Broadword Implementation of Parenthesis Queries

Sebastiano Vigna
Dipartimento di Scienze dell'Informazione
Università degli Studi di Milano, Italy

Abstract

We continue the line of research started in [Vig08] proposing *broadword* (a.k.a. SWAR—"SIMD Within A Register") algorithms for finding matching closed parentheses and the k-th far closed parenthesis. Our algorithms work in time $O(\log w)$ on a word of w bits, and contain no branch and no test instruction. On 64-bit (and wider) architectures, these algorithms make it possible to avoid costly tabulations, while providing a very significant speedup with respect to for-loop implementations.

1 Introduction

A *succinct* data structure (e.g., a succinct tree) provides the same operations of its standard counterpart (and sometimes more), but occupies space that is asymptotically near to the information-theoretical lower bound. A classical example is the $(2n + 1)$-bit representation of a binary tree with n internal nodes proposed by Jacobson [Jac89]. Recent years have witnessed a growing interest in succinct data structures, mainly because of the explosive growth of information in various types of text indexes (e.g., large XML trees).

In this paper we discuss practical implementations of two basic building blocks: given a string of w bits, where w is the machine word, representing open (1) and closed (0) parentheses, we are interested in solving the following two problems:

- assuming the first bit is a one, finding the matching closed parenthesis;
- finding the k-th far closed parenthesis in the string (a parenthesis is *far* if its matching parenthesis is not in the string).

Trivial solutions require scanning the string in $O(w)$ time. For the necessities of data structures supporting operations on balanced parenthesis, usually representing trees (see, e.g., [Jac89, MR01, JSS07, GRRR06]), the two operations can be implemented by tables that in principle use $o(n)$ bits for a structure with n parentheses. However, the tables are actually very big, unless n is very large, and they do not usually fit the processor cache.

In this paper we push further the work started in [Vig08], where we argued that on modern 64-bit architecture a much more efficient approach uses *broadword programming*. The term "broadword" has been introduced by Don Knuth in the fascicle on bitwise manipulation techniques of the fourth volume of *The Art of Computer Programming* [Knu07]. Broadword programming uses large (say, more than 64-bit wide) registers as small parallel computers, processing several pieces of information at a time. An alternative, more traditional name for similar techniques is SWAR ("SIMD Within A Register"), a term coined by Fisher and Dietz [FD99]. One of the first techniques for manipulating several bytes in parallel were actually proposed by Lamport [Lam75]. The famous HAKMEM memo [BGS72] contains several examples of broadword programming.

We are also very careful of avoiding tests whenever possible. Branching is a very expensive operation that disrupts speculative execution, and should be avoided when possible. All broadword algorithms we discuss contain no test and no branching.

While broadword programming and careful consideration of testing and cache side-effects are by now quite common in practical implementations of succinct data structures (see, e.g., [DR06]), to the best of our knowledge no one has proposed broadword algorithms for the problems we study. See [Gog09] for other applications of the same ideas.

We concentrate on 64-bit and wider architecture, but we cast all our algorithms in a 64-bit framework to avoid excessive notation: the modification for wider registers are trivial. We have in mind modern processors (in particular, the very common Opteron processor) in which multiplications are extremely fast (actually, because the clock is slowed down in favour of multicores), so we use them occasionally. They can be safely replaced by $O(\log w)$ basic operation, but in practice experiments show that on the Opteron replacing multiplications by shifts and additions, *even in very small number*, is not competitive.

The C++/Java code implementing all data structures in this paper is available under the terms of the GNU Lesser General Public License at `http://sux.dsi.unimi.it/`.

2 Notation

Consider a string $\boldsymbol{s}$ of n bits numbered from 0. We write s_i for the bit of index i. When can view $\boldsymbol{s}$ as a string of parentheses by stipulating that 1 represent an open parenthesis, and 0 a closed parenthesis. We define the *closed excess function*

$$E_{\boldsymbol{s}}(i) = |\{s_j \mid j < i \wedge s_j = 0\}| - |\{s_j \mid j < i \wedge s_j = 1\}|,$$

which represent the excess of closed w.r.t. open parentheses at position i (excluded). The string $\boldsymbol{s}$ is *balanced* if the excess function is always negative, except for 0 and n, where it is zero.

