Cargèse Fall School on Random Graphs
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Introduction to random graphs

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Two main random graph models

The binomial random graph $G(n, p)$

$G(n, p)$ is the (random) graph on vertices $\{1, 2, \ldots, n\}$ in which each of $\binom{n}{2}$ possible pairs appears as an edge independently with probability p.

The uniform random graph $G(n, M)$

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What does it mean?

Given a graph G with vertex set $[n]$:

$$\Pr(G(n,p) = G) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}.$$

while

$$\Pr(G(n,M) = G) = \begin{cases} 0 & \text{if} \quad e(G) \neq M \\ 1/\binom{\binom{n}{2}}{M} & \text{if} \quad e(G) = M \end{cases}$$

Asymptotics

Typically, we are interested only in the asymptotic behaviour of $G(n, M)$ for very large n, where $M = M(n)$.

For a given function $M = M(n)$, we say that a property holds for $G(n, M)$ aas if the probability that it holds for $G(n, M)$ tends to 1 as $n \to \infty$.

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(Most of) asymptotic properties of $G(n, M)$ and $G(n, p)$ are very similar, provided $p = M / \binom{n}{2}$.

Observation

Results on $G(n, M)$ are, in a way, more precise, since

$$\Pr(G(n, M) = G) = \Pr(G(n, p) = G | e(G(n, p)) = M),$$

i.e., roughly speaking,

$$G(n, M) = G(n, p) | \{ e(G(n, p)) = M \} .$$

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Let us start with something easy

Theorem Erdős, Rényi'59

Let $p(n) = \frac{1}{n}(\ln n + \gamma(n))$. Then

$$\lim_{n\to\infty} \Pr(G(n,p) \text{ is connected}) = \begin{cases} 0 & \text{if } \gamma(n) \to -\infty, \\ 1 & \text{if } \gamma(n) \to \infty. \end{cases}$$

. . . OR SOMETHING EVEN EASIER

Theorem Erdős, Rényi'59

Let $p(n) = \frac{1}{n}(\ln n + \gamma(n))$. Then

$$\lim_{n\to\infty} \Pr(\delta(G(n,p)) > 0) = \begin{cases} 0 & \text{if } \gamma(n) \to -\infty, \\ 1 & \text{if } \gamma(n) \to \infty. \end{cases}$$

THE FIRST MOMENT METHOD

MARKOV INEQUALITY

Let X be a non-negative, integer-valued random variable. Then

$$\Pr(X > 0) = \Pr(X \geq 1) \leq \mathbb{E}X \ .$$

The first moment method

Theorem Erdős, Rényi'59

Let $p(n) = \frac{1}{n}(\ln n + \gamma(n))$ and $\gamma(n) \to \infty$.
Moreover, let X count isolated vertices in $G(n, p)$.
Then $\Pr(X > 0) \to 0$ as $n \to \infty$.

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Proof Note that $X = \sum_{i=1}^{n} X_i$, where

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$$\mathbb{E}X_i = P(X_i = 1)$$

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In our case

$$\begin{aligned} \mathbb{E}X_i = (1-p)^{n-1} &= \exp\big(-(n-1)\log(1-p)\big) \\ &= \exp(-np + O(p + p^2 n)). \end{aligned}$$

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If $p(n) = \frac{1}{n}(\ln n + \gamma(n))$, then

$$\begin{aligned}\mathbb{E}X = \sum_{i=1}^{n} \mathbb{E}X_i &= n \exp(-np + O(p + p^2 n)) \\ &= (1 + o(1))e^{-\gamma},\end{aligned}$$

and so, for $\gamma(n) \to \infty$, we get

$$\Pr(X > 0) \leq \mathbb{E}X \to 0. \quad \square$$

Remark

If we apply the first moment method to the random variable Y which counts non-trivial components in $G(n, p)$ we get a much stronger result.

Theorem Erdős, Rényi'59

If $p(n) = \frac{1}{n}(\ln n + \gamma(n))$, where $\gamma(n) \to \infty$, then $G(n, p)$ is aas connected.

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Back to isolated vertices

If If $p(n) = \frac{1}{n}(\ln n + \gamma(n))$, where $\gamma(n) \to -\infty$, then

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$$\Pr(X > 0) \to 1, \quad \text{i.e.} \quad \Pr(X = 0) \to 0?$$

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Quite often (but by no means always) it is the case!

The second moment method

Observation

If X counts structures which are "mostly weakly-dependent", then the expected number of ordered pairs of such structures is roughly $(\mathbb{E}X)^2$, i.e.

$$\mathbb{E}X(X-1) = (1+o(1))(\mathbb{E}X)^2 .$$

Then, for the variance of X, we have

$$\mathrm{Var}X = \mathbb{E}X(X-1) + \mathbb{E}X - (\mathbb{E}X)^2 = o\big((\mathbb{E}X)^2\big) .$$

Chebyshev's and Cauchy's inequalities

Let us assume that $\mathbb{E}X \to \infty$, $\mathbb{E}X(X-1) = (1+o(1))(\mathbb{E}X)^2$, and so $\mathrm{Var}X = o\big((\mathbb{E}X)^2\big)$.

