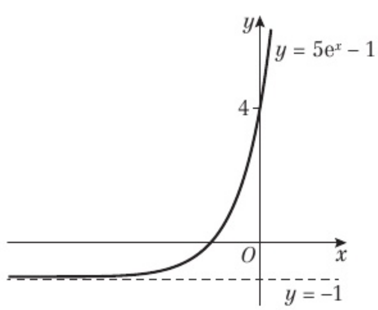
d In Model 2 the car will always be worth at least 1000 Dirhams. This could be the value of the car as scrap metal.

Chapter review 5

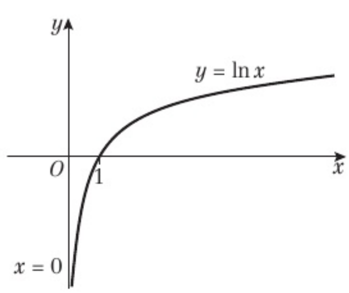
1 a



b



c



2 a $2\ln p + \ln q$ **b** $\ln p = 4$, $\ln q = 1$

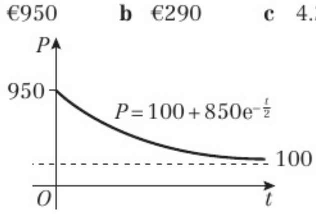
3 a $-e^{-x}$ **b** $11e^{11x}$ **c** $30e^{5x}$

4 a $\frac{e^8 + 5}{2}$ **b** $\frac{\ln 5}{4}$ **c** $-\frac{1}{2}\ln 14$ **d** $\frac{3 + \sqrt{13}}{2}$

e 0 **f** $\frac{e^4}{2}$

5 a €950 **b** €290 **c** 4.28 years **d** €100

e



f A good model. The computer will always be worth something.

6 a $y = \left(\frac{2}{\ln 4}\right)x$

b (0, 0) satisfies the equation of the line.

c 2.43

7 a We cannot go backward in time

b 75 °C

c 5 minutes

d The exponential term will always be positive, so the overall temperature will be greater than 20 °C.

8 a $S = aV^b$

$\log S = \log(aV^b)$

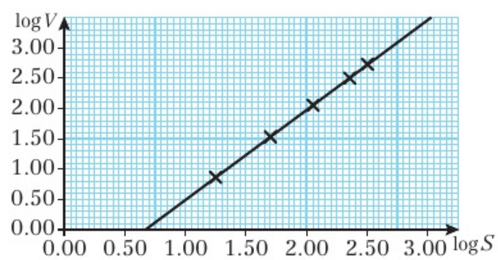
$= \log a + \log(V^b)$

$= \log a + b\log V$

b

$\log S$	1.26	1.70	2.05	2.35	2.50
$\log V$	0.86	1.53	2.05	2.49	2.72

c



d The gradient is approximately 1.5; $a \approx 0.09$

9 a They exponentiated the two terms on the LHS separately rather than combining them first.

b $x = 2 + 2\sqrt{2}$

10 a $\log_{10} P = 0.01t + 2$

b 100, initial population

c 1.023

d Accept answers from 195 to 200

Challenge

$y = 5.8 \times 0.9^x$

CHAPTER 6

Prior knowledge check

1 a $6x - 5$ **b** $-\frac{2}{x^2} - \frac{1}{2\sqrt{x}}$ **c** $8x - 16x^3$

2 $y = -6x + 17$

3 0.58, 3.73 (3 s.f. each)

Exercise 6A

1 a $-2\sin x$ **b** $\cos\left(\frac{1}{2}x\right)$

c $8\cos 8x$ **d** $4\cos\left(\frac{2}{3}x\right)$

2 a $11\sin x$ **b** $-5\sin\left(\frac{5}{6}x\right)$

c $-2\sin\left(\frac{1}{2}x\right)$ **d** $-6\sin 2x$

3 a $2\cos 2x - 3\sin 3x$ **b** $-8\sin 4x + 4\sin x - 14\sin 7x$

c $2x - 12\sin 3x$ **d** $-\frac{1}{x^2} + 10\cos 5x$

4 (0.41, −0.532), (1.68, 2.63), (2.50, 1.56)

5 8

6 (0.554, 2.24), (2.12, −2.24)

7 $y = -5x + 5\pi - 1$

8 $\frac{dy}{dx} = 4x - \cos x$

At $x = \pi$, $y = 2\pi^2$, $\frac{dy}{dx} = 4\pi - \cos\pi = 4\pi + 1$

Gradient of normal $= -\frac{1}{4\pi + 1}$

Equation of normal:

$y - 2\pi^2 = -\frac{1}{4\pi + 1}(x - \pi)$

$(4\pi + 1)y - 2\pi^2(4\pi + 1) = -x + \pi$

$x + (4\pi + 1)y - 8\pi^3 - 2\pi^2 - \pi = 0$

$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$

9 Let $f(x) = \sin x$

$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$

$= \lim_{h\to 0}\left[\left(\frac{\cos h - 1}{h}\right)\sin x + \left(\frac{\sin h}{h}\right)\cos x\right]$

Since $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$ the expression inside the limit $\to (0 \times \sin x + 1 \times \cos x)$

So $\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x$

Hence the derivative of $\sin x$ is $\cos x$

Challenge

Let $f(x) = \sin kx$

$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{\sin(kx + kh) - \sin kx}{h}\right)$

$= \lim_{h\to 0}\left(\frac{\sin kx\cos kh + \cos kx\sin kh - \sin kx}{h}\right)$

$= \lim_{h\to 0}\left(\left(\frac{\cos kh - 1}{h}\right)\sin kx + \left(\frac{\sin kh}{h}\right)\cos kx\right)$

As $h \to 0$, $\left(\frac{\sin kh}{h}\right) \to k$ and $\left(\frac{\cos kh - 1}{h}\right) \to 0$ as given,

so $f'(x) = 0\sin kx + k\cos kx = k\cos kx$

Exercise 6B

1 a $28e^{7x}$ b $3^x\ln 3$ c $\left(\frac{1}{2}\right)^x\ln\frac{1}{2}$ d $\frac{1}{x}$

e $4\left(\frac{1}{3}\right)^x\ln\frac{1}{3}$ f $\frac{3}{x}$ g $3e^{3x} + 3e^{-3x}$ h $-e^{-x} + e^x$

2 a $3^{4x}4\ln 3$ b $\left(\frac{3}{2}\right)^{2x}2\ln\frac{3}{2}$

c $2^{4x}8\ln 2$ d $2^{3x}3\ln 2 - 2^{-x}\ln 2$

3 323.95

4 $4y = 15\ln 2(x-2) + 17$

5 $\frac{dy}{dx} = 2e^{2x} - \frac{1}{x}$ At $x = 1$, $y = e^2$, $\frac{dy}{dx} = 2e^2 - 1$

Equation of tangent: $y - e^2 = (2e^2 - 1)(x - 1)$

$\Rightarrow y = (2e^2 - 1)x - 2e^2 + 1 + e^2 \Rightarrow y = (2e^2 - 1)x - e^2 + 1$

6 −9.07 millicuries/day

7 a $P_0 = 37\,000$, $k = 1.01$ (2 d.p.) b 1178

c The rate of change of the population in the year 2000

8 The student has treated '$\ln kx$' as if it is 'e^{kx}' – they have applied the incorrect standard differential.

Correct differential is: $\frac{1}{x}$

9 Let $y = a^{kx} \Rightarrow y = e^{\ln a^{kx}} = e^{kx\ln a}$

$\frac{dy}{dx} = k\ln a\, e^{kx\ln a} = k\ln a\, e^{\ln a^{kx}} = a^{kx}k\ln a$

10 a $2e^{2x} - \frac{2}{x}$

b $2e^{2a} - \frac{2}{a} = 2 \Rightarrow 2ae^{2a} - 2 = 2a \Rightarrow a(e^{2a} - 1) = 1$

11 a $5\sin(3 \times 0) + 2\cos(3 \times 0) = 0 + 2 = 2 = y$

When $x = 0$, $y = 2$, therefore $(0, 2)$ lies on C.