We use $a \setminus b$ to denote integer division of a by b, $\gg$ and $\ll$ to denote right and left (zero-filled) shifting, $\gg^{+}$ denotes right shifting with sign extension, &, | and $\oplus$ to denote bit-by-bit not, and, or, and xor; $\overline{x}$ denotes the bit-by-bit complement of x. We pervasively use precedence to avoid excessive parentheses, and we use the same precedence conventions of the C programming language: arithmetic operators come first, ordered in the standard way, followed by shifts, followed by logical operators; $\oplus$ sits between | and &.

We use L_k to denote the constant whose ones are in position 0, k, $2k$, ... that is, the constant with the *lowest* bit of each k-bit subword set (e.g, $L_8 = 0x0101010101010101$). This constant is very useful both to spread values (e.g., $0x12{*}L_8 = 0x1212121212121212$)

and to sum them up, as it generates cumulative sums of k-bit subwords if the values contained in each k-bit subword, when added, do not exceed k bits. (e.g., 0x030702 $* L_8 =$ 0x30A0C0C0C0C0C0C0902—look carefully at the three rightmost bytes). We use H_k to denote $L_k \ll k-1$, that is, the constant with the *highest* bit of each k-bit subword set (e.g, $H_8 = 0x8080808080808080$).

We use the notation

$$\mu_k := \left(2^{2^w} - 1\right) \setminus \left(2^{2^k} + 1\right),$$

where \ denotes integer division. More intuitively, $\mu_0 =$ 0x5555 ... 5555, $\mu_1 =$ 0x3333 ... 3333, $\mu_1 =$ 0x0F0F ... 0F0F, $\mu_2 =$ 0x00FF ... 00FF, and so on.

Our model is a RAM machine with w-bit words that performs logic operations, additions and subtractions in unit time using 2-complement arithmetic. In our algorithms we also use a constant number of multiplications, which can be substituted with $O(\log w)$ shifts and adds without altering the running time.

3 Basic operations

We recall the expression for computing in parallel the differences modulo 2^k of each k-bit subword (see [Knu07]):

$$x -_k y := ((x \mid H_k) - (y \mathbin{\&} \overline{H_k})) \oplus ((x \oplus \overline{y}) \mathbin{\&} H_k).$$

If we know in advance that the blocks in x and y contain positive entries, this simplifies to

$$((x \mid H_k) - y) \oplus H_k.$$

Another important operation we will use is blockwise nonzero test:

$$x \neq_k 0 := \Big(\big(((x \mid H_k) - L_k) \mid x\big)\Big) \mathbin{\&} H_k.$$

Finally, truncated difference of positive entries:

$$x \mathbin{\dot{-}_k} y = (x -_k y) \mathbin{\&} ((x -_k y) \gg k-1) -_k 1$$

The subexpression after the & is simply a mask that cancels out every block in which a negative result was obtained. The common subexpression $x -_k y$ should be, of course, computed just once.

4 Matching open parentheses

Assume we have a string $\boldsymbol{s}$ such that $s_0 = 1$. We would like find the associated matching closed parenthesis, if it lies in $\boldsymbol{s}$, or get some special value otherwise. The general strategy to obtain this result in $O(\log w)$ time and $O(1)$ additional space is to consider the excess function, as clearly we are interested in computing

$$\min_{0 \le j < w} E_{\boldsymbol{s}}(j) = 0.$$

We operate in the following manner: we will *sample* E_s each $2^{\lceil \log\log w \rceil}$ positions. Then, we will scan linearly in parallel each of the resulting $w/2^{\lceil \log\log w \rceil}$ blocks from the end, recording whether in some block the function crosses zero, and where this happens. Finally, we find the first block that hit a zero and return the corresponding position.

Let us first consider a 64-bit sampling phase on input x; blocks are just bytes this case. We start with a small variant of the standard broadword algorithm for sideways additions:

$b = x - (x \,\&\, \mathrm{0xAAAAAAAAAAAAAAAA}) \gg 1$
$b = (b \,\&\, \mathrm{0x3333333333333333}) + ((b \gg 2) \,\&\, \mathrm{0x3333333333333333})$
$b = (b + (b \gg 4)) \,\&\, \mathrm{0x0F0F0F0F0F0F0F0F0}$
$b = (b * L_8) \ll 1$