Chebyshev's inequality

$$\Pr(X=0) \le \Pr(|X-\mathbb{E}X| \le \mathbb{E}X) \le \frac{\mathrm{Var}X}{(\mathbb{E}X)^2} \to 0\,.$$

Cauchy's inequality

If X is an integer-valued, non-negative random variable, then

$$\Pr(X>0) = \Pr(X \ge 1) \ge \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} = \frac{(\mathbb{E}X)^2}{\mathbb{E}X(X-1)+\mathbb{E}X} \to 1\,.$$

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Chebyshev's vs. Cauchy's

The left hand side of Chebyshev's inequality can be larger than one while Cauchy's bound is always strictly positive!

Revenons à nos moutons

Let X be the number of isolated vertices in $G(n,p)$, where $p(n) = \frac{1}{n}(\log n + \gamma(n))$ and $\gamma \to -\infty$.

Then $\mathbb{E}X = (1+o(1)e^{-\gamma} \to \infty$. What about $\mathbb{E}X(X-1)$?

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$$\mathbb{E}X(X-1) = n(n-1)(1-p)^{2(n-1)-1} = \frac{n-1}{n(1-p)}\left[n(1-p)^{n-1}\right]^2$$
$$= (1+o(1))(\mathbb{E}X)^2.$$

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Thus, $\Pr(X > 0) \to 1$. □

Erdős-Rényi Theorem is finally shown!

Theorem Erdős, Rényi'59

Let $p(n) = \frac{1}{n}(\ln n + \gamma(n))$. Then

(I) If $\gamma \to -\infty$, then aas $G(n, p)$ contains isolated vertices (and so aas it is not connected);

(II) If $\gamma \to \infty$, then aas $G(n, p)$ is connected (and so contains no isolated vertices).

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Can we define (and prove) even stronger result which relates connectivity to the absence of isolated vertices?

The hitting time

The random graph process

$G(n, M)$ can be viewed as the $(M + 1)$th stage of a Markov chain $\{G(n, M) : 0 \leq M \leq \binom{n}{2}\}$, where we add edges to a graph in a random order.

The hitting time

Let $h_1 = \min\{M : \delta(G(n, M)) \geq 1\}$ and $h_{\text{conn}} = \min\{M : G(n, M) \text{ is connected}\}$.
Note that both h_1 and h_{conn} are random variables!

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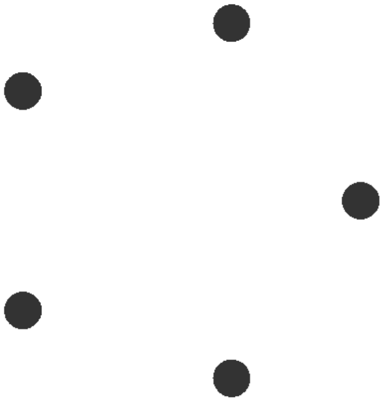
$\{G(n,p) : 0 \leq p \leq 1\}$

THE RANDOM GRAPH PROCESS (FOR $G(n,p)$)

$G(n,p)$ can also be viewed as a stage of a Markov process $\{G(n,M) : 0 \leq p \leq 1\}$.

$\{G(n, p) : 0 \leq p \leq 1\}$



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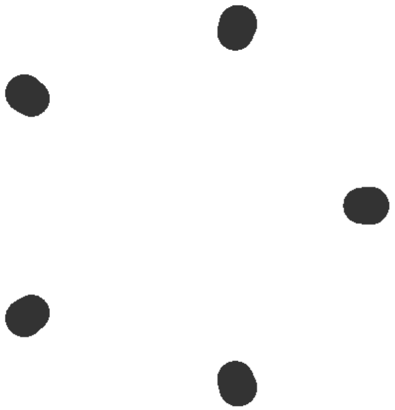

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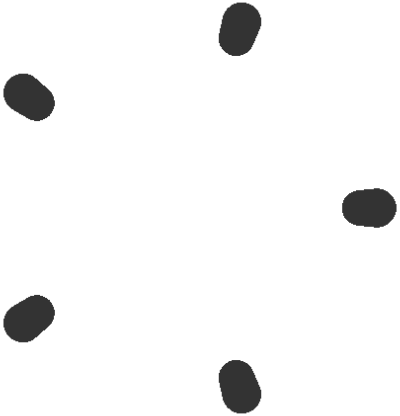

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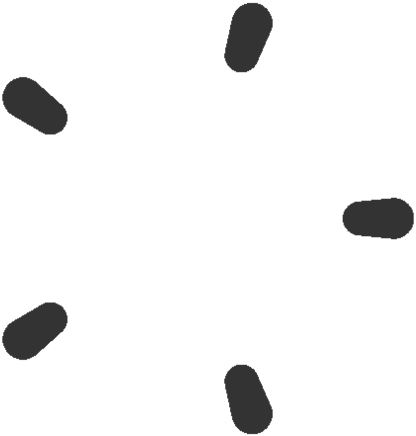