b $y = -\frac{1}{15}x + 2$

12 $y = -\frac{1}{648\ln 3}x + \frac{1}{648\ln 3} + 162$

Challenge

$y = 3x - 2\ln 2 + 2$

Exercise 6C

1 a $8(1+2x)^3$ b $20x(3-2x^2)^{-6}$

c $2(3 + 4x)^{-\frac{1}{2}}$ d $7(6 + 2x)(6x + x^2)^6$

e $-\frac{2}{(3 + 2x)^2}$ f $-\frac{1}{2\sqrt{7 - x}}$

g $128(2 + 8x)^3$ h $18(8 - x)^{-7}$

2 a $-\sin x\, e^{\cos x}$ b $-2\sin(2x - 1)$ c $\frac{1}{2x\sqrt{\ln x}}$

d $5(\cos x - \sin x)(\sin x + \cos x)^4$

e $(6x - 2)\cos(3x^2 - 2x + 1)$

f $\cot x$ g $-8\sin 4x\, e^{\cos 4x}$ h $-2e^{2x}\sin(e^{2x} + 3)$

3 −1

4 $y = -54x + 81$

5 $12e^{-3}$

6 a $\frac{1}{2y + 1}$ b $\frac{1}{e^y + 4}$ c $\frac{1}{2}\sec 2y$ d $\frac{4y}{1 + 3y^3}$

7 $\frac{1}{10}$

8 $\frac{16}{3}$

9 a $e^y = \frac{dx}{dy}$

b $y = \ln x$, $e^y = x$

Differentiate with respect to y using part **a**

$e^y = \frac{dx}{dy} \Rightarrow \frac{1}{e^y} = \frac{dy}{dx}$

Since $x = e^y$, $\frac{dy}{dx} = \frac{1}{x}$

10 a $4\cos 2\left(\frac{\pi}{6}\right) = 4\left(\frac{1}{2}\right) = 2$

When $y = \frac{\pi}{6}$, $x = 2$, therefore $\left(2, \frac{\pi}{6}\right)$ lies on C.

b $\frac{dx}{dy} = -8\sin 2y$

At $Q\left(2, \frac{\pi}{6}\right)$: $\frac{dx}{dy} = -8\sin 2\left(\frac{\pi}{6}\right) = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$

So, $\frac{dy}{dx} = -\frac{1}{4\sqrt{3}}$

c $4\sqrt{3}\,x - y - 8\sqrt{3} + \frac{\pi}{6} = 0$

11 a $6\sin 3x\cos 3x$ b $2(x + 1)e^{(x+1)^2}$ c $-2\tan x$

d $\frac{2\sin 2x}{(3 + \cos 2x)^2}$ e $-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

12 $3125x - 100y - 9371 = 0$

13 $9\ln 3$

Challenge

a $\dfrac{\cos\sqrt{x}}{4\sqrt{x}\sin\sqrt{x}}$ **b** $9e^{\sin^3(3x+4)}\cos(3x+4)\sin^2(3x+4)$

Exercise 6D

1 **a** $(3x+1)^4(18x+1)$ **b** $2(3x^2+1)^2(21x^2+1)$
c $16x^2(x+3)^3(7x+9)$ **d** $3x(5x-2)(5x-1)^{-2}$

2 **a** $-4(x-3)(2x-1)^4\,e^{-2x}$
b $2\cos 2x\cos 3x - 3\sin 2x\sin 3x$
c $e^x(\sin x + \cos x)$ **d** $5\cos 5x\ln(\cos x) - \tan x\sin 5x$

3 **a** 52 **b** 13 **c** $\frac{3}{25}$

4 $(2, 0), \left(-\frac{1}{3}, \frac{343}{27}\right)$

5 $\dfrac{5\pi^4}{256}$

6 $\sqrt{2\pi}$ $(\pi-4)x + 8y - \pi\sqrt{2}\left(\dfrac{\pi-2}{2}\right) = 0$

7 $6x(5x-3)^3 + 3x^2[3(5x-3)^2(5)]$
$= 6x(5x-3)^3 + 45x^2(5x-3)^2$
$= 3x(5x-3)^2(2(5x-3)+15x) = 3x(5x-3)^2(10x-6+15x)$
$= 3x(5x-3)^2(25x-6) \Rightarrow n = 2, A = 3, B = 25, C = -6$

8 **a** $(x+3)(3x+11)e^{3x}$ **b** $85e^6$

9 **a** $(3\sin x + 2\cos x)\ln(3x) + \dfrac{2\sin x - 3\cos x}{x}$
b $x^3(7x+4)e^{7x-3}$

10 21.25

Challenge

a $-e^x\sin x(\sin^2 x - \cos x\sin x - 2\cos^2 x)$
b $-(4x-3)^5(4x-1)^8(256x^2-148x+3)$

Exercise 6E

1 **a** $\dfrac{5}{(x+1)^2}$ **b** $-\dfrac{4}{(3x-2)^2}$ **c** $-\dfrac{5}{(2x+1)^2}$
d $-\dfrac{6x}{(2x-1)^3}$ **e** $\dfrac{15x+18}{(5x+3)^{\frac{3}{2}}}$

2 **a** $\dfrac{e^{4x}(\sin x + 4\cos x)}{\cos^2 x}$ **b** $\dfrac{1}{x(x+1)} - \dfrac{\ln x}{(x+1)^2}$
c $\dfrac{e^{-2x}((2xe^{4x}-2x)\ln x - e^{4x} - 1)}{x(\ln x)^2}$
d $\dfrac{(e^x+3)^2((e^x+3)\sin x + 3e^x\cos x)}{\cos^2 x}$
e $\dfrac{2\sin x\cos x}{\ln x} - \dfrac{\sin^2 x}{x(\ln x)^2}$

3 $\dfrac{1}{16}$

4 $\dfrac{2}{25}$

5 $(0.5, 2e^4)$

6 $y = \frac{1}{3}e$

7 $\dfrac{6\sqrt{3} - 2\pi\ln\left(\frac{\pi}{9}\right)}{\pi}$

8 **a** $\left(\frac{1}{3}, 0\right)$ **b** $y = -\frac{1}{9}x + \frac{1}{27}$

9 $\dfrac{x^3(3x\sin 3x + 4\cos 3x)}{\cos^2 3x}$

10 **a** $\dfrac{(x-2)^2(2e^{2x}) - e^{2x}[2(x-2)]}{(x-2)^4} = \dfrac{2(x-2)^2e^{2x} - 2e^{2x}(x-2)}{(x-2)^4}$
$= \dfrac{2(x-2)e^{2x} - 2e^{2x}}{(x-2)^3} = \dfrac{2e^{2x}(x-2-1)}{(x-2)^3} = \dfrac{2e^{2x}(x-3)}{(x-2)^3}$
$A = 2, B = 1, C = 3$
b $y = 4e^2x - 3e^2$

11 **a** $\dfrac{2x}{x+5} + \dfrac{6x}{(x+5)(x+2)} = \dfrac{2x(x+2)}{(x+5)(x+2)} + \dfrac{6x}{(x+5)(x+2)}$
$= \dfrac{2x(x+2+3)}{(x+5)(x+2)} = \dfrac{2x(x+5)}{(x+5)(x+2)} = \dfrac{2x}{(x+2)}$
b $\dfrac{4}{25}$

12 **a** Using the quotient rule:
$f'(x) = \dfrac{e^{2-x}(-4\sin 2x) - 2\cos 2x(-e^{2-x})}{(e^{2-x})^2}$
At the turning points, $f'(x) = 0$
Thus, $e^{2-x}[(-4\sin 2x) + 2\cos 2x] = 0$
$\Rightarrow -4\sin 2x + 2\cos 2x = 0$
$-4\sin 2x = -2\cos 2x$
$\tan 2x = \frac{2}{4} = \frac{2}{4}$
b Range is $y \in \mathbb{R}$

Exercise 6F

1 **a** $3\sec^2 3x$ **b** $12\tan^2 x\sec^2 x$ **c** $\sec^2(x-1)$
d $\frac{1}{2}x^2\sec^2\frac{1}{2}x + 2x\tan\frac{1}{2}x + \sec^2\left(x - \frac{1}{2}\right)$