At this point, each byte of b contains *twice* the number of open parentheses appearing up to that block, included. Note that the excess function satisfies

$$E_s(j) = |\{s_j \mid j < i \wedge s_j = 0\}| - |\{s_j \mid j < i \wedge s_j = 1\}| = j - 2|\{s_j \mid j < i \wedge s_j = 1\}|,$$

so getting a sample of E_s each 8 bits just requires parallel subtraction with a suitable constant:

$$b = (H_8 \mid \mathrm{0x4038302820181008}) -_8 b.$$

Note the presence of H_8, which avoid propagation of the sign bit, and in practice let us represent each sample in two's complement in the seven lower bits of each byte. We now set up an *update mask* u that contains, for each byte of b, zero, if the byte is nonzero in the lower seven bits, but 0x7F otherwise:

$$u = ((((b \mid H_8) - L_8) \gg 7 \,\&\, L_8) | H_8) - L_8$$

Using u we set up our last variable z, that throughout the computation will contain, for each byte of b, either zero, if the byte was never equal to zero (in the lower seven bits), or a counter expressing the position of the parenthesis that caused the excess function to go to zero. If we find a zero byte initially, the position is clearly 7:

$$z = (H_8 \gg 1 \mid L_8 * 7) \,\&\, u.$$

We now update b, modifying the values of the excess function *two bits at a time*: this is correct, as a balanced string has necessarily even length, so the excess function cannot go to zero at an odd position. In the first round we thus compute

$$b = b - (L_8 * 2 - ((x \gg 6 \,\&\, L_8 \ll 1) + (x \gg 5 \,\&\, L_8 \ll 1)))$$

We now recompute u as above, but update z as follows:

$$z = z \,\&\, \overline{u} \mid (H_8 \gg 1 \mid L_8 * 5) \,\&\, u.$$

Due to the update rule, even nonzero bytes of z will be updated. This is correct, as we want to find the zero of the excess function that is closer to the first bit. We continue in this way until we have completed scanning each byte: the next update of b is thus

$$b = b - (L_8 * 2 - ((x \gg 4 \,\&\, L_8 \ll 1) + (x \gg 3 \,\&\, L_8 \ll 1))),$$

and so on. Finally, we gather our result by locating the relevant block using an LSB operator (e.g., Brodal's [Knu07]), which we assume to return -1 in case no bit is set:

$$p = \mathrm{LSB}(z \gg 6 \,\&\, L_8)$$
$$((p + (z \gg p \,\&\, \mathrm{0x3F})) \mid (p \gg 8))) \,\&\, \mathrm{0x7F}$$

The last line contains the expression returned (we will return 127 in case no matching parenthesis exists).

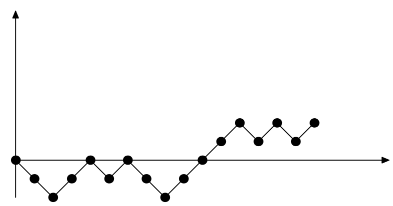
The algorithm is best followed on an example: consider the first two bytes of a 64-bit string:

$$\boxed{1\,1\,0\,0\,1\,0\,1\,1} \quad \boxed{0\,0\,0\,0\,1\,0\,1\,0}\ldots$$

Note that the left most bit is bit zero. We are representing the string of parentheses

$$\boxed{(\,(\,)\,)\,(\,)\,(\,(} \quad \boxed{)\,)\,)\,)\,(\,)\,(\,)}\ldots$$

The excess function behaves as follows:



In the first computation step, we sample the excess function at each byte, so the first bytes of b (in two's complement) are -2 and +2. No result is thus stored in z. However, in the first update we modify the samples of the excess function by subtracting the contribution of the underlined parentheses:

$$\boxed{1\,1\,0\,0\,1\,0\,\underline{1}\,\underline{1}} \quad \boxed{0\,0\,0\,0\,1\,0\,\underline{1}\,\underline{0}}\ldots$$

Now the first byte of b changes to 0, so we store in z our result as follows:

$$\boxed{0\,1\,0\,0\,0\,1\,0\,1} \quad \boxed{0\,0\,0\,0\,0\,0\,0\,0}\ldots$$

The sixth bit records that there is a value, and for the time being the candidate result is 5. Note that the current result is spurious, because there is another zero to be found.