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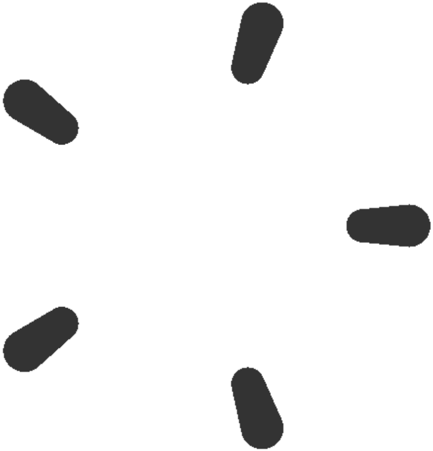

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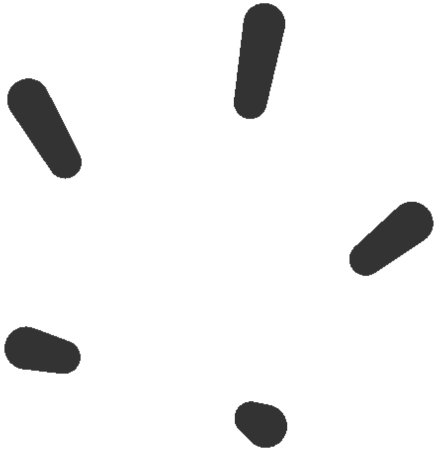

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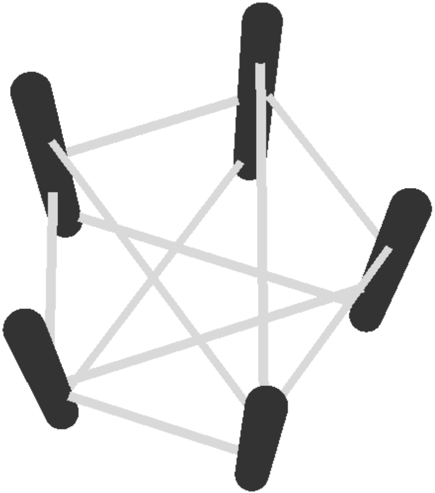

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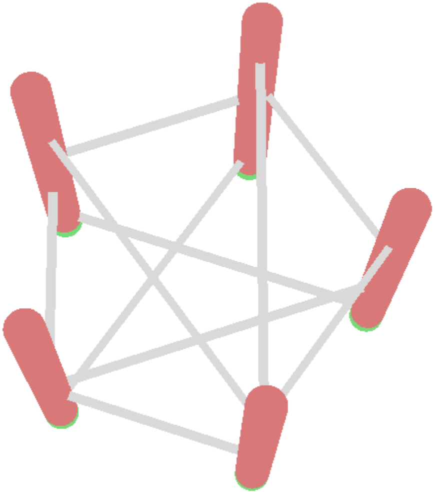

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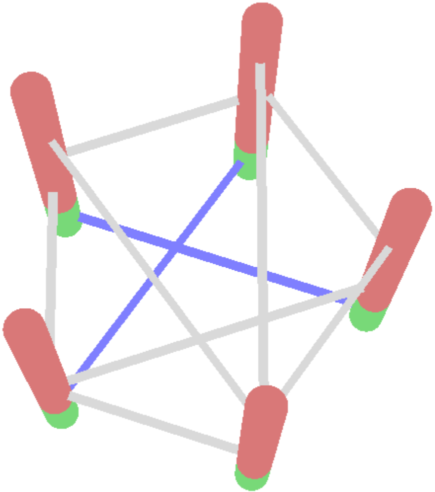

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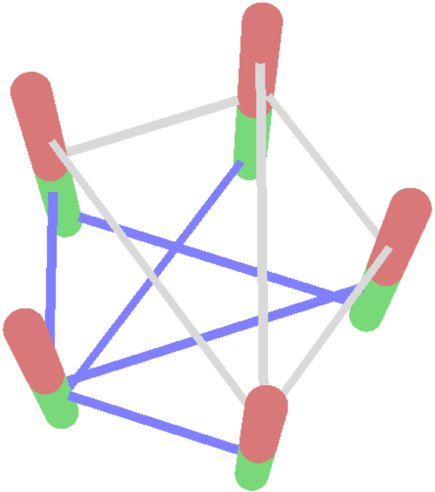

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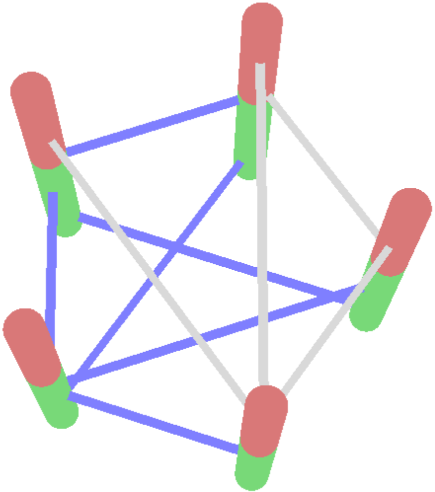