2 **a** $-4\,\text{cosec}^2 4x$ **b** $5\sec 5x\tan 5x$
c $-4\,\text{cosec}\,4x\cot 4x$ **d** $6\sec^2 3x\tan 3x$
e $\cot 3x - 3x\,\text{cosec}^2 3x$ **f** $\dfrac{\sec^2 x(2x\tan x - 1)}{x^2}$
g $-6\,\text{cosec}^3 2x\cot 2x$
h $-4\cot(2x-1)\,\text{cosec}^2(2x-1)$

3 **a** $\frac{1}{2}(\sec x)^{\frac{1}{2}}\tan x$ **b** $-\frac{1}{2}(\cot x)^{-\frac{1}{2}}\,\text{cosec}^2 x$
c $-2\,\text{cosec}^2 x\cot x$ **d** $2\tan x\sec^2 x$
e $3\sec^3 x\tan x$ **f** $-3\cot^2 x\,\text{cosec}^2 x$

4 **a** $2x\sec 3x + 3x^2\sec 3x\tan 3x$
b $\dfrac{2x\sec^2 2x - \tan 2x}{x^2}$ **c** $\dfrac{2x\tan x - x^2\sec^2 x}{\tan^2 x}$
d $e^x\sec 3x(1 + 3\tan 3x)$ **e** $\dfrac{\tan x - x\sec^2 x\ln x}{x\tan^2 x}$
f $e^{\tan x}\sec x(\tan x + \sec^2 x)$

5 **a** $\dfrac{1}{\cos^2 x} - \dfrac{1}{\sin^2 x}$ **b** 2
c $24x - 9y + 12\sqrt{3} - 8\pi = 0$

6 $y = \dfrac{1}{\cos x}, \dfrac{dy}{dx} = \dfrac{\cos x \times 0 - 1 \times -\sin x}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x}$
$= \sec x\tan x$

7 $y = \dfrac{1}{\tan x}$
$\dfrac{dy}{dx} = \dfrac{\tan x \times 0 - 1 \times \sec^2 x}{\tan^2 x} = -\dfrac{\sec^2 x}{\tan^2 x} = \dfrac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\,\text{cosec}^2 x$

8 **a** Let $y = \arccos x \Rightarrow \cos y = x \Rightarrow \dfrac{dx}{dy} = -\sin y$
$\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$

b Let $y = \arctan x$
Then, $\tan y = x$
$$\frac{dx}{dy} = \sec^2 y$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

9 **a** $\frac{-2}{\sqrt{1 - 4x^2}}$ **b** $\frac{2}{4 + x^2}$
c $\frac{3}{\sqrt{1 - 9x^2}}$ **d** $\frac{-1}{1 + x^2}$
e $\frac{1}{x\sqrt{x^2 - 1}}$ **f** $\frac{-1}{x\sqrt{x^2 - 1}}$
g $\frac{-1}{(x - 1)\sqrt{1 - 2x}}$ **h** $\frac{-2x}{\sqrt{1 - x^4}}$
i $e^x\left(\arccos x - \frac{1}{\sqrt{1 - x^2}}\right)$ **j** $\frac{\cos x}{\sqrt{1 - x^2}} - \sin x \arcsin x$
k $x\left(2 \arccos x - \frac{x}{\sqrt{1 - x^2}}\right)$ **l** $\frac{e^{\arctan x}}{1 + x^2}$

10 **a** $$\frac{dy}{dx} = \frac{x \times 2\frac{1}{1 + (2x)^2} - \arctan 2x}{x^2}$$
$$= \frac{\frac{2x}{1 + 4x^2} - \arctan 2x}{x^2} = \frac{2}{x(1 + 4x^2)} - \frac{\arctan 2x}{x^2}$$
$x = \frac{\sqrt{3}}{2}$, then
$$\frac{dy}{dx} = \frac{2}{\frac{\sqrt{3}}{2}\left(1 + 4\left(\frac{\sqrt{3}}{2}\right)^2\right)} - \frac{\arctan 2\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{2}{2\sqrt{3}} - \frac{4\pi}{9} = \frac{\sqrt{3}}{3} - \frac{4\pi}{9} = \frac{3\sqrt{3} - 4\pi}{9}$$

b $x = \frac{\sqrt{3}}{2}$, $y = \frac{2\pi\sqrt{3}}{9}$
Equation of normal:
$$y - \frac{2\pi\sqrt{3}}{9} = -\frac{9}{3\sqrt{3} - 4\pi}\left(x - \frac{\sqrt{3}}{2}\right)$$
$$y = -\frac{9}{3\sqrt{3} - 4\pi}x + \frac{9\sqrt{3}}{6\sqrt{3} - 8\pi} + \frac{2\pi\sqrt{3}}{9}$$

11 $$\frac{dx}{dy} = 2 \arccos y \times -\frac{1}{\sqrt{1 - y^2}} = -\frac{2 \arccos y}{\sqrt{1 - y^2}}$$
$$\frac{dy}{dx} = -\frac{\sqrt{1 - y^2}}{2 \arccos y} = -\frac{\sqrt{1 - \cos^2\sqrt{x}}}{2\sqrt{x}}$$

12 **a** $\frac{-1}{5 \cot 5y \operatorname{cosec} 5y}$ **b** $-\frac{1}{5x\sqrt{x^2 - 1}}$

Chapter review 6

1 **a** $\frac{2}{x}$ **b** $2x \sin 3x + 3x^2 \cos 3x$

2 **a** $2\frac{dy}{dx} = 1 - \sin x \frac{d}{dx}(\cos x) - \frac{d}{dx}(\sin x) \cos x$
$= 1 + \sin^2 x - \cos^2 x = 2 \sin^2 x$
So $\frac{dy}{dx} = \sin^2 x$

b $\left(\frac{\pi}{2}, \frac{\pi}{4}\right), \left(\pi, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)$

3 **a** $\frac{x \cos x - \sin x}{x^2}$ **b** $-\frac{2x}{x^2 + 9}$

4 **a** $k = \sqrt{2}$ **b** $(0, 0), \left(\pm\sqrt{6}, \pm\frac{\sqrt{3}}{4\sqrt{2}}\right)$

5 **a** $x > 0$ **b** $(\sqrt[3]{256}, 32 \ln 2 + 16)$

6 $\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right), \left(\frac{3\pi}{2}, -1\right)$

7 Maximum is when $\frac{dy}{dx} = 0$
$$\frac{dy}{dx} = \sqrt{\sin x} + x\left(\cos x \times \frac{1}{2\sqrt{\sin x}}\right) = \frac{2 \sin x + x \cos x}{2\sqrt{\sin x}} = 0$$
So $2 \sin x + x \cos x = 0 \Rightarrow 2 \sin x = -x \cos x \Rightarrow 2 \tan x = -x$
$\therefore 2 \tan x + x = 0$

8 **a** $f'(x) = 0.5e^{0.5x} - 2x$
b $f'(6) = -1.957\ldots < 0$, $f'(7) = 2.557\ldots > 0$
So there exists $p \in [6, 7]$ such that $f'(p) = 0$
$\therefore$ there is a stationary point for some $x = p$, $6 < p < 7$

9 **a** $\left(\frac{3\pi}{8}, \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}\right), \left(\frac{7\pi}{8}, -\frac{e^{\frac{7\pi}{4}}}{\sqrt{2}}\right)$
b $f''(x) = 2e^{2x}(-2 \sin 2x + 2 \cos 2x) + 4e^{2x}(\cos 2x + \sin 2x)$
$= 4e^{2x}(-\sin 2x + \cos 2x + \cos 2x + \sin 2x)$
$= 8e^{2x} \cos 2x$
c $\left(\frac{3\pi}{8}, \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}\right)$ is a maximum; $\left(\frac{7\pi}{8}, -\frac{e^{\frac{7\pi}{4}}}{\sqrt{2}}\right)$ is a minimum.
d $\left(\frac{\pi}{4}, e^{\frac{\pi}{2}}\right), \left(\frac{3\pi}{4}, -e^{\frac{3\pi}{2}}\right)$

10 $x + 2y - 8 = 0$

11 **a** $x = \frac{1}{3}$ **b** $y = -\frac{1}{2}x + 1\frac{1}{2}$

12 **a** $f'(x) = e^{2x}(2 \cos x - \sin x)$
$2 \cos x - \sin x = 0 \Rightarrow \tan x = 2$
b $y = 2x + 1$