We now update again b, subtracting another pair of parentheses:

$$\boxed{1\,1\,0\,0\,\underline{1}\,\underline{0}\,\underline{1}\,\underline{1}} \quad \boxed{0\,0\,0\,0\,\underline{1}\,\underline{0}\,\underline{1}\,\underline{0}}\ldots$$

The first byte of b is now 0, the second byte $+2$. Thus, z is updated as follows:

$$\boxed{0\,1\,0\,0\,0\,0\,1\,1} \quad \boxed{0\,0\,0\,0\,0\,0\,0\,0}\ldots$$

In the last update, the *second* byte of b becomes 0. The final value of z is thus

$$\boxed{0\,1\,0\,0\,0\,0\,1\,1} \quad \boxed{0\,1\,0\,0\,0\,0\,0\,1}\ldots$$

Now the LSB operator detects that the first zero is in the first byte, and the correct value (3) is extracted from z and returned.

The construction of the first sample requires $O(\log\log w)$ instructions and a single multiplication (which could be substituted with $O(\log w)$ operations). The parallel linear scan clearly requires $O(\log w)$ operations.

5 Finding far closed parentheses by index

In this section we discuss the problem of finding the p-th far closed parenthesis. The simple combinatorial idea at the heart of the algorithm is the following, easily proved statement:

Proposition 1 *Let $\boldsymbol{t}$, $\boldsymbol{u}$ be bit strings, $\#_{\text{open}}$/$\#_{\text{closed}}$ the operators returning the number of far open/closed parenthesis in a string, and $\dot{-}$ truncated subtraction. Then:*

$$\#_{\text{open}}\boldsymbol{tu} = (\#_{\text{open}}\boldsymbol{t} \mathbin{\dot{-}} \#_{\text{closed}}\boldsymbol{u}) + \#_{\text{open}}\boldsymbol{u}$$
$$\#_{\text{closed}}\boldsymbol{tu} = (\#_{\text{closed}}\boldsymbol{u} \mathbin{\dot{-}} \#_{\text{open}}\boldsymbol{t}) + \#_{\text{closed}}\boldsymbol{u}$$

Since it is easy to compute the number of far open/closed parenthesis in 2-bit blocks using masking, and it is also easy to do parallel truncated subtraction, using additional $O(\log w)$ words we can compute the number of far open/closed parentheses in blocks of length 2^i, $2 \le i < \log w$. At that point, we can use the above property backwards: if we are searching from the k-th far closed parenthesis in $\boldsymbol{tu}$, this must be either in $\boldsymbol{t}$, if $k < \#_{\text{closed}}\boldsymbol{t}$, or in $\boldsymbol{u}$, but in position $k - \#_{\text{closed}}\boldsymbol{t} + \#_{\text{open}}\boldsymbol{t}$. We will assume for the time being that a p-th far closed parenthesis *does* exist in the string. Results will be unpredictable otherwise.

In general, each 2^k-bit block of the variables o_k and c_k will keep track of the number of far open/closed parentheses in the corresponding 2^k-bit block of the input x. We bootstrap our computation by filling o_1 and c_1:

$b_0 = x \mathbin{\&} \text{0x5555555555555555}$
$b_1 = (x \mathbin{\&} \text{0xAAAAAAAAAAAAAAAA}) \gg 1$
$l = (b0 \oplus b1) \mathbin{\&} b1$
$o_1 = (b0 \mathbin{\&} b1) \ll 1 \mid l$
$c_1 = ((b0 \mid b1) \oplus \text{0x5555555555555555}) \ll 1 \mid l$

These operations implements the mappings

$$\begin{array}{lll} 00 \to 00 & \qquad & 00 \to 10 \\ 01 \to 01 & & 01 \to 01 \\ 10 \to 00 & & 10 \to 00 \\ 11 \to 10 & & 11 \to 00 \end{array}$$

They send each 2-bit substring to the number of far open, or closed, respectively, parentheses.