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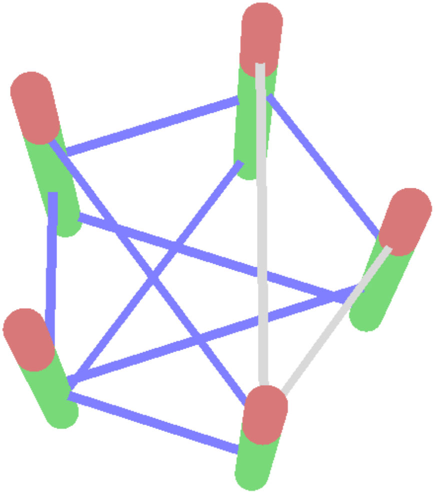

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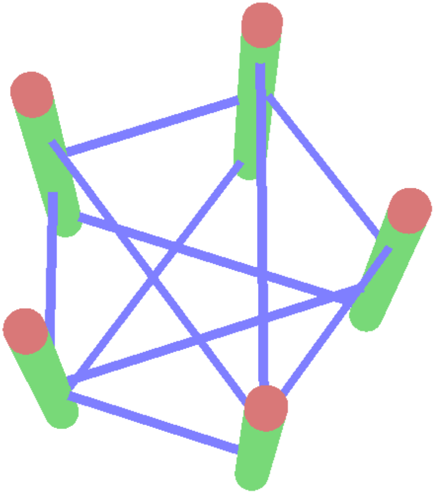

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The hitting times for $G(n,p)$

We can define $\hat{h}_1 = \min\{p : \delta(G(n,p)) \geq 1\}$ and $\hat{h}_{\mathrm{conn}} = \min\{p : G(n,p) \text{ is connected}\}$.
As in the case of h_1 and h_{conn} both $\hat{h}_1$ and $\hat{h}_{\mathrm{conn}}$ are random variables, but they take values in the interval $[0, 1]$.

The hitting times

However, the statement that aas $h_1 = h_{\mathrm{conn}}$ is clearly equivalent to the statement that aas $\hat{h}_1 = \hat{h}_{\mathrm{conn}}$.

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The random graph process: coupling

Since we can view $G(n, M)$ as the stage of the random graph process, for $M_1 \leq M_2$ we have

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In a similar way, for $p_1 \leq p_2$ we have

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THE EVOLUTION OF THE RANDOM GRAPH

If $M = o(\sqrt{n})$ then aas $G(n, p)$ consists of isolated vertices and isolated edges.

If $M = o(n^{(k-1)/k})$ then aas all components of $G(n, p)$ are trees with at most k vertices.

If $M = o(n)$ then aas all components of $G(n, p)$ are trees of size $o(\log n)$.

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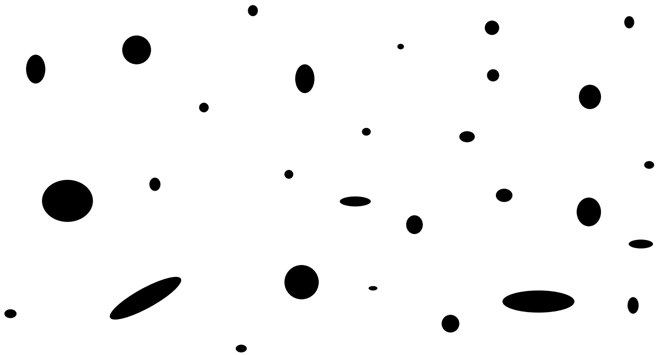
THE SUBCRITICAL PHASE



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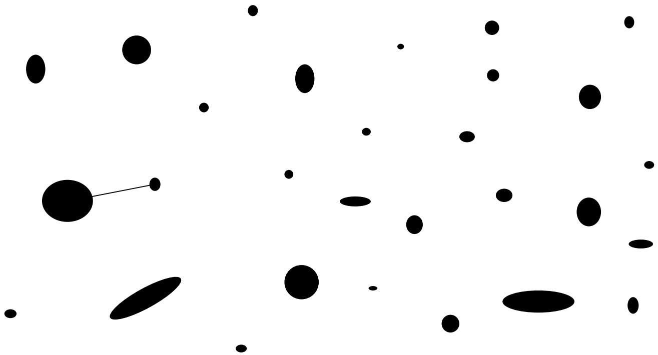

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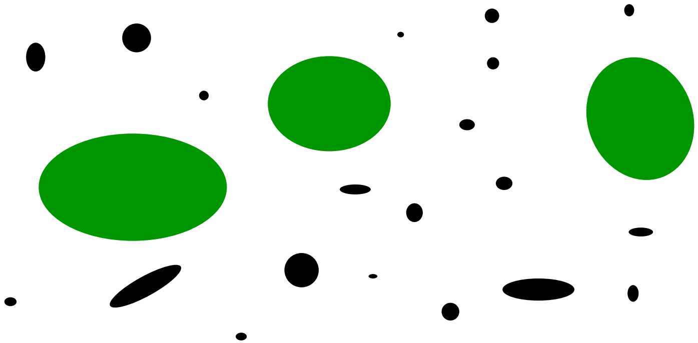