13 **a** $y + 2y \ln y$ **b** $\frac{1}{3e}$

14 **a** $e^{-x}(-x^3 + 3x^2 + 2x - 2)$
b $f'(0) = -2 \Rightarrow$ gradient of normal $= \frac{1}{2}$
Equation of normal is $y = \frac{1}{2}x$
$(x^3 - 2x)e^{-x} = \frac{1}{2}x \Rightarrow 2x^3 - 4x = xe^x \Rightarrow 2x^2 = e^x + 4$

Challenge

a $1 + x + (1 + 2x) \ln x$
b $1 + x + (1 + 2x) \ln x = 0 \Rightarrow x = e^{-\frac{1 + x}{1 + 2x}}$

CHAPTER 7

Prior knowledge check

1 **a** $12(2x - 7)^5$ **b** $5 \cos 5x$ **c** $\frac{1}{3}e^{\frac{x}{3}}$

2 **a** $y = \frac{16}{3}x^{\frac{3}{2}} - 12x^{\frac{1}{2}}$ **b** $\frac{268}{3}$

3 6 units^2

Exercise 7A

1 **a** $3 \tan x + 5 \ln |x| - \frac{2}{x} + c$ **b** $5e^x + 4 \cos x + \frac{x^4}{2} + c$
c $-2 \cos x - 2 \sin x + x^2 + c$ **d** $3 \sec x - 2 \ln |x| + c$
e $5e^x + 4 \sin x + \frac{2}{x} + c$ **f** $\frac{1}{2} \ln |x| - 2 \cot x + c$
g $\ln |x| - \frac{1}{x} - \frac{1}{2x^2} + c$ **h** $e^x - \cos x + \sin x + c$
i $-2 \operatorname{cosec} x - \tan x + c$ **j** $e^x + \ln |x| + \cot x + c$

2 **a** $\tan x - \frac{1}{x} + c$ **b** $\sec x + 2e^x + c$
c $-\cot x - \operatorname{cosec} x - \frac{1}{x} + \ln|x| + c$
d $-\cot x + \ln|x| + c$ **e** $-\cos x + \sec x + c$
f $\sin x - \operatorname{cosec} x + c$ **g** $-\cot x + \tan x + c$
h $\tan x + \cot x + c$ **i** $\tan x + e^x + c$
j $\tan x + \sec x + \sin x + c$

3 **a** $2e^7 - 2e^3$ **b** $\frac{95}{72}$ **c** -5 **d** $2 - \sqrt{2}$

4 $a = 2$

5 $a = 7$

6 $b = 2$

7 **a** $x = 4$ **b** $\frac{1}{20}x^{\frac{5}{2}} - 4\ln|x| + c$
c $\frac{31}{20} - 4\ln 4$

Exercise 7B

1 **a** $-\frac{1}{2}\cos(2x+1) + c$ **b** $\frac{3}{2}e^{2x} + c$
c $4e^{x+5} + c$ **d** $-\frac{1}{2}\sin(1-2x) + c$
e $-\frac{1}{3}\cot 3x + c$ **f** $\frac{1}{4}\sec 4x + c$
g $-6\cos\left(\frac{1}{2}x + 1\right) + c$ **h** $-\tan(2-x) + c$
i $-\frac{1}{2}\operatorname{cosec} 2x + c$ **j** $\frac{1}{3}(\sin 3x + \cos 3x) + c$

2 **a** $\frac{1}{2}e^{2x} + \frac{1}{4}\cos(2x-1) + c$ **b** $\frac{1}{2}e^{2x} + 2e^x + x + c$
c $\frac{1}{2}\tan 2x + \frac{1}{2}\sec 2x + c$
d $-6\cot\left(\frac{x}{2}\right) + 4\operatorname{cosec}\left(\frac{x}{2}\right) + c$
e $-e^{3-x} + \cos(3-x) - \sin(3-x) + c$

3 **a** $\frac{1}{2}\ln|2x+1| + c$ **b** $-\frac{1}{2(2x+1)} + c$
c $\frac{(2x+1)^3}{6} + c$ **d** $\frac{3}{4}\ln|4x-1| + c$
e $-\frac{3}{4}\ln|1-4x| + c$ **f** $\frac{3}{4(1-4x)} + c$
g $\frac{(3x+2)^6}{18} + c$ **h** $\frac{3}{4(1-2x)^2} + c$

4 **a** $-\frac{3}{2}\cos(2x+1) + 2\ln|2x+1| + c$
b $\frac{1}{5}e^{5x} - \frac{(1-x)^6}{6} + c$
c $-\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| - \frac{1}{2(1+2x)} + c$
d $\frac{(3x+2)^3}{9} - \frac{1}{3(3x+2)} + c$

5 **a** 1 **b** $\frac{7}{4}$ **c** $\frac{2\sqrt{3}}{9}$ **d** $\frac{5}{2}\ln 3$

6 $b = 6$

7 $k = 24$

Challenge
$a = 4, b = -3$ or $a = 8, b = -6$

Exercise 7C

1 **a** $-\cot x - x + c$ **b** $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$
c $-\frac{1}{8}\cos 4x + c$ **d** $\frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + c$
e $\frac{1}{3}\tan 3x - x + c$ **f** $-2\cot x - x + 2\operatorname{cosec} x + c$
g $x - \frac{1}{2}\cos 2x + c$ **h** $\frac{1}{8}x - \frac{1}{32}\sin 4x + c$
i $-2\cot 2x + c$ **j** $\frac{3}{2}x + \frac{1}{8}\sin 4x - \sin 2x + c$

2 **a** $\tan x - \sec x + c$ **b** $-\cot x - \operatorname{cosec} x + c$
c $2x - \tan x + c$ **d** $-\cot x - x + c$
e $-2\cot x - x - 2\operatorname{cosec} x + c$
f $-\cot x - 4x + \tan x + c$ **g** $x + \frac{1}{2}\cos 2x + c$
h $-\frac{3}{2}x + \frac{1}{4}\sin 2x + \tan x + c$ **i** $-\frac{1}{2}\operatorname{cosec} 2x + c$

3 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$
$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{8} + \frac{1}{4} = \frac{2+\pi}{8}$

4 **a** $\frac{4\sqrt{3}}{3}$ **b** $\frac{9\sqrt{3} - 10 - \pi}{8}$ **c** $2\sqrt{2} - \frac{\pi}{4}$ **d** $\frac{\sqrt{2}-1}{2}$

5 **a** $\sin(3x+2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$
$\sin(3x-2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$
Adding gives $\sin 5x + \sin x = 2\sin 3x \cos 2x$
b So $\int \sin 3x \cos 2x \, dx = \int \frac{1}{2}(\sin 5x + \sin x)\, dx$
$= \frac{1}{2}(-\frac{1}{5}\cos 5x - \cos x) + c = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$

6 **a** $5\sin^2 x + 7\cos^2 x = 5 + 2\cos^2 x$
$= 6 + (2\cos^2 x - 1)$
$= \cos 2x + 6$
b $\frac{1}{2}(1 + 3\pi)$

7 **a** $\cos^4 x = (\cos^2 x)^2$
$= \left(\frac{1+\cos 2x}{2}\right)^2$
$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$
$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)$
$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$
b $\frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + c$

Exercise 7D

1 **a** $\frac{1}{2}\ln|x^2+4| + c$ **b** $\frac{1}{2}\ln|e^{2x}+1| + c$
c $-\frac{1}{4}(x^2+4)^{-2} + c$ **d** $-\frac{1}{4}(e^{2x}+1)^{-2} + c$
e $\frac{1}{2}\ln|3+\sin 2x| + c$ **f** $\frac{1}{4}(3+\cos 2x)^{-2} + c$
g $\frac{1}{2}e^{x^2} + c$ **h** $\frac{1}{10}(1+\sin 2x)^5 + c$
i $\frac{1}{3}\tan^3 x + c$ **j** $\tan x + \frac{1}{3}\tan^3 x + c$