The k-th phase, $1 < k < \log w$, records in temporary variables e_o and e_c the number of far open and far closed parentheses in each half of 2^{k+1}-bit blocks. These numbers are then combined using Proposition 1:

$$
\begin{aligned}
& e_o = o_k \mathbin{\&} \mu_k \\
& e_c = (c_k \mathbin{\&} \mu_k \ll 2^k) \gg 2^k \\
& o_{k+1} = ((o_k \mathbin{\&} \mu_k \ll 2^k) \gg 2^k) + (e_o \mathbin{\dot{-}_8} e_c) \\
& c_{k+1} = (c_k \mathbin{\&} \mu_k) + (e_c \mathbin{\dot{-}_8} e_o)
\end{aligned}
$$

Finally, we work backwards, isolating the part of the string containing the required parenthesis. At the k-th step, $k = \log w - 1, \log w - 2, \ldots, 1$ we operate as follows:

$$
\begin{aligned}
& b = ((p - (c_k \gg s \mathbin{\&} 2^{2^k} - 1)) \gg^{+} w - 1) - 1 \\
& m = b \mathbin{\&} 2^{2^k} - 1 \\
& p \mathrel{-}= c_k \mathbin{\&} m \\
& p \mathrel{+}= o_k \mathbin{\&} m \\
& s \mathrel{+}= 2^k \mathbin{\&} b
\end{aligned}
$$

The variable s keeps track of the left (i.e., lowest) extreme of the interval of width 2^{k+1} in which we are performing our binary search. Initially, s is zero and $k = \log w - 1$, which means that we are searching for the p-th far closed parenthesis in the whole string $\boldsymbol{s}$.

In each phase, we first of all set b so that it is 0 if the p-th far closed parenthesis appears in the block of length 2^k starting at position s, 0 otherwise. Note that we can do this because the far closed parentheses in the *first* half are true, global far closed parentheses. We then set up our mask m, which will be used to update p: if m is zero, there is no update to do—we just have to restrict our search interval. Otherwise, we have to decrease p by $c_k \mathbin{\&} m$ (as we are skipping $c_k \mathbin{\&} m$ far closed parentheses) and increase it by $o_k \mathbin{\&} m$ (as there are $o_k \mathbin{\&} m$ far open parentheses before the block we're moving in, so we must offset p). Finally, s must be updated and moved forward by 2^k in case $b \neq 0$.

In the last phase, we are left with a two-bits string and a value p. It is easy to check that the following hand-crafted expression gives the correct result:

$$s + p + ((x \gg s \mathbin{\&} ((p \ll 1) \mid 1)) \ll 1).$$

Finally, it is easy to see that be performing an additional phase in the first part of the algorithm we can obtain the overall number of far closed parentheses in the whole string, making it easy to return a special value in case the requested parenthesis does not exist.

6 Experiments

We performed a number of experiments on a Linux-based system sporting a 64-bit Opteron processor running at 2814.501 MHz with 1 MiB of first-level cache. The tests show that on 64-bit architectures broadword programming provides significant performance improvements. We compiled using `gcc` 4.1.2 and options `-O9`.

Our previous experience with similar code shows that testing in isolation very tight code can produce paradoxical results. It is much more informative to embed the code in a typical simple application: in our case, we implemented Jacobson's classical $O(n)$ balanced parentheses representation [Jac89] and performed tests measuring the time required to find a matching closed parenthesis using our broadword algorithms and a tuned for-loop implementation.

The experimental setting for benchmarking operations that require nanoseconds must be set up carefully. We generate at random bit arrays containing correctly parenthesised strings, and store a million test positions. During the tests, the positions are read with a linear scan, producing minimal interference; generating random positions during the tests causes instead a significant perturbation of the results, mainly due to the slowness of the modulo operator. The tests are repeated ten times and averaged. We measure user time using the system function `getrusage()`.

Generating random balanced strings of parenthesis requires some attention. We use Arnold and Sleep's classical algorithm [AS80], but with a *twist*. The algorithm chooses at each step whether to add a closed parenthesis with probability

$$P_{r,k} = \frac{1}{2}\frac{r(k+r+2)}{k(r+1)},$$

where r is the number of open parentheses still to be closed, and k the remaining number of symbols to be generated. Note that when $k = r$ we have $P_{r,k} = 1$, so we just generate closed parentheses.