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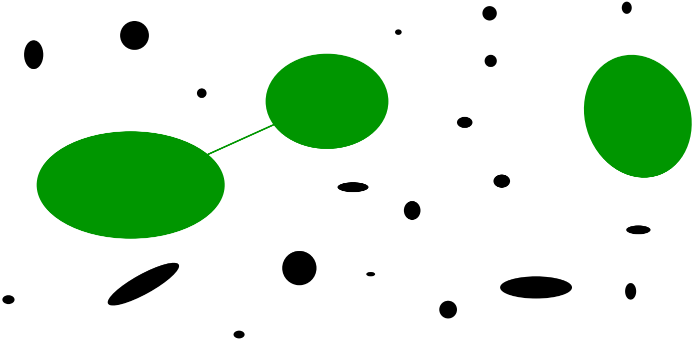

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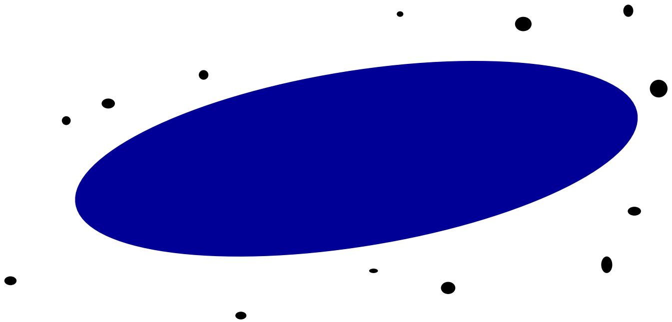

THE RIGHT SCALING

THEOREM Erdős, Rényi'60

The "coagulation phase" takes place when $M = (1/2 + o(1))n$.

Thus, for instance, the largest component of $G(n, 0.4999n)$ has aas $\Theta(\log n)$ vertices, while the size of the largest component of $G(n, 0.5001n)$ is aas $\Theta(n)$.

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The components start to merge when they are of size $\Theta(n^{2/3})$. It happens when $M = n/2 + \Theta(n^{2/3})$.

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If $np \to 0$, then aas $G(n, p)$ contains no triangles.
If $np \to \infty$, then aas $G(n, p)$ contains triangles.

This can be easily proved using the 1st and 2nd moment method we mastered ten minutes ago.

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How fast does the probability $\Pr(G(n, p) \not\supseteq K_3)$ tends to 0 for $np \to \infty$?

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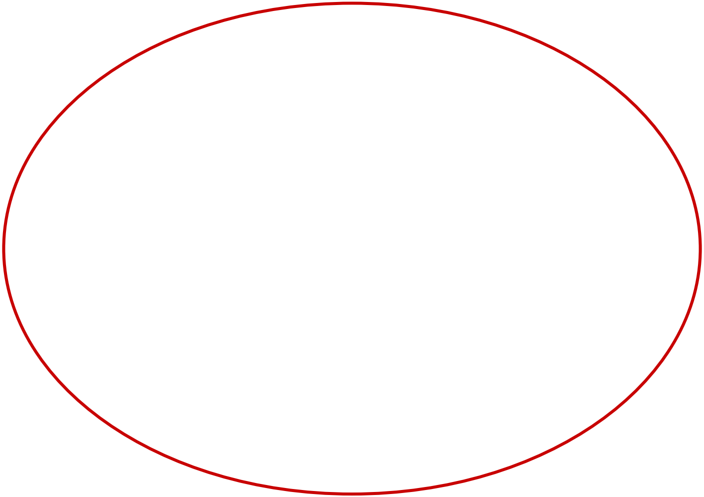
Large deviation inequalities

Let us consider a partition $\mathcal{P}$ of the set of all edges of K_n into small sets.



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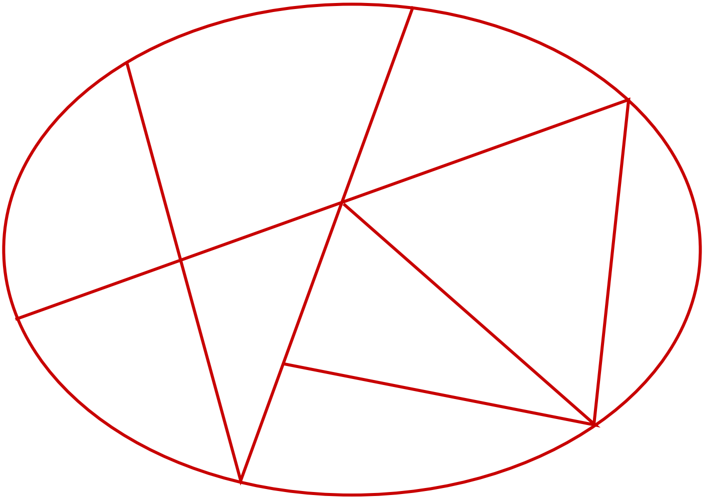

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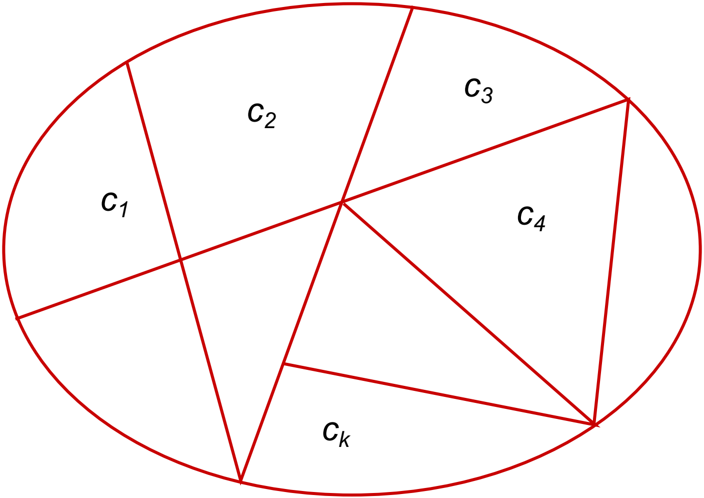
Take any graph parameter A and compute for each part of the partition its “Lipschitz constant”.