2 **a** $\frac{1}{10}(x^2+2x+3)^5 + c$ **b** $-\frac{1}{4}\cot^2 2x + c$
c $\frac{1}{18}\sin^6 3x + c$ **d** $e^{\sin x} + c$
e $\frac{1}{2}\ln|e^{2x}+3| + c$ **f** $\frac{1}{5}(x^2+1)^{\frac{5}{2}} + c$
g $\frac{2}{3}(x^2+x+5)^{\frac{3}{2}} + c$ **h** $2(x^2+x+5)^{\frac{1}{2}} + c$
i $-\frac{1}{2}(\cos 2x+3)^{\frac{1}{2}} + c$ **j** $-\frac{1}{4}\ln|\cos 2x+3| + c$

3 **a** 468 **b** $2\ln 3$ **c** $\frac{1}{2}\ln\left(\frac{16}{5}\right)$ **d** $\frac{1}{4}(e^4 - 1)$

4 $k = 2$

5 $\theta = \frac{\pi}{2}$

6 **a** $\ln|\sin x| + c$
b $\int \tan x \, dx = -\ln|\cos x| + c$
$= \ln\left|\frac{1}{\cos x}\right| + c$
$= \ln|\sec x| + c$

Chapter review 7

1 **a** $\frac{1}{16}(2x-3)^8 + c$ **b** $\frac{1}{40}(4x-1)^{\frac{5}{2}} + \frac{1}{24}(4x-1)^{\frac{3}{2}} + c$
c $\frac{1}{3}\sin^3 x + c$ **d** $\frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$

e $-\frac{1}{4}\ln|\cos 2x| + c$ f $-\frac{1}{4}\ln|3 - 4x| + c$

2 a $-\frac{995\,085}{4}$ b $\frac{1}{4}\pi - \frac{1}{2}\ln 2$ c $\frac{992}{5} - 2\ln 4$

d $\frac{\sqrt{3} - 1}{4}$ e $\frac{1}{4}\ln\left(\frac{35}{19}\right)$ f $\ln\left(\frac{4}{3}\right)$

3 a $\int \frac{1}{x^2}\ln x\,dx = (\ln x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$

$= -\frac{\ln x}{x} + \int\frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + c$

$\int_1^e \frac{1}{x^2}\ln x\,dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^e = \left(-\frac{1}{e} - \frac{1}{e}\right) - (0 - 1) = 1 - \frac{2}{e}$

b $\frac{1}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1} \Rightarrow A = -\frac{1}{3}, B = \frac{2}{3}$

$\int_1^p \frac{1}{(x+1)(2x-1)}\,dx = \int_1^p\left(-\frac{1}{3(x+1)} + \frac{2}{3(2x-1)}\right)dx$

$= \left[-\frac{1}{3}\ln(x+1) + \frac{1}{3}\ln(2x-1)\right]_1^p = \left[\frac{1}{3}\ln\left(\frac{2x-1}{x+1}\right)\right]_1^p$

$= \left(\frac{1}{3}\ln\left(\frac{2p-1}{p+1}\right)\right) - \left(\frac{1}{3}\ln\left(\frac{1}{2}\right)\right)$

$= \frac{1}{3}\ln\left(\frac{2(2p-1)}{p+1}\right) = \frac{1}{3}\ln\left(\frac{4p-2}{p+1}\right)$

4 $b = 2$

5 $\theta = \frac{\pi}{3}$

Challenge

$k = \frac{1}{2}$

CHAPTER 8

Prior knowledge check

1 a 3.25 b 11.24

2 a $f'(x) = \frac{3}{2\sqrt{x}} + 8x + \frac{15}{x^4}$ b $f'(x) = \frac{5}{x+2} - 7e^{-x}$

c $f'(x) = x^2\cos x + 2x\sin x + 4\sin x$

3 $u_1 = 2, u_2 = 2.5, u_3 = 2.9$

Exercise 8A

1 a $f(-2) = -1 < 0, f(-1) = 5 > 0$
Sign change implies root.

b $f(3) = -2.732 < 0, f(4) = 4 > 0$
Sign change implies root.

c $f(-0.5) = -0.125 < 0, f(-0.2) = 2.992 > 0$
Sign change implies root.

d $f(1.65) = -0.294 < 0, f(1.75) = 0.195 > 0$
Sign change implies root.

2 a $f(1.8) = 0.408 > 0, f(1.9) = -0.249$
Sign change implies root.

b $f(1.8635) = 0.0013 > 0, f(1.8645) = -0.0053 < 0$
Sign change implies root.

3 a $h(1.4) = -0.0512\ldots < 0, h(1.5) = 0.0739\ldots > 0$
Sign change implies root.

b $h(1.4405) = -0.0005 < 0, h(1.4415) = 0.0006 > 0$
Sign change implies root.

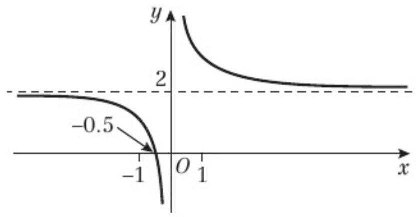
4 a $f(2.2) = 0.020 > 0, f(2.3) = -0.087$
Sign change implies root.

b $f(2.2185) = 0.00064\ldots > 0, f(2.2195) = -0.00041\ldots < 0$
There is a sign change in the interval.
$2.2185 < x < 2.2195$, so $\alpha = 2.219$ correct to 3 decimal places.

5 a $f(1.5) = 16.10\ldots > 0, f(1.6) = -32.2\ldots < 0$
Sign change implies root.

b There is an asymptote in the graph of $y = f(x)$ at $x = \frac{\pi}{2} \approx 1.57$. So there is not a root in this interval.

6



Alternatively: $\frac{1}{x} + 2 = 0 \Rightarrow \frac{1}{x} = -2 \Rightarrow x = -\frac{1}{2}$

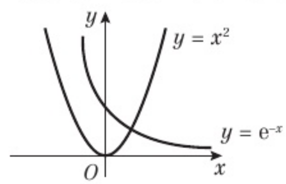
7 a $f(0.2) = -0.4421\ldots, f(0.8) = -0.1471\ldots$

b There are either no roots or an even number of roots in the interval $0.2 < x < 0.8$

c $f(0.3) = 0.01238\ldots > 0, f(0.4) = -0.1114\ldots < 0, f(0.5) = -0.2026\ldots < 0, f(0.6) = 0, f(0.7) = -0.2710\ldots > 0$

d There exists at least one root in the interval $0.2 < x < 0.3$, $0.3 < x < 0.4$ and $0.7 < x < 0.8$
Additionally $x = 0.6$ is a root. Therefore there are at least four roots in the interval $0.2 < x < 0.8$

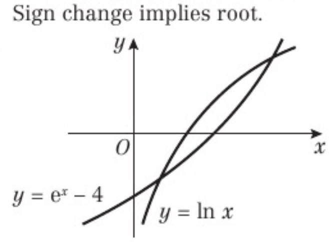
8 a



b One point of intersection, so one root.

c $f(0.7) = 0.0065\ldots > 0, f(0.71) = -0.0124\ldots < 0$
Sign change implies root.

9 a



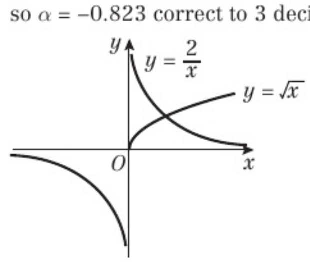
b 2

c $f(x) = \ln x - e^x + 4$. $f(1.4) = 0.2812\ldots < 0$, $f(1.5) = -0.0762\ldots < 0$. Sign change implies root.

10 a $h'(x) = 2\cos 2x + 4e^{4x}$, $h'(-0.9) = -0.3451\ldots < 0$, $h'(-0.8) = 0.1046\ldots > 0$. Sign change implies slope changes from decreasing to increasing over interval, which implies turning point.

b $h'(-0.8235) = -0.003839\ldots. < 0$, $h'(-0.8225) = 0.00074\ldots > 0$. Sign change implies α lies in the range $-0.8235 < \alpha < -0.8225$, so $\alpha = -0.823$ correct to 3 decimal places.