To estimate better the behaviour of our algorithms, we introduce a *twist*, that is, a number $0 \leq t \leq 1$ that shifts the probability so that open parentheses are more likely to be generated. In other words,

$$P_{r,k,t} = \begin{cases} 1 & \text{if } P_{r,k} = 1; \\ t\,P_{r,k} & \text{otherwise.} \end{cases}$$

The result is that when $t < 1$ we will tend to generate strings with deeper nesting. We are interested in experimenting with the behaviour at different deepness levels because trivial (for-loop) solutions behave very well on random strings because most open parentheses are near, and moreover their matching parenthesis is a few bits away. But if you consider a typical application, for instance, binary search trees, then a search going down into a large tree has to find a far matching parenthesis for most of the search. More precisely, it is not difficult to see that for a complete binary tree the *average* (over all paths going from the root to a leaf) distance between the open and closed parenthesis of a query is $\Theta(n/\log n)$ (assuming the binary tree is mapped to a forest using the inverse of the first-child/next-sibling isomorpshism, and that the forest is represented using balanced parentheses in the standard way). To simulate this fact, we use a skewed distribution: we plan to enlarge, however, our test set with more realistic large search trees or XML trees.

	1	.75	.50	.25
1 Ki	62.10/89.90	68.50/116.80	76.50/130.40	86.50/142.60
4 Ki	62.90/95.40	68.80/115.00	77.30/123.60	87.70/152.20
16 Ki	63.10/100.30	68.70/113.50	78.00/127.20	87.10/153.20
64 Ki	63.70/100.70	69.40/113.10	79.00/128.50	88.20/154.30
256 Ki	69.20/105.40	75.50/119.50	86.20/134.70	96.00/161.60
1 Mi	78.70/116.30	87.50/130.30	97.00/144.90	109.30/173.50
4 Mi	179.20/213.20	190.20/231.60	211.80/261.50	237.40/301.20
16 Mi	246.30/278.20	281.50/320.60	327.50/376.10	424.30/489.30

Table 1: Timings in nanoseconds for a parenthesis matching operation in Jacobson's data structure. The first value is obtained used the broadword algorithms presented in this paper, whereas the second value is obtained using a for-loop implementation. Column labels show the amount of twisting, whereas row labels show the number of parentheses in the string.

We compare our structures against tuned for-loop implementations: results are shown in Table 1 and Figure 1, which clearly show the advantage of the broadword implementation, in particular for longer matchings (e.g., for low twist). We expect, of course, that figures will improve as w gets larger.

7 Conclusions

Extending some previous work of ours [Vig08], we have introduced some two new broadword algorithms that implement two basic operations typical of succinct static data structures for balanced parentheses. We have also presented experiments that compares our results with a for-loop baseline. We discussed our algorithms in the case of closed parentheses, but they can be immediately modified to find matching open or far open parentheses.

We leave for future work experimentation with tabulated implementations. The latter tend to be, of course, very fast when tested, but they engage the processor cache significantly, and their global impact cannot be measured easily. For-loop implementations have a cache footprint similar to that of our broadword versions, so they are first natural candidate for comparisons.

A Java version of this code is currently distributed by the Sux4J project[1] as part of a highly compressed implementation of a monotone minimal perfect hash function (see [BBPV09]).

[1] `http://sux4j.dsi.unimi.it/`.