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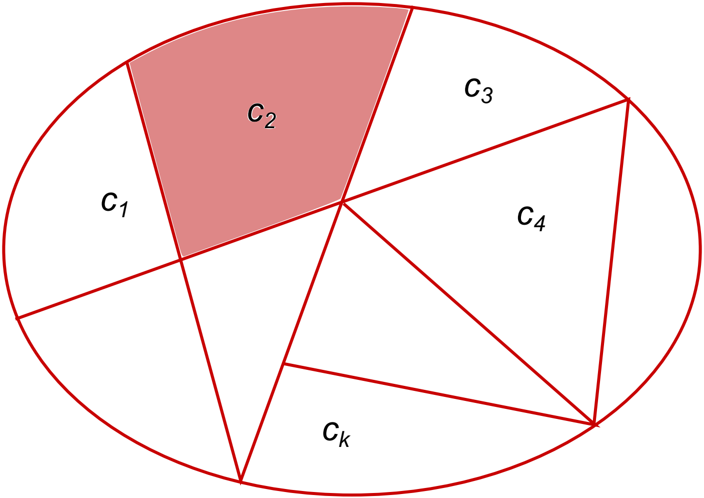

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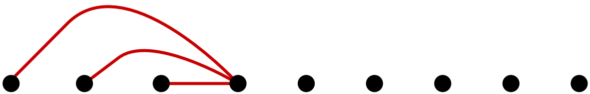

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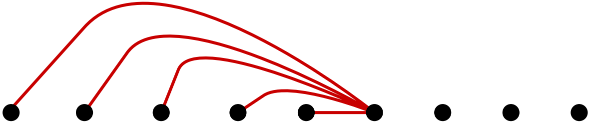

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Let $\mathcal{P}$ be a partition, A be a graph parameter, and $c_1, \ldots, c_k$ denote Lipschitz constants for $\mathcal{P}$ and A. Consider the random variable $X = A(G(n,p))$ for some p. Then, for every t,

$$\Pr\left(|X - \mathbb{E}X| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sum_i c_i^2}\right).$$

In particular,

$$\Pr(X = 0) \leq 2\exp\left(-\frac{(\mathbb{E}X)^2}{2\sum_i c_i^2}\right).$$

THE INDEPENDENCE AND CHROMATIC NUMBERS

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We want to replace the full sum $\sum_{i=1}^k c_i^2$ by a partial sum of c_i's.

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$$\Pr\left(|X - \mu X| \geq t\right) \leq 4\exp\left(-\frac{t^2}{4w}\right),$$

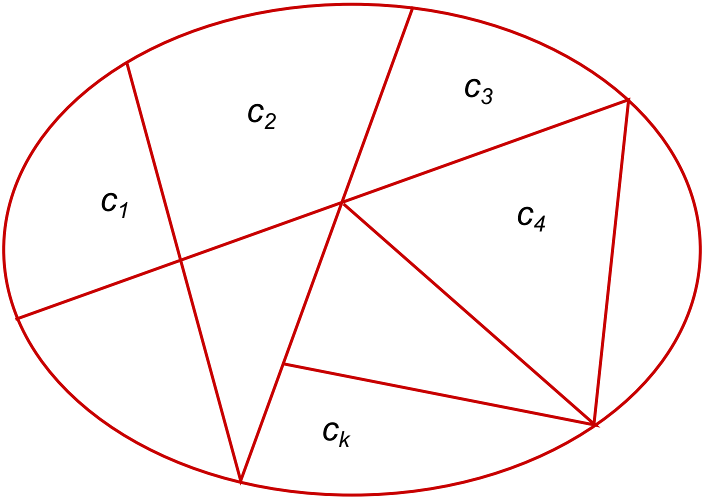
where μX is the median of X and

$$w = \max_{\Lambda}\left\{\sum_{i\in\Lambda} c_i^2\right\}$$

where the maximum is taken over all certificates Λ for A.