11 a



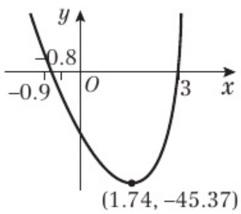
b 1 point of intersection $\Rightarrow$ 1 root

c $f(1) = -1, f(2) = 0.414\ldots$ d $p = 3, q = 4$ e $4^{\frac{1}{3}}$

12 a f(–0.9) = 1.5561 > 0, f(–0.8) = –0.7904 < 0
There is a change of sign in the interval [–0.9, –0.8], so there is at least one root in this interval.

b (1.74, –45.37) to 2 d.p. **c** $a = 3, b = 9$ and $c = 6$

d



Exercise 8B

1 a i $x^2 - 6x + 2 = 0 \Rightarrow 6x = x^2 + 2 \Rightarrow x = \dfrac{x^2+2}{6}$

ii $x^2 - 6x + 2 = 0 \Rightarrow x^2 = 6x - 2 \Rightarrow x = \sqrt{6x-2}$

iii $x^2 - 6x + 2 = 0 \Rightarrow x - 6 + \dfrac{2}{x} = 0 \Rightarrow x = 6 - \dfrac{2}{x}$

b i $x = 0.354$ **ii** $x = 5.646$ **iii** $x = 5.646$

c $a = 3, b = 7$

2 a i $x^2 - 5x - 3 = 0 \Rightarrow x^2 = 5x + 3 \Rightarrow x = \sqrt{5x+3}$

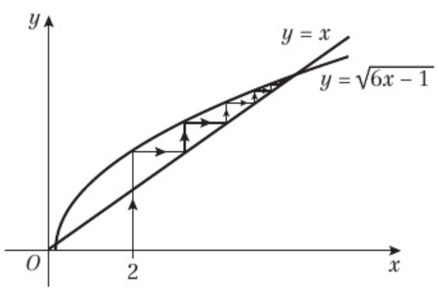
ii $x^2 - 5x - 3 = 0 \Rightarrow x^2 - 3 = 5x \Rightarrow x = \dfrac{x^2-3}{5}$

b i 5.5 (1 d.p.) **ii** –0.5 (1 d.p.)

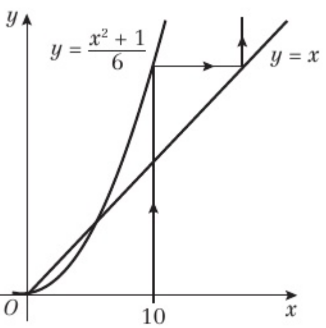
3 a $x^2 - 6x + 1 = 0 \Rightarrow x^2 = 6x - 1 \Rightarrow x = \sqrt{6x-1}$

c The graph shows there are two roots of f(x) = 0

b, d



e



4 a $xe^{-x} - x + 2 = 0 \Rightarrow e^{-x} = \dfrac{x-2}{x} \Rightarrow e^x = \dfrac{x}{x-2}$

$\Rightarrow x = \ln\left|\dfrac{x}{x-2}\right|$

b $x_1 = -1.10,\ x_2 = -1.04,\ x_3 = -1.07$

5 a i $x^3 + 5x^2 - 2 = 0 \Rightarrow x^3 = 2 - 5x^2 \Rightarrow x = \sqrt[3]{2-5x^2}$

ii $x^3 + 5x^2 - 2 = 0 \Rightarrow x + 5 - \dfrac{2}{x^2} = 0 \Rightarrow x = \dfrac{2}{x^2} - 5$

iii $x^3 + 5x^2 - 2 = 0 \Rightarrow 5x^2 = 2 - x^3 \Rightarrow x^2 = \dfrac{2-x^3}{5}$

$\Rightarrow x = \sqrt{\dfrac{2-x^3}{5}}$

b $x = -4.917$ **c** $x = 0.598$

d It is not possible to take the square root of a negative number over ℝ.

6 a $x^4 - 3x^3 - 6 = 0 \Rightarrow \dfrac{1}{3}x^4 - x^3 - 2 = 0$

$\Rightarrow \dfrac{1}{3}x^4 - 2 = x^3 \Rightarrow x = \sqrt[3]{\dfrac{1}{3}x^4 - 2} \Rightarrow p = \dfrac{1}{3}, q = -2$

b $x_1 = -1.256, x_2 = -1.051, x_3 = -1.168$

c f(–1.1315) = –0.014… < 0, f(–1.1325) = 0.0024… > 0
There is a sign change in this interval, which implies $\alpha = -1.132$ correct to 3 decimal places.

7 a $3\cos(x^2) + x - 2 = 0 \Rightarrow \cos(x^2) = \dfrac{2-x}{3}$

$\Rightarrow x^2 = \arccos\left(\dfrac{2-x}{3}\right) \Rightarrow x = \left[\arccos\left(\dfrac{2-x}{3}\right)\right]^{\frac{1}{2}}$

b $x_1 = 1.109, x_2 = 1.127, x_3 = 1.129$

c f(1.12975) = 0.000423… > 0,
f(1.12985) = –0.0001256… < 0. There is a sign change in this interval, which implies $\alpha = 1.1298$ correct to 4 decimal places.

8 a f(0.8) = 0.484…, f(0.9) = –1.025… There is a change of sign in the interval, so there must exist a root in the interval, since f is continuous over the interval.

b $\dfrac{4\cos x}{\sin x} - 8x + 3 = 0 \Rightarrow 8x = \dfrac{4\cos x}{\sin x} + 3$

$\Rightarrow x = \dfrac{\cos x}{2\sin x} + \dfrac{3}{8}$

c $x_1 = 0.8142,\ x_2 = 0.8470,\ x_3 = 0.8169$

d f(0.8305) = 0.0105… > 0, f(0.8315) = –0.0047… < 0
There is a change of sign in the interval, so there must exist a root in the interval.

9 a $e^{x-1} + 2x - 15 = 0 \Rightarrow e^{x-1} = 15 - 2x$

$\Rightarrow x - 1 = \ln(15 - 2x)$

$\Rightarrow x = \ln(15 - 2x) + 1$

b $x_1 = 3.1972,\ x_2 = 3.1524,\ x_3 = 3.1628$

c f(3.155) = –0.062… < 0, f(3.165) = 0.044… > 0
There is a sign change in this interval, which implies $\alpha = 3.16$ correct to 2 decimal places.

10 a $A(0, 0)$ and $B(\ln 4, 0)$

b $f'(x) = xe^x + e^x - 4 = e^x(x + 1) - 4$

c f′(0.7) = –0.5766... < 0, f′(0.8) = 0.0059… > 0
There is a sign change in this interval, which implies f′(x) = 0 in this range. f′(x) = 0 at a turning point.

d $e^x(x + 1) - 4 = 0 \Rightarrow e^x = \dfrac{4}{x+1} \Rightarrow x = \ln\left(\dfrac{4}{x+1}\right)$

e $x_1 = 1.386, x_2 = 0.517, x_3 = 0.970, x_4 = 0.708$

Chapter review 8

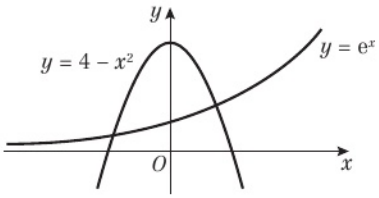
1 a $x^3 - 6x - 2 = 0 \Rightarrow x^3 = 6x + 2$

$\Rightarrow x^2 = 6 + \dfrac{2}{x} \Rightarrow x = \pm\sqrt{6 + \dfrac{2}{x}};\ a = 6, b = 2$

b $x_1 = 2.6458, x_2 = 2.5992, x_3 = 2.6018, x_4 = 2.6017$

c $f(2.6015) = (2.6015)^3 - 6(2.6015) - 2 = -0.0025\ldots < 0$
$f(2.6025) = (2.6025)^3 - 6(2.6025) - 2 = 0.0117 > 0$
There is a sign change in the interval $2.6015 < x < 2.6025$, so this implies there is a root in the interval.

2 **a**



b 2 roots: 1 positive and 1 negative

c $x^2 + e^x - 4 = 0 \Rightarrow x^2 = 4 - e^x \Rightarrow x = \pm(4 - e^x)^{\frac{1}{2}}$

d $x_1 = -1.9659$, $x_2 = -1.9647$, $x_3 = -1.9646$, $x_4 = -1.9646$

e You would need to take the square root of a negative number.