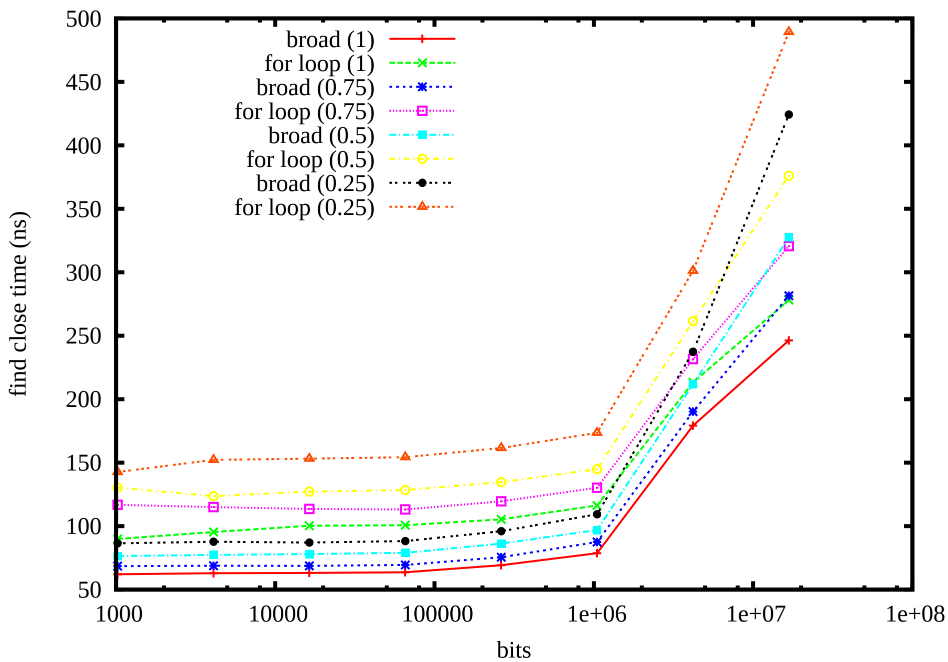


Figure 1: A graph displaying the data shown in Table 1. Up to around one million bit the timings remain constant even in practice; after that, memory access becomes significant and size has a significant effect on speed (as in the case of rank/select queries—see [GGMN05]).

References

[AS80] D. B. Arnold and M. R. Sleep. Uniform random generation of balanced parenthesis strings. *ACM Trans. Program. Lang. Syst.*, 2(1):122–128, 1980.

[BBPV09] Djamal Belazzougui, Paolo Boldi, Rasmus Pagh, and Sebastiano Vigna. Theory and practise of monotone minimal perfect hashing. In *Proceedings of the Tenth Workshop on Algorithm Engineering and Experiments (ALENEX)*, pages 132–144. SIAM, 2009.

[BGS72] Michael Beeler, Ralph William Gosper, and Rich Schroeppel. HAKMEM. Report A. I. MEMO 239, Massachusetts Institute of Technology, A.I. Lab., Cambridge, Massachusetts, 1972.

[DR06] O'Neil Davion Delpratt and Rajeev Raman. Engineering the louds succinct tree representation. In *Proceedings of the 5th International Workshop on Experimental Algorithms*, pages 134–145. Springer, 2006.

[FD99] Randall J. Fisher and Henry G. Dietz. Compiling for SIMD within a register. In Siddhartha Chatterjee, Jan Prins, Larry Carter, Jeanne Ferrante, Zhiyuan Li, David C. Sehr, and Pen-Chung Yew, editors, *Languages and Compilers for Parallel Computing, (11th LCPC'98)*, number 1656 in Lecture Notes in Computer Science, pages 290–304. Springer–Verlag, 1999.

[GGMN05] R. Gonzàlez, S. Grabowski, V. Mäkinen, and G. Navarro. Practical implementation of rank and select queries. In *Poster Proceedings Volume of 4th Workshop on Efficient and Experimental Algorithms (WEA'05)*, pages 27–38. CTI Press and Ellinika Grammata, 2005.

[Gog09] Simon Gog. Broadword computing and fibonacci code speed up compressed suffix arrays. In *Experimental Algorithms: 8th International Symposium Sea 2009, Dortmund, Germany, June 4-6, 2009, Proceedings*, page 161. Springer-Verlag New York Inc, 2009.

[GRRR06] R.F. Geary, N. Rahman, R. Raman, and V. Raman. A simple optimal representation for balanced parentheses. *Theoretical Computer Science*, 368(3):231–246, 2006.

[Jac89] Guy Jacobson. Space-efficient static trees and graphs. In *30th Annual Symposium on Foundations of Computer Science (FOCS '89)*, pages 549–554, Research Triangle Park, North Carolina, 1989. IEEE Computer Society Press.

[JSS07] Jesper Jansson, Kunihiko Sadakane, and Wing-Kin Sung. Ultra-succinct representation of ordered trees. In *SODA '07: Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 575–584, Philadelphia, PA, USA, 2007. Society for Industrial and Applied Mathematics.

[Knu07] Donald E. Knuth. The Art of Computer Programming. Pre-Fascicle 1A. Draft of Section 7.1.3: Bitwise Tricks and Techniques, 2007.

[Lam75] Leslie Lamport. Multiple byte processing with full-word instructions. *Communications of the ACM*, 18(8):471–475, 1975.

[MR01] J.I. Munro and V. Raman. Succinct representation of balanced parentheses and static trees. *SIAM Journal on Computing*, 31(3):762–776, 2001.

[Vig08] Sebastiano Vigna. Broadword implementation of rank/select queries. In Catherine C. McGeoch, editor, *Experimental Algorithms. 7th International Workshop, WEA 2008*, number 5038 in Lecture Notes in Computer Science, pages 154–168. Springer–Verlag, 2008.