Certificates

Take any graph parameter A and find the set of partitions which can certify that $A \geq r$



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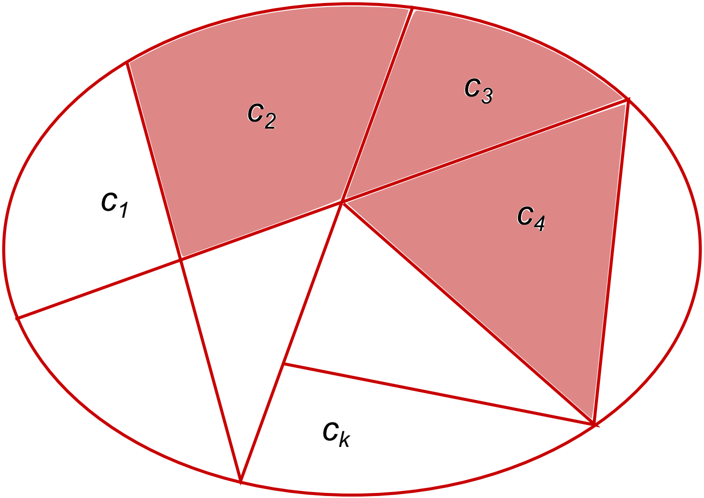

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(ii) There are no small certificates that $\chi(G) \geq r$.

(iii) The size of the certificate that the number of triangles is larger than r is, of course, $3r$.

The independence number

Let $X = \alpha(G(n,p)$ and $k = 2\mathbb{E}X$. Then random variable $\bar{X} = \min\{X, k\}$ has roughly the same expectation (and median) as X, but its certificate is at most $2\mathbb{E}X$.

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From Azuma's inequality we get

$$\Pr(|X - \mathbb{E}X| \geq t) \leq 2\exp\left(-t^2/(2n)\right),$$

while from Talagrand's inequality, applied to $\bar{X}$, we get roughly

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which is typically much stronger inequality.

In particular, for every $\gamma \to \infty$,

$$\Pr(|X - \mathbb{E}X| \geq \gamma\sqrt{\mathbb{E}X}) \to 0.$$

The probability that there are no triangles

Let X denote the number of triangles in $G(n,p)$ and $\bar{X}$ be the maximum size of the family of edge-disjoint triangles.
Let $\hat{X} = \min\{\bar{X}, 2\mathbb{E}X\}$.

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$$\begin{aligned}\Pr(X=0) = \Pr(\hat{X}=0) &\leq \Pr(|\hat{X}-\mathbb{E}\hat{X}| \geq \mathbb{E}\hat{X}) \\ &\leq 4\exp\left(-\frac{(\mathbb{E}\hat{X})^2}{12\mathbb{E}X}\right) \leq 4\exp\left(-\frac{\mathbb{E}X}{108}\right).\end{aligned}$$

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On the other hand, from FKG inequality we get

$$\begin{aligned} \Pr(X = 0) &\geq (1 - p^3)^{\binom{n}{3}} = e^{-(1+o(1))\binom{n}{3}p^3} \\ &= \exp(\big(- (1 + o(1))\mathbb{E}X\big) . \end{aligned}$$

Remarks

Theorem Janson, Łuczak, Ruciński '90

Let $X(H)$ count the number of copies of H in $G(n,p)$. Then, for every H, we have

$$\Pr(X(H)=0)=\exp\left(-\Theta(\min_{F\subseteq H}\mathbb{E}X(F))\right).$$

Although we know that

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Proof Let Y count the number of subsets E of $0.01M$ edges such that $G(n, M) \setminus E$ contains no triangles. Then

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The first factor can be bounded from above by $\exp(-cM)$, the second one, by our theorem and the equivalence results, is smaller than $\exp(-c'M)$ and it turns out that $c' > c$. Hence $\mathbb{E}Y \to 0$ and the assertion follows from the first moment method. $\square$

Maker-Breaker Game $MB(n, q, H)$

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Maker wins if his graph contains a copy of H
otherwise the win comes to Breaker.

Threshold bias

The threshold bias $\bar{q}(n) = \bar{q}_{\mathcal{A}}(n)$ is the maximum q so that Maker can win $MB(n, q, \mathcal{A})$.

i.e. Maker has a winning strategy to build a graph with $\binom{n}{2}/(q+1)$ edges which has property $\mathcal{A}$.

$MB(n, q, K_3)$

CLAIM FOLKLORE

In $MB(n, q, K_3)$, when Maker tries to build a triangle, the threshold bias is $\Theta(\sqrt{(}n))$.
More specifically:

- Maker has a winning strategy if $q < \sqrt{n}$,
- Breaker has a winning strategy if $q > 2\sqrt{n}$.

Our aim

Claim folklore

The threshold bias for $MB(n, q, K_3)$
lies in the interval $[\sqrt{n}, 2\sqrt{n}]$.

We aim into the following exciting result.

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If you are not very much impressed...

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I can understand it...

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I can understand it...

but you should know that the method we shall present (introduced by BEDNARSKA, ŁUCZAK'99) is the only known method which gives the right order of bias for every H!

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However, since the choice is random, with a very large probability fewer than 1% of edges of $\hat{F} = G(n, M)$ have been claimed by Breaker, i.e. more than 99% of edges of $\hat{F}$ are in Maker's graph!

PROOF

But we know that aas $G(n, M)$ has the property that it contains a triangle in every subgraph which have at least $0.99M$ edges! Thus, the blind random strategy of Maker aas brings him a win!

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This is the end (Adele’12)!
Since only one of the player can have a winning strategy, if Maker has got a strategy that wins sometimes, he has also got a strategy which wins always (since Breaker cannot have it). □

The independence number

Problem

What is the independence number of $G(n, p)$, say, for $p = \log n/n$?