3 **a** $g(1) = -10 < 0$, $g(2) = 16 > 0$. The sign change implies there is a root in this interval.

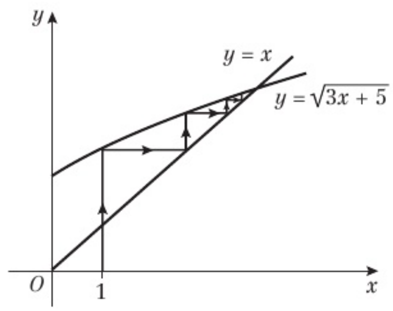
b $g(x) = 0 \Rightarrow x^5 - 5x - 6 = 0$
$\Rightarrow x^5 = 5x + 6 \Rightarrow x = (5x + 6)^{\frac{1}{5}}$
$p = 5$, $q = 6$, $r = 5$

c $x_1 = 1.6154$, $x_2 = 1.6971$, $x_3 = 1.7068$

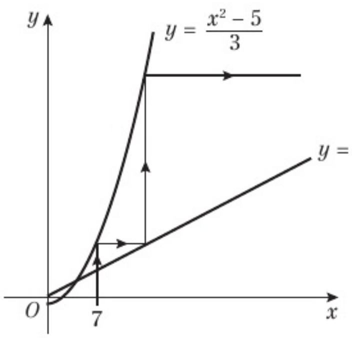
d $g(1.7075) = -0.0229\ldots < 0$, $g(1.7085) = 0.0146\ldots > 0$
The sign change implies there is a root in this interval.

4 **a** $g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$
$\Rightarrow x^2 = 3x + 5 \Rightarrow x = \sqrt{3x + 5}$

b, c



d

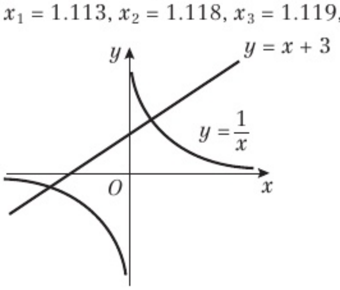


5 **a** $f(1.1) = -0.0648\ldots < 0$, $f(1.15) = 0.0989\ldots > 0$
The sign change implies there is a root in this interval.

b $5x - 4\sin x - 2 = 0 \Rightarrow 5x = 4\sin x + 2$
$\Rightarrow x = \frac{4}{5}\sin x + \frac{2}{5} \Rightarrow p = \frac{4}{5}$, $q = \frac{2}{5}$

c $x_1 = 1.113$, $x_2 = 1.118$, $x_3 = 1.119$, $x_4 = 1.120$

6 **a**



b 2

c $\frac{1}{x} = x + 3 \Rightarrow 0 = x + 3 - \frac{1}{x}$, let $f(x) = x + 3 - \frac{1}{x}$
$f(0.30) = -0.0333\ldots < 0$, $f(0.31) = 0.0841\ldots > 0$
Sign change implies root.

d $\frac{1}{x} = x + 3 \Rightarrow 1 = x^2 + 3x \Rightarrow 0 = x^2 + 3x - 1$

e 0.303

Challenge

a $f(x) = x^6 + x^3 - 7x^2 - x + 3$
$f'(x) = 6x^5 + 3x^2 - 14x - 1$
$f''(x) = 30x^4 + 6x - 14$
$f''(x) = 0 \Rightarrow 15x^4 + 3x - 7 = 0$

i $15x^4 + 3x - 7 = 0 \Rightarrow 3x = 7 - 15x^4 \Rightarrow x = \frac{7 - 15x^4}{3}$

ii $15x^4 + 3x - 7 = 0 \Rightarrow 15x^4 + 3x = 7$
$\Rightarrow x(15x^3 + 3) = 7 \Rightarrow x = \frac{7}{15x^3 + 3}$

iii $15x^4 + 3x - 7 = 0 \Rightarrow 15x^4 = 7 - 3x$
$\Rightarrow x^4 = \frac{7 - 3x}{15} \Rightarrow x = \sqrt[4]{\frac{7 - 3x}{15}}$

b Using formula **iii**, root = 0.750 (3 d.p.)

c Formula **iii** gives the positive fourth root, so cannot be used to find a negative root.

Review exercise 2

1 **a** $k = -1$, $A(0, 2)$

b ln 3

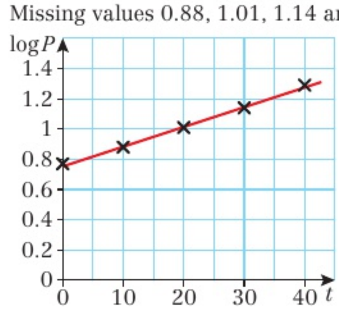
2 **a** 425 °C **b** 7.49 minutes **c** 1.64 °C/minute

d The temperature can never go below 25 °C.

3 **a** $x = 2$ **b** $x = \ln 3$ or $x = \ln 1 = 0$

4 **a** Missing values 0.88, 1.01, 1.14 and 1.29

b



c $P = ab^t$
$\log P = \log(ab^t) = \log a + t\log b$
This is a linear relationship. The gradient is $\log b$ and the intercept is $\log a$

d $a = 5.9$, $b = 1.0$

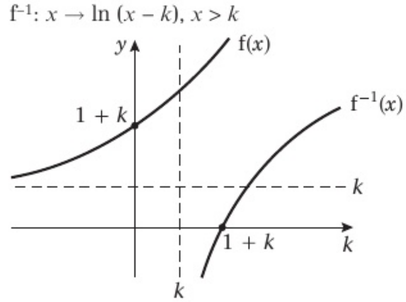
5 **a** $\frac{e^x + 2}{5}$ **b** $x \in \mathbb{R}$ **c** 1.878

6 **a** $f(x) > k$

b $2k$

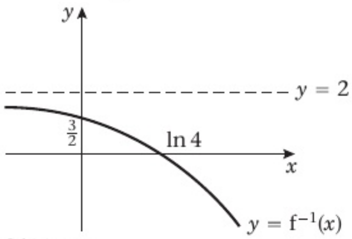
c $f^{-1}: x \to \ln(x - k)$, $x > k$

d



7 **a** $f^{-1}: x \to \frac{4 - e^x}{2}$

b

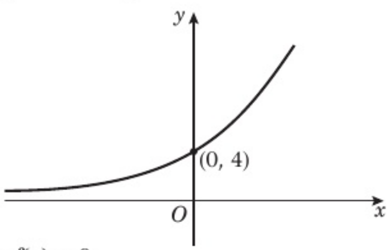


c $f^{-1}(x) < 2$

d 3

8 **a** $gf: x \to 4e^{4x}, x \in \mathbb{R}$

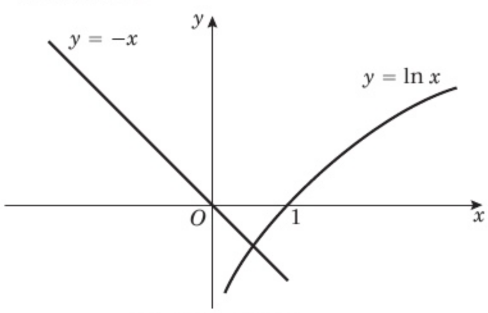
b



c $gf(x) \geqslant 0$

d −0.418 (3 s.f.)

9 **a**



b $x = -\ln x \Rightarrow \frac{2x - \ln x}{3} = \frac{2x + x}{3} = x$

c 0.56714 (5 d.p.)

10 $\frac{dy}{dx} = x - 4\sin x$

$x = \frac{\pi}{2}, \frac{dy}{dx} = \frac{\pi}{2} - 4, y = \frac{\pi^2}{8}, m_n = -\frac{1}{\frac{\pi}{2} - 4}$

$y - \frac{\pi^2}{8} = -\frac{1}{\frac{\pi}{2} - 4}\left(x - \frac{\pi}{2}\right)$

$\Rightarrow 8y(8 - \pi) - 16x + \pi(\pi^2 - 8\pi + 8) = 0$

11 $\frac{dy}{dx} = 3e^{3x} - \frac{2}{x}, x = 2, y = e^6 - \ln 4, \frac{dy}{dx} = 3e^6 - 1$

$y - e^6 + \ln 4 = (3e^6 - 1)(x - 2)$

$\Rightarrow y - (3e^6 - 1)x - 2 + \ln 4 + 5e^6 = 0$

12 **a** $\frac{dy}{dx} = 4(2x - 3)(e^{2x}) + 2(2x - 3)^2(e^{2x})$

$= 2(e^{2x})(2x - 3)(2x - 1)$

b $\left(\frac{3}{2}, 0\right)$ and $\left(\frac{1}{2}, 4e\right)$

13 **a** $\frac{dy}{dx} = \frac{(x - 1)(2\sin x + \cos x - x\cos x)}{\sin^2 x}$

b $x = \frac{\pi}{2}, y = \left(\frac{\pi}{2} - 1\right)^2, \frac{dy}{dx} = 2\left(\frac{\pi}{2} - 1\right)$

$y - \left(\frac{\pi}{2} - 1\right)^2 = (\pi - 2)\left(x - \frac{\pi}{2}\right)$

$\Rightarrow y = (\pi - 2)x + \left(1 - \frac{\pi^2}{4}\right)$

14 **a** $y = \operatorname{cosec} x = \frac{1}{\sin x}$

$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$

b $\frac{dy}{dx} = -\frac{1}{6x\sqrt{x^2 - 1}}$

15 $y = \arcsin x \Rightarrow x = \sin y$

$\frac{dx}{dy} = \cos y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

16 $a = 1$

17 **a** $\cos 7x + \cos 3x = \cos(5x + 2x) + \cos(5x - 2x)$

$= \cos 5x \cos 2x - \sin 5x \sin 2x + \cos 5x \cos 2x +$
$\sin 5x \sin 2x = 2\cos 5x \cos 2x$

b $\frac{3}{7}\sin 7x + \sin 3x + c$

18 $m = 3$

19 **a** $A = \frac{1}{2}, B = 2, C = -1$

b $\frac{1}{2}\ln|x| + 2\ln|x - 1| = \frac{1}{x - 1} + c$

c $\int_4^9 f(x)\,dx = \left[\frac{1}{2}\ln|x| = 2\ln|x - 1| + \frac{1}{x - 1}\right]_4^9$

$= \left(\frac{1}{2}\ln 9 + 2\ln 8 + \frac{1}{8}\right) - \left(\frac{1}{2}\ln 4 + 2\ln 3 + \frac{1}{3}\right)$

$= \left(\ln 3 + \ln 64 + \frac{1}{8}\right) - \left(\ln 2 + \ln 9 + \frac{1}{3}\right)$

$= \ln\left(\frac{3 \times 64}{2 \times 9}\right) - \frac{5}{24} = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$

20 **a** $\frac{5x + 3}{(2x - 3)(x - 2)} \equiv \frac{3}{2x - 3} + \frac{1}{x + 2}$

b ln 54

21 $\frac{1}{9}(2e^3 + 10)$

22 **a** g(1.4) = −0.216 < 0, g(1.5) = 0.125 > 0
Sign change implies root.

b g(1.4655) = −0.00025… < 0
g(1.4655) = 0.00326… > 0
Sign change implies root.

23 **a** p(1.7) = 0.0538… > 0, p(1.8) = 0.0619… < 0
Sign change implies root.

b p(1.7455) = 0.00074… > 0
p(1.7465) = −0.00042… < 0
Sign change implies root.

24 **a** $e^{x-2} - 3x + 5 = 0 \Rightarrow e^{x-2} = 3x - 5$

$\Rightarrow x - 2 = \ln(3x - 5) \Rightarrow x = \ln(3x - 5) + 2$

b $x_0 = 4, x_1 = 3.9459, x_2 = 3.9225, x_3 = 3.9121$

25 **a** f(0.2) = −0.01146… < 0, f(0.3) = 0.1564… > 0
Sign change implies root.

b $\frac{1}{(x - 2)^3} + x^2 = 0 \Rightarrow \frac{1}{(x - 2)^3} = -4x^2$

$\Rightarrow \frac{-1}{4x^2} = (x - 2)^3 \Rightarrow \sqrt[3]{\frac{-1}{4x^2}} + 2x$

c $x_0 = 1, x_1 = 1.3700, 75, x_2 = 1.4893,$
$x_3 = 1.5170, x_4 = 1.5228$

d f(1.5235) = 0.0412… > 0, f(1.5245) = −0.0050… < 0

Challenge

1 $8x + 36y + 19 = 0$

2 **a** $f(0) = 0^3 - k(0) + 1 = 1$; $g(0) = e^{2(0)} = 1$; $P(0, 1)$

b $\frac{1}{2}$

3 $\frac{18}{r}$

Exam practice

1 $\dfrac{x+7}{2x-1}$

2 a $200 < V \leq 2000$

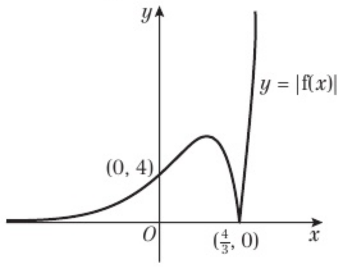
b After 15 years the value of Maria's saxophone is decreasing at 30 euros per year.

c $10\ln\left(\dfrac{4}{3}\right)$

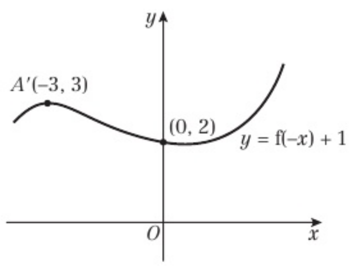
3 a $A = \left(\dfrac{1}{5}, \dfrac{5}{2}e^{\frac{1}{2}}\right)$

b $0 < f(x) < \dfrac{5}{3}e^{\frac{1}{2}}$

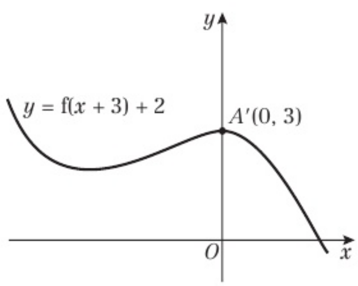
c



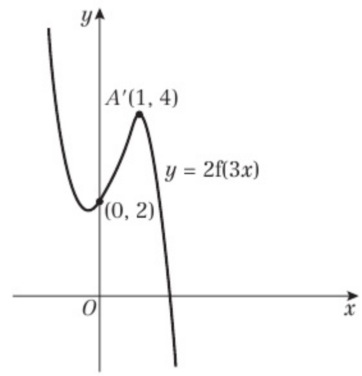
4 a



b



c



5 a $f(x) = \sin^2 x + 2(\sin^2 x + \cos^2 x) = \sin^2 x + 2$

$\sin^2 x = \dfrac{1-\cos 2x}{2} \Rightarrow f(x) = \dfrac{1-\cos 2x}{2} + 2 = \dfrac{5-\cos 2x}{2}$

b $\dfrac{5\pi - 2}{8}$

6 a $\dfrac{dy}{dx} = 2x + \dfrac{\pi}{2}\cos\left(\dfrac{\pi}{2}x\right)$

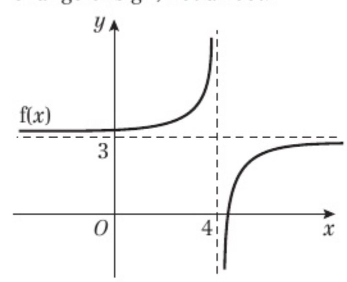
b $y = \dfrac{x+1}{2}$

7 $k = \dfrac{1}{12}$

8 $f(x) = 3x + 2 + \dfrac{4}{x-2} + \dfrac{1}{(x-2)^2}$

9 a $f(3.9) = 13$, $f(4.1) = -7$

b There is an asymptote at $x = 4$ which causes the change of sign, not a root.



c $\alpha = \frac{13}{3}$

10 a $\dfrac{e^{4x+3}}{4} + c$

b $-\dfrac{e^{-\sin 4x}}{4} + c$

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