Fact

Let $p = \log n/n$, $\epsilon > 0$ and $k = n \log\log n/\log n$. Then, aas $\alpha(G(n,p)) \leq (2+\epsilon)k$.

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Proof The first moment method. Estimate $\mathbb{E}X$, where X is the number of independent subsets of size $(2+\epsilon)k$. Then

$$\mathbb{E}X = \binom{n}{(2+\epsilon)k}(1-p)^{\binom{(2+\epsilon)k}{2}} \to 0\,.$$

The independence number

Fact

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Proof Surprisingly, this result can also be proved by the first moment method. Estimate $\mathbb{E}Y$, where Y is the number of covering subsets of size $(1-\epsilon)k$. Then

$$\mathbb{E}Y = \binom{n}{(1-\epsilon)k}\left(1-(1-p)^{(1-\epsilon)k}\right)^{n-(1-\epsilon)k} \to 0\,.$$

The independence number

Fact

Let $p = \log n/n$, $\epsilon > 0$ and $k = n \log\log n/\log n$. Then, aas

$$(1-\epsilon)k \leq \alpha(G(n,p)) \leq (2+\epsilon)k \ .$$

Talagrand's inequality

$$P\left(\left|\alpha(G(n,p)) - \mathbb{E}\alpha(G(n,p))\right| \geq t\right) \leq 4\exp\left(-t^2/9k\right) .$$

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The second moment method

Our aim

Let $p = \log n/n$, $\epsilon > 0$ and $k = n \log\log n/\log n$. Then, aas

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Let $p = \log n/n$, $\epsilon > 0$ and $k = n \log\log n/\log n$. Then, aas

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Let X count independent sets of size $(2-\epsilon)k$.

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Let X count independent sets of size $(2-\epsilon)k$.
Two random sets of this size share $\Theta(k^2/n)$ vertices, so we cannot expect that the existence of one set in such a pair is "almost independent" from the existence of the second one.
After some (fairly long) calculations one can show that

$$\mathbb{E}X(X-1) \geq (\mathbb{E}X)^2 \exp\left(\frac{2k}{(\log\log n)^3}\right) .$$

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$$\Pr(X=0) \leq \frac{\mathrm{Var} X}{(\mathbb{E}X)^2} \quad \text{but} \quad \frac{\mathrm{Var} X}{(\mathbb{E}X)^2} \gg 1 \quad \text{(sic!)}$$

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$$\Pr(X>0) \geq \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} \geq \exp\left(-\frac{3k}{(\log\log n)^3}\right)$$

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It seems that the 2nd moment method
is completely useless in this case!

FRIEZE'S IDEA: COMBINE CAUCHY AND TALAGRAND!

The main idea of Frieze's argument

We want to show that aas $\alpha(G(n,p)) \geq (2-3\epsilon)k$.

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Thus, it is enough to show that $\mathbb{E}\alpha(G(n,p))$ is close to $2k$!

Let us assume that this is not the case, i.e. that

$$\mathbb{E}\alpha(G(n,p)) \leq (2-2\epsilon)k$$

and hope to get a contradiction.

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If X counts independent sets of size $(2-\epsilon)k$, then $\Pr(X>0) \geq \exp\big(-3k/(\log\log n)^3\big)$.

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$$
\begin{aligned}
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This is the contradiction we have been hoping for! □

Triangles: some further remarks

(Easy) corollary of large deviation inequalities

If $M = n^{3/2}$, then aas we cannot destroy all triangles in $G(n, M)$ by removing $0.01M$ edges.

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Here is a much harder result.

Theorem Haxell, Kohayakawa, Łuczak'96

If $M = n^{3/2}$, then aas we cannot destroy all triangles in $G(n, M)$ by removing $0.49M$ edges.

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or hypergraph containers (by Saxton, Thomasson and Balogh, Morris, Samotij).

Although this talk was brought to you completely commercial-free...

Theorem Erdős, Rényi'60

If $np \to 0$, then aas $G(n,p)$ contains no triangles.
If $np \to \infty$, then aas $G(n,p)$ contains triangles.

Theorem Erdős, Rényi'59

Let $p(n) = \frac{1}{n}(\ln n + \gamma(n))$. Then

$$\lim_{n\to\infty} \Pr(G(n,p) \text{ is connected}) = \begin{cases} 0 & \text{if } \gamma(n) \to -\infty, \\ 1 & \text{if } \gamma(n) \to \infty. \end{cases}$$

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We say that the property "$G \not\supseteq K_3$" has a coarse threshold, while the property "*G is connected*" has a sharp threshold.

THRESHOLDS

PROBLEM

Can we (combinatorially) characterize graph properties which have sharp thresholds?

THEOREM FRIEDGUT

A property $\mathcal{A}$ has a coarse threshold if it is ‘local’.

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but there exists a nice application to random groups.

Thank you!

Further readings

If you are interested in the subject, there are three books on random graphs you might want to read.

B. Bollobás, *Random graphs*, Cambridge University Press, 2nd edition, 2011.

S. Janson, T. Łuczak, A. Ruciński, *Random graphs*, Wiley, 2000.

A. Frieze, M. Karoński, *Introduction to random graphs*, Cambridge University Press, to be published this